Correa [illegible] [illegible]

Bought March [illegible] [illegible]

[redacted]

AN

ELEMENTARY TREATISE

ON THE

DIFFERENTIAL AND INTEGRAL

Calculus.

BY

S. F. LACROIX.

TRANSLATED FROM THE FRENCH.

WITH

AN APPENDIX AND NOTES.

Cambridge:

Printed by J. Smith, Printer to the University;

FOR J. DEIGHTON AND SONS, CAMBRIDGE;

AND LAW & WHITTAKER, NO. 13, AVE-MARIA-LANE, LONDON.

1816.

ADVERTISEMENT.

THE work of Lacroix, of which a Translation is now presented to the Public, forms one of a series of Elementary Treatises, by that distinguished Author, on the different branches of the Pure Mathematics. It may be considered as an abridgement of his great work on the Differential and Integral Calculus, although in the demonstration of the first principles, he has substituted the method of limits of D'Alembert, in the place of the more correct and natural method of Lagrange, which was adopted in the former. The first part of this Treatise, which is devoted to the exposition of the principles of the Differential Calculus, was translated by Mr. Babbage. The translation of the second part, which treats of the Integral Calculus, was executed by Mr. G. Peacock, of Trinity College, and by Mr. Herschel, of St. John's College, in nearly equal proportions. The Appendix of Lacroix, on the Calculus of Differences and Series, has been replaced by an original Treatise, by Mr. Herschel, in which many important subjects are included, which had been either entirely omitted, or very imperfectly considered in the other.

The first twelve of the Notes were written by Mr. Peacock, and were principally designed to enable the Student to make use of the principle of Lagrange, adopting those statements and examples of our author, which do not involve the theory of limits. The others were written by Mr. Herschel.

It is intended to publish a sequel to this Work, containing a collection of examples and results connected with the different subjects considered in it, omitting the entire operations by which they are deduced, and merely indicating such steps in the processes as cannot be expected to be discovered by an ordinary student. The work is already begun, and it is expected to be ready for publication in the course of a few months.

Cambridge,
Dec. 12, 1816.

CONTENTS.

PART I.

DIFFERENTIAL CALCULUS.

Part II.

INTEGRAL CALCULUS.

PART I.

DIFFERENTIAL CALCULUS.

Preliminary Notions, and the Principles of the Differentiation of Functions of one Variable.

1. THE subject of this branch of Analysis is the passage of one or more quantities through different states of magnitude, and the changes which consequently take place in other quantities, whose value depends on that of these first.

2. In order to indicate that a quantity depends on one or several others, either by operations of any kind, or by other relations, which it is impossible to assign algebraically, but whose existence is determined by certain conditions, we call the first quantity a *function* of the others. The use of this word will best illustrate its signification.

3. Any quantity which is considered as changing its value, or as being capable of changing it, is called *Variable*. The name of *Constant* is applied to any quantity which is supposed always to preserve the same value throughout the course of the calculation. From this it is evident, that it is the nature of the question which is proposed that determines what quantities ought to be considered as variable, and what as constant.

4. To illustrate this, we will subjoin a few examples; suppose $u = ax$, a being considered as constant: u is a

function of x of the simplest kind, since it is a quantity proportionable to that variable. If we suppose that x becomes $x+h$, and if we represent by u' the new value of u, we shall have $u'=ax+ah$, from which $u'-u=ah$; and by dividing both sides of the equation by h, it becomes $\frac{u'-u}{h}=a$; that is to say, the ratio of the increment of the function to that of the variable, is independent of their particular values.

Let us take a function a little more complicated, $u=ax^2$; putting $x+h$ for x, it becomes $u'=a(x^2+2hx+h^2)$, and subtracting the first equation from the second, we have $u'-u=2axh+ah^2$: if we then divide both sides by h, it becomes $\frac{u'-u}{h}=2ax+ah$. Here the ratio of the increment of the function to that of the variable is composed of two parts; one of these is independent of the particular value of the quantity h by which x is increased, the other not so. If we conceive this quantity continually to diminish, the result will continually approach to $2ax$, and will only become equal to it by supposing $h=0$; so that $2ax$ is the limit of the ratio $\frac{u'-u}{h}$, or it is, *the value towards which this ratio tends in proportion as the quantity* h *diminishes, and to which it may approach as near as we choose to make it.*

It is easy to perceive, that the difference $u'-u$ is always equal to nothing at the same time with h, inasmuch as it is to this latter quantity only that the former owes its existence; their ratio however is not annihilated: it is one of that species of quantities mentioned in N° 70. of the Elements of Algebra.*

---

* "The Author here refers to his Elemens d' Algebre, which form a part of his Elementary System of Analysis."

Again, making $u=ax^3$, we have by the substitution of $x+h$ instead of x,

$$u'=a(x+h)^3=ax^3+3ax^2h+3axh^2+ah^3;$$

subtracting the first equation from the second, we find

$$u'-u=3ax^2h+3axh^2+ah^3,$$

and taking the ratio of the increments,

$$\frac{u'-u}{h}=3ax^2+3axh+ah^2.$$

We here also find a term independent of the particular value of the increments, and towards which their ratio continually tends while h diminishes; so that this ratio has also a limit.

This first term, or this limit, is not peculiar to the functions we have just examined. It will be met with in every function.

The respective increments of a function and its variable preserve in vanishing the same ratio to which they have gradually approximated; and there exists between this latter and the function from which it is derived a mutual dependance, which determines one from the other, and reciprocally. These assertions will be illustrated and confirmed in a very satisfactory manner by the consideration of curves, as we shall see hereafter (61, 62.)*

5. We shall first make known the signs, by which we express the new relations which the preceding notions establish among magnitudes. To explain the principles upon which they are assumed, let us resume the function $u=ax^3$, already considered in N° 4.

In substituting in it $x+h$ for x, and subtracting ax^3 from the result, we have obtained in the expression

$$u'-u=3ax^2h+3axh^2+ah^3,$$

the developement of the *difference* of the two states of the function u, arranged according to the powers of the quantity

---

* See Note A.

h, by which we have supposed the variable x to be increased; and the limit of the ratio of the increment $u'-u$ and h only depends on the first term of this difference (4.) This first term, which is only a part of the difference, is called the *Differential*, and is denoted by du, using the letter d as a characteristic; we have therefore, in the example proposed, $du = 3ax^2h$.

To pass from this to $3ax^2$, which is the limit required, we must divide by h, and we shall thus get $\frac{du}{h} = 3ax^2$; but when we consider a simple variable x only, as this quantity is changed into $x'=x+h$, we have $x'-x=h$; the difference and the differential are here the same thing: consequently we replace the quantity h by dx, in order to preserve uniformity in our calculations, and there arises

$$du = 3ax^2dx, \quad \frac{du}{dx} = 3ax^2;$$

the first expression will be the differential of u, or of ax^3, and the second, which expresses the limit of the ratio of the simultaneous changes of the function and its variable, will take the name of the *Differential Coefficient*; because the quantity which it expresses is nothing else than the multiplier of the differential dx, in the expression for du. From hence it follows, that *the limit of the ratio of the increments, or the differential coefficient, will be obtained by dividing the differential of the function by that of the variable;* and reciprocally, *we shall obtain the differential, by multiplying the limit of the ratio of the increments, or the differential coefficient, by the differential of the variable.*

This remark is important, because there are some functions whose differential coefficients can be found more readily than their differentials. In fact, to arrive immediately at this last, *we must write* $x+dx$ *instead of* x *in the proposed function, develope the result according to the powers of* dx, *stopping at the term affected by the first power of it, and then subtract from the result the primitive expression.* It is

obvious, that this method supposes that we are acquainted with some means of expanding the proposed function; this may possibly require some foreign aid, with which the consideration of limits most frequently dispenses.

According to these various considerations, *the differential calculus is the finding the limit of the ratios of the simultaneous increments of a function, and of the variable on which it depends.*

6. We must take care not to confound the differential with the difference $u'-u$. In the example N° 4. the one is $3ax^2h$, and the other

$$3ax^2h + 3axh^2 + ah^3;$$

but when the quantity h is very small, the differential $3ax^2h$ forms the most considerable part of the difference $u'-u$, and the differential approaches more and more nearly to the difference in proportion as h diminishes. In general, *the error arising from taking the differential instead of the difference, will be so much the less, the smaller we suppose the increment of the variable to be.* The same consequence may also be drawn from the consideration of limits; for if the ratio of the simultaneous increments $u'-u$ and h has for its limit a function p, it will continually approach towards it; and the equation $\frac{u'-u}{h}=p$ will be so much more exact as the increment h is smaller, and on this hypothesis $u'-u=ph$. *

It is proper to remark, that when the result of the substitution $x+h$ is expanded according to the powers of h in the form

$$U + ph + qh^2 + \&c.$$

the first term U is the primitive value of the proposed function, since it is to this term, that the above expression

* It is on this principle, that Leibnitz founded the Differential Calculus, considering differentials as infinitely small differences.

is reduced, when we make $h=0$, which is nothing more than supposing that x has not changed.

7. It is easy to perceive, that two equal functions have equal differentials; for whilst two functions are equal, whatever may be the value of the variable on which they depend, it must necessarily happen, that the respective changes which they receive in consequence of that which is attributed to this variable, must also be equal. If for example, u and v are two functions, such that $u=v$ whatever may be the value of x, and that when x becomes $x+dx$, u is changed into u', and v into v', we shall have $u'=v'$: subtracting the former equation from this, there will result

$$u'-u=v'-v;$$

then dividing by dx, we shall get

$$\frac{u'-u}{dx}=\frac{v'-v}{dx}.$$

If then p and q denote the limits of these ratios, we have $p=q$, from which we may conclude that $pdx=qdx$, and $du=dv$, by observing that according to N° 5. pdx and qdx are the differentials of the functions u and v.

The inverse of this proposition is not generally true, and we should be wrong in affirming, that two equal differentials belong to equal functions. In fact, if we had $a+bx$, and should substitute $x+dx$ for x, we should obtain $a+bx+bdx$; subtracting $a+bx$ from this, we should find bdx; a result in which there is no trace of the constant quantity a. The differential bdx belongs then equally to $a+bx$, and to bx; and in general it belongs to all the different values presented by the function $a+bx$, which arise from assigning to a every possible value: from hence it may be seen, that in differentiating any function whatever, all the constant quantities combined with it either by addition or by subtraction disappear. With respect to those which are connected with it either by multiplication or division, they always remain as coefficients or divisors.

8. Before we proceed to find the differentials of quantities by means of their limits, it must be remarked, 1st. *that the limit of the product of two quantities which vary together is the product of their corresponding limits*; 2dly, *that the limit of the quotient of the same quantities is also the quotient of their limits.*

Let P and Q be the two quantities proposed, and p and q their corresponding limits; the former of these when considered in their general state may be represented by $p+\alpha$ and $q+\beta$, α and β denoting quantities which vanish at the same time after having passed through every stage of successive diminution (4): we have then in general,

$$PQ=(p+\alpha)(q+\beta)=pq+p\beta+q\alpha+\alpha\beta$$

the second member of this equation is reduced to pq, when in order to take the limits we make $\alpha=0$, $\beta=0$. It may also be observed, that by assigning to α and β proper values we may reduce to as small a quantity as we please the difference

$$PQ-pq=p\beta+q\alpha+\alpha\beta.$$

The quotient $$\frac{P}{Q}=\frac{p+\alpha}{q+\beta}$$

being put under the form

$$\frac{P}{Q}=\frac{p}{q}+\frac{p+\alpha}{q+\beta}-\frac{p}{q}$$

becomes, by the reduction of the two last fractions to the same denominator,

$$\frac{P}{Q}=\frac{p}{q}+\frac{q\alpha-p\beta}{q(q+\beta)}.$$

The numerator of the last fraction of this result vanishes when α and β become nothing, after passing through every degree of diminution, whilst the denominator constantly approaches to q^2. Thus the limit of $\frac{P}{Q}$ is reduced to $\frac{p}{q}$,

and the difference

$$\frac{P}{Q} - \frac{p}{q} = \frac{q\alpha - p\beta}{q(q+\beta)}$$

may be rendered as small as we please.

9. By means of the preceding remarks, we may obtain the differential coefficient of a function with respect to a variable on which it does not immediately depend. If for example, three quantities v, u, x, which are such that the first is a function of the second, and this also is a function of the third, pass simultaneously to the new states of magnitude represented by v', u', x', or take the respective increments

$$v'-v, \quad u'-u, \quad x'-x,$$

the ratios of these increments being

$$\frac{v'-v}{u'-u}, \quad \frac{u'-u}{x'-x},$$

and their limits

$$\frac{dv}{du} = p, \quad \frac{du}{dx} = q,$$

it may be concluded from the first of the preceding remarks, that the limit of

$$\frac{v'-v}{u'-u} \times \frac{u'-u}{x'-x} \quad \text{or of} \quad \frac{v'-v}{x'-x}$$

is pq, and that consequently

$$\frac{dv}{dx} = pq = \frac{dv}{du} \times \frac{du}{dx}.$$

When the increment $u'-u$ is successively compared to $x'-x$, and to $v'-v$, and instead of the ratios

$$\frac{u'-u}{x'-x}, \quad \text{and} \quad \frac{u'-u}{v'-u}$$

we have the limits

$$\frac{du}{dx} = p, \quad \frac{du}{dv} = q,$$

we may conclude from the second remark, that the limit of

$$\frac{\frac{u'-u}{x'-x}}{\frac{u'-u}{v'-v}}, \text{ or of } \frac{v'-v}{x'-x},$$

is $\frac{p}{q}$; and consequently, that

$$\frac{dv}{dx} = \frac{p}{q} = \frac{\frac{du}{dx}}{\frac{du}{dv}}.$$

When two quantities u and x are connected together by a mutual dependance, we may indifferently call u a function of x, or x a function of u, according as u is considered as determined by x, or x by u. The differential coefficient may also be presented under each of these different points of view. If in the first case we have $\frac{du}{dx} = p$, it is evident that we ought in the second to have $\frac{dx}{du} = \frac{1}{p}$.

10. Let us now apply what has preceded, to the finding of the differentials of those functions which occur in the Elements of Algebra, such as the sums, differences, products, quotients, powers and roots of algebraic quantities. In the first place, when several quantities dependent on x, whose differentials we know how to find, are joined together by addition or subtraction, as in this example $u + v - w$, if the substitution of $x + dx$, instead of x would change u into $u + \alpha$, v into $v + \beta$, and w into $w + \gamma$, the expression $u + v - w$ will become

$$u + v - w + \alpha + \beta - \gamma.$$

The change which it has undergone, consisting of the terms $\alpha + \beta - \gamma$, when compared with the increment dx of the variable x, gives

$$\frac{\alpha}{dx} + \frac{\beta}{dx} - \frac{\gamma}{dx},$$

a quantity whose limit will be

$$p + q - r,$$

denoting by p, q and r the limits of the respective ratios $\frac{\alpha}{dx}$, $\frac{\beta}{dx}$, and $\frac{\gamma}{dx}$; and multiplying by dx the quantity $p + q - r$, the result is $p\,dx + q\,dx - r\,dx$, which will be the differential of the proposed function: but $p\,dx$, $q\,dx$, and $r\,dx$ are respectively the differentials of the functions u, v, and w, and denoting them by du, dv, and dw, we find

$$d(u + v - w) = du + du - dw,$$

that is to say, *the differential of a function of* x, *composed of several terms may be obtained by taking the differential of each term with the sign which belongs to that term.*

11. Secondly, if in the product of two functions u and v; u is changed into $u + \alpha$, and v into $v + \beta$, this product becomes

$$uv + u\beta + v\alpha + \alpha\beta;$$

and its increment

$$u\beta + v\alpha + \alpha\beta,$$

being compared with dx, gives the expression

$$u\frac{\beta}{dx} + \frac{\alpha}{dx} + \frac{\alpha}{dx}\beta.$$

Denoting, as before, the respective limits of the ratio $\frac{\alpha}{dx}$, and $\frac{\beta}{dx}$ by p and q; and observing that the increment β vanishes at the same time with dx, of which the quantities u and v are in other respects independent; it is obvious that the limit of the term $\frac{\alpha}{dx}\beta$ is equal to nothing (8), whilst that of the two other terms is

$$uq + vp.$$

From this we may conclude, that the differential of uv is

$$uq\,dx + vp\,dx;$$

but $q\,dx$ and $p\,dx$ are equivalent to dv and du: therefore $d\,.\,uv = u\,dv + v\,du$ (*).

From this formula we learn, *that to obtain the differential of the product of two functions, we must multiply each one by the differential of the other and add together the two results.*

If we divide each side of the equation,

$$d\,.\,uv = u\,dv + v\,du$$

by the original function uv, we have

$$\frac{d\,.\,uv}{uv} = \frac{du}{u} + \frac{dv}{v};$$

This easily leads to the expression for the differential of a product composed of any number of factors. For suppose $v = ts$, there results

$$\frac{dv}{v} = \frac{d\,.\,ts}{ts} = \frac{dt}{t} + \frac{ds}{s},$$

and consequently,

$$\frac{d\,.\,uts}{uts} = \frac{du}{u} + \frac{dt}{t} + \frac{ds}{s}$$

and in the same manner we should find

$$\frac{d\,.\,utsr\ldots\ \&c.}{utsr\ldots\ \&c.} = \frac{du}{u} + \frac{dt}{t} + \frac{ds}{s} + \frac{dr}{r} + \&c.$$

If we take away the denominators from the equation

$$\frac{d\,.\,uts}{uts} = \frac{du}{u} + \frac{dt}{t} + \frac{ds}{s},$$

we find $d\,.\,uts = ts\,du + us\,dt + ut\,ds$; and we shall easily see, *that whatever may be the number of the factors, the differential of the product will always be equal to the sum of the products of the differential of each multiplied by all the others.*

---

* When a point is placed after the characteristic d it signifies, that the operation is to be performed on all that immediately follows it; thus $d\,.\,uv$ is the same thing as $d\,(uv)$, and $d\,.\,x^n$ the same as $d\,(x^n)$.

12. The differential of $\frac{u}{v}$ may be obtained by making $\frac{u}{v} = t$; for $u = vt$, and from what precedes, $du = vdt + tdv$; eliminating the value of dt, and substituting instead of t the fraction $\frac{u}{v}$, we shall have $dt = \frac{du}{v} - \frac{udv}{v^2}$, or reducing the fractions to the same denominator

$$dt = \frac{vdu - udv}{v^2};$$

hence this rule: *to find the differential of a fraction, multiply the denominator by the differential of the numerator, subtract from this product that of the numerator multiplied by the differential of the denominator, and divide the whole by the square of the denominator.*

When the numerator of the proposed fraction is constant, u being independent of x, has no differential; that is to say, $du = 0$; and there remains only

$$dt = -\frac{udv}{v^2}.$$

13. The function x^n denoting, when n is a whole positive number, the product of a number (n) of factors, each equal to x, we may deduce from No. 11.

$$\frac{d.x^n}{x^n} = \frac{d.xxx\ldots}{xxx\ldots}$$

$$= \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x} + \&c.$$

the number of factors on the first side of the equation being n, the second will be composed of an equal number of terms, each equal to $\frac{dx}{x}$; we have therefore

$$\frac{d.x^n}{x^n} = \frac{ndx}{x},$$

from which we get $d.x^n = nx^{n-1}dx$.

If the number n be fractional, let it be represented by $\frac{r}{s}$, and make $x^{\frac{1}{s}} = v$, whence $x = v^s$; then successively $u = x^r$, and $u = v^r$, the numbers r and s being supposed integral and we have from the preceding,

$$\frac{du}{dx} = r x^{r-1}, \text{ and } \frac{du}{dv} = s v^{s-1};$$

from which we find, by No. 9.

$$\frac{dv}{dx} = \frac{r x^{r-1}}{s v^{s-1}} = \frac{r x^{r-1}}{s x^{\frac{1}{s}(s-1)}},$$

and by reduction,

$$\frac{dv}{dx} = \frac{r}{s} x^{\frac{r}{s}-1},$$

and consequently,

$$dv = \frac{r}{s} x^{\frac{r}{s}-1} dx,$$

which is in fact the same as $d \,.\, x^n = n x^{n-1} dx$, n being equal to $\frac{r}{s}$.

If the number n be negative, we have $x^{-n} = \frac{1}{x^n}$ from which we find by the principle in No. 12.

$$d.x^{-n} = \frac{-d.x}{x^{2n}}$$

and since we have just proved that $d \,.\, x^n = n x^{n-1} dx$, in all cases where n is positive, we have

$$d \,.\, x^{-n} = \frac{-n x^{n-1} dx}{x^{2n}} = -n x^{-n-1} dx.$$

From this enumeration we may conclude, *that in order to differentiate any power whatever of a variable quantity, we must multiply it by its exponent, diminish the exponent by unity and multiply the result by the differential of the variable.**

14. The rules delivered in Nos. 10, 11, 12, 13, are sufficient for differentiating all functions in which the variable is only involved by addition, subtraction, multiplication, division, or by powers whose indices are integral or fractional, positive or negative. Functions which result from algebraical operations are from that circumstance called algebraical functions. We shall differentiate a few such, as exemplifications of these rules.

In the first place, let $u = a + b\sqrt{x} - \frac{c}{x}$; taking the differential of each term of this function separately, the first disappears, because it is constant (7); the second put under the form $b\,x^{\frac{1}{2}}$, gives, (7) by the application of the rule in No. 13, $\frac{1}{2}b\,x^{\frac{1}{2}-1}\,dx$, or $\frac{b\,dx}{2\sqrt{x}}$; the third or $-\frac{c}{x}$, leads to $+\frac{c\,dx}{x^2}$ (13): collecting together these results, we find

$$du = \left(\frac{b}{2\sqrt{x}} + \frac{c}{x^2}\right)dx, \text{ and } \frac{du}{dx} = \frac{b}{2\sqrt{x}} + \frac{c}{x^2},$$

Secondly, let

$u = a + \frac{b}{\sqrt[3]{x^2}} - \frac{c}{x\sqrt[3]{x}} + \frac{e}{x^3}$: if we put this function

---

* This rule might have been immediately deduced from the developement of the binomial $(x + dx)^n$, since that developement being $x^n + n\,x^{n-1}dx +$ &c. if we subtract x^n, the first term of the difference will be $n\,x^{n-1}\,dx$; but it is more convenient not to pre-suppose the demonstration of the formula for the binomial, because the differential calculus furnishes a very general and a very simple one.

into the form

$$u = a + b x^{-\frac{2}{3}} - c x^{-1-\frac{1}{3}} + e x^{-2},$$

the application of the rule in No. 13. will give

$$d u = -\frac{2}{3}\frac{b d x}{x^{\frac{5}{3}}} + \frac{4}{3}\frac{c d x}{x^{\frac{7}{3}}} - \frac{2 e d x}{x^3},$$

which becomes

$$d u = -\frac{2 b d x}{3 x \sqrt[3]{x^2}} + \frac{4 c d x}{3 x^2 \sqrt[3]{x}} - \frac{2 e d x}{x^3}.$$

15. The examples in the preceding No. consist of collections of single terms; but there are many functions which cannot, without a previous developement, be reduced to this form; such is the function $u = (a + b x^m)^n$. In this case put $a + b x^m = z$; from which we have $u = z^n$; and observing that $d . z^n = n z^{n-1} d z$ (13),

we shall get $\quad \dfrac{d u}{d x} = \dfrac{d u}{d z} \times \dfrac{d z}{d x} = n z^{n-1} \dfrac{d z}{d x};$

but $\quad d z = d (a + b x^m) = d . b x^m = m b x^{m-1} d x;$

therefore $\quad \dfrac{d u}{d x} = n (a + b x^m)^{n-1} \times m b x^{m-1},$

and $\quad d u = n m b x^{m-1} d x (a + b x^m)^{n-1}.$

It is proper to remark, that this method is nothing more than *first differentiating the expression for* u *relative to* z, *and then substituting the values of* z *and* d z, *in terms of* x *and* d x.

If we had $u = \sqrt{a + b x + c x^2}$, we should consider the trinomial $a + b x + c x^2$, as a particular function z, and the differential of $\sqrt{z}$ or of $z^{\frac{1}{2}}$ being $\frac{1}{2} z^{-\frac{1}{2}} d z$, or $\dfrac{d z}{2\sqrt{z}}$, there would result

$$d u = \frac{d . (a + b x + c x^2)}{2\sqrt{a + b x + c x^2}} = \frac{b d x + 2 c x d x}{2\sqrt{a + b x + c x^2}}.$$

As we have frequent occasion to differentiate radicals of the second degree, it is well to notice that, according to the formula $\frac{dz}{2\sqrt{z}}$, *the differential of a radical of the second degree is obtained by dividing that of the quantity contained under the sign by double the radical itself.*

16. The rule given (11) for differentiating products, being applied to the function

$u = x(a^2 + x^2)\sqrt{a^2 - x^2}$, leads to

$$du = dx\,(a^2 + x^2)\sqrt{a^2 - x^2} + x\sqrt{a^2 - x^2}\,.\,d\,.\,(a^2 + x^2)$$
$$+ x(a^2 + x^2)\,d\,.\sqrt{a^2 - x^2}\,.$$

The two last terms of this expression include operations which are only indicated, but which may be easily effected, by observing that

$$d(a^2 + x^2) = d\,.\,x^2 = 2x\,dx,$$

$$d\sqrt{a^2 - x^2} = \frac{d(a^2 - x^2)}{2\sqrt{a^2 - x^2}} = \frac{-x\,dx}{\sqrt{a^2 - x^2}};$$

and we thus find

$$du = \left\{(a^2 + x^2)\sqrt{a^2 - x^2} + 2x^2\sqrt{a^2 - x^2} - \frac{x^2(a^2 + x^2)}{\sqrt{a^2 - x^2}}\right\} dx:$$

or, by reducing all the terms to the same denominator,

$$du = \frac{(a^4 + a^2x^2 - 4x^4)\,dx}{\sqrt{a^2 - x^2}}.$$

The rule for the differentiation of fractions being applied to the function $u = \frac{a^2 - x^2}{a^4 + a^2x^2 + x^4}$, gives immediately,

$$du = \frac{(a^4 + a^2x^2 + x^4)\,d\,.(a^2 - x^2) - (a^2 - x^2)\,d\,.\,(a^4 + a^2x^2 + x^4)}{(a^4 + a^2x^2 + x^4)^2}$$

whence we deduce

$$du = \frac{-2x(2a^4 + 2a^2x^2 - x^4)\,dx}{(a^4 + a^2x^2 + x^4)^2}.$$

We shall terminate these examples with the function

$$u = \sqrt[4]{\left(a - \frac{b}{\sqrt{x}} + \sqrt[3]{(c^2 - x^2)^2}\right)^3},$$

which involves several algebraical operations to be effected successively. To facilitate the differentiation, we may put $\dfrac{b}{\sqrt{x}} = y$, $\sqrt[3]{(c^2 - x^2)^2} = z$, then

$$u = \sqrt[4]{(a - y + z)^3} = (a - y + z)^{\frac{3}{4}};$$

which, by No. 13, gives

$$du = \tfrac{3}{4}(a - y + z)^{\frac{3}{4}-1}\,d.(a - y + z)$$

$$= \tfrac{3}{4}(a - y + z)^{-\frac{1}{4}}(-dy + dz)$$

$$= \frac{-3\,dy + 3\,dz}{4\sqrt[4]{a - y + z}}:$$

we also have

$$dy = d.\frac{b}{\sqrt{x}} = -b\,\frac{d.\sqrt{x}}{x} = \frac{-b\,dx}{2x\sqrt{x}},$$

$$dz = d.(c^2 - x^2)^{\frac{2}{3}} = \frac{2}{3}(c^2 - x^2)^{\frac{2}{3}-1}d\,(c^2 - x^2)$$

$$= \frac{2}{3}(c^2 - x^2)^{-\frac{1}{3}} \times -2x\,dx = \frac{-4x\,dx}{3\sqrt[3]{c^2 - x^2}};$$

and substituting these values and also those of y and z in the expression for du, we shall find

$$du = \left\{\frac{\dfrac{3b}{2x\sqrt{x}} - \dfrac{4x}{\sqrt[3]{c^2 - x^2}}}{4\sqrt[4]{a - \dfrac{b}{\sqrt{x}} + \sqrt[3]{(c^2 - x^2)^2}}}\right\}dx^*.$$

---

* See note (B.)

On successive Differentiations.

17. The differential coefficient being a new functio of x, may itself be differentiated, and we may find from th limit of the ratio of its increment to that of the variable a its own differential coefficient, which will also be a functio of x. By continuing these differentiations one after an other, we deduce from the proposed function a series c limits or of differential coefficients, which are distinguishe into orders, according to the number of differentiation which have taken place in order to obtain them.

If we make $\frac{du}{dx} = p$, $\frac{dp}{dx} = q$, $\frac{dq}{dx} = r$, &c.

p will represent the differential coefficient of the first orde of the proposed function; q represents that of p, or it i the differential coefficient of the second order of the pro posed function; r is that of q, or the coefficient c the third order of the proposed function, &c.; and i should be observed, that the coefficients q, r, &c. are de duced from the successive differentials of du, taken o the hypothesis of the increment dx being constant. Thes differentials are thus written:

$$d(du) = ddu = d^2u, \quad d(d^2u) = d^3u, \text{ \&c.}$$

the exponent which accompanies the characteristic d indi cates the repetition of an operation, and not a power c the letter d, which is never considered as a quantity, bu merely as a sign. This being supposed, the equations

$$\frac{du}{dx} = p, \quad \frac{dp}{dx} = q, \quad \frac{dq}{dx} = r, \text{ \&c.}$$

will give

$$du = pdx, \quad dp = qdx, \quad dq = rdx, \text{ \&c.}$$

differentiating again the first of these without supposin dx variable, it becomes $d^2u = dp\,dx$, and putting fo

dp its value deduced from the second, we find $d^2u = q\,dx^2$*, whence $q = \frac{d^2u}{dx^2}$. Differentiating the equation $d^2u = q\,dx^2$ again, we have $d^3u = dq\,dx^2$, and since $dq = r\,dx$, there results $d^3u = r\,dx^3$, or $r = \frac{d^3u}{dx^3}$: we find therefore that

$$p = \frac{du}{dx}, \quad q = \frac{d^2u}{dx^2}, \quad r = \frac{d^3u}{dx^3}, \text{ \&c.}$$

18. If, for example, the proposed function was ax^n, we should find $d.ax^n = nax^{n-1}dx$ (13); the factors na and dx being considered as constant in the first differential $nax^{n-1}dx$, it is sufficient (7) in order to obtain the second differential, to differentiate x^{n-1} and to multiply the result by $na\,dx$; but $d.x^{n-1} = (n-1)\,x^{n-2}dx$: we have therefore $d^2.ax^n = n(n-1)\,ax^{n-2}dx^2$.

In a similar manner we should find

$$d^3.ax^n = n(n-1)(n-2)\,ax^{n-3}dx^3,$$
$$d^4.ax^n = n(n-1)(n-2)(n-3)\,ax^{n-4}dx^4,$$
&c. &c.

and the differential coefficients would have the following values:

$$\frac{d.ax^n}{dx} = nax^{n-1}$$

$$\frac{d^2.ax^n}{dx^2} = n(n-1)\,ax^{n-2}$$

$$\frac{d^3.ax^n}{dx^3} = n(n-1)(n-2)\,ax^{n-3}$$

$$\frac{d^4.ax^n}{dx^4} = n(n-1)(n-2)(n-3)\,ax^{n-4}$$

&c. &c.

---

* We must remember that the expressions dx^2, dx^3, .. &c. are equivalent to $(dx)^2$, $(dx)^3$, &c. and not to $d.x^2$, $d.x^3$, &c. (See note, page 11.)

It is obvious, that when the exponent n is a whole positive number, the function $a x^n$ has only a limited number of differentials, the highest of which is

$$d^n . a x^n = n(n-1)(n-2) \dots 2 . 1 . a d x^n.$$

This expression is no longer capable of differentiation, since it no longer involves a variable quantity: we have then for the last differential coefficient

$$\frac{d^n . a x^n}{d x^n} = n(n-1)(n-2) \dots 2 . 1 . a,$$

which is a constant quantity.

19. This remark furnishes a very simple method of developing in a series arranged according to the whole positive powers of x, any function u of that variable, provided the developement can be effected. If we assume the equation

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$$

by differentiating, we find

$$\frac{du}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \&c.$$

$$\frac{d^2u}{dx^2} = 1.2C + 2.3Dx + 3.4Ex^2 + \&c.$$

$$\frac{d^3u}{dx^3} = 1.2.3D + 2.3.4Ex + \&c.$$

&c.

And if we also have expressions in terms of x, for the following quantities

$$u, \quad \frac{du}{dx}, \quad \frac{d^2u}{dx^2}, \quad \frac{d^3u}{dx^3}, \quad \&c.$$

then denoting by U, U', U'', U''', &c. their values when $x = 0$, we shall deduce from the preceding equations by making in them also $x = 0$,

$$A = U,\ B = \frac{1}{1} U',\ C = \frac{1}{1.2} U'',\ D = \frac{1}{1.2.3} U''', \&c.$$

whence

$$u = U + U' \frac{x}{1} + U'' \frac{x^2}{1.2} + U''' \frac{x^3}{1.2.3} + \&c.$$

20. If we take $u = (a + x)^n$, we shall have

$$\frac{du}{dx} = n(a+x)^{n-1}, \quad \frac{d^2u}{dx^2} = n(n-1)(a+x)^{n-2},$$

$$\frac{d^3u}{dx^3} = n(n-1)(n-2)(a+x)^{n-3}, \text{ \&c.}$$

and making $x = 0$, we find

$$U = a^n, \; U' = n a^{n-1}, \; U'' = n(n-1) a^{n-2},$$

$$U''' = n(n-1)(n-2) a^{n-3}, \text{ \&c.}$$

from whence we have

$$(a+x)^n = a^n + \frac{n}{1} a^{n-1} x + \frac{n(n-1)}{1 \cdot 2} a^{n-2} x^2$$

$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3} x^3 + \text{\&c.}$$

The principles of differentiation having been deduced without assuming the developement of $(a + x)^n$, we may consider this developement as proved for all cases, whether the exponent be integral or fractional, positive or negative.

21. The same method leads us to express, by means of the differential coefficients, the value which any function u assumes when we substitute $x + h$ instead of x. In this case the function u when it is changed into u' by this substitution, may be considered as a function of h, and we shall have, from what has preceded,

$$u' = U + U\frac{h}{1} + U' \frac{h^2}{1 \cdot 2} + U''' \frac{h^3}{1 \cdot 2 \cdot 3} + \text{\&c.}$$

if U, U', U'', U''', &c.

denote the values of

$$u', \; \frac{du'}{dh}, \; \frac{d^2u'}{dh^2}, \; \frac{d^3u'}{dh^3}, \text{ \&c.}$$

when we make $h = 0$.

It is evident, that u' becomes u, when $h = 0$, and consequently, that $U = u$; but, moreover, the differential

coefficients formed by considering h as variable, and x as constant, are the same as those which would be found by treating x as variable, and h as constant. In order to prove this, let $x+h=x'$; the function u' will be composed of x', just in the same manner as u is of x: we have therefore $du'=p'dx'$, p' being a function of x', and $dx'=d(x+h)$. If we suppose h alone to vary, we shall have

$$dx'=dh, \quad du'=p'dh \text{ and } \frac{du'}{dh}=p'.$$

If x alone be supposed to vary, we obtain

$$dx'=dx, \quad du'=p'dx \text{ and } \frac{du'}{dx}=p':$$

therefore $\frac{du'}{dh}=\frac{du'}{dx}$. Again, the function p' being itself a function of x', we have also

$$\frac{dp'}{dh}=\frac{dp'}{dx}, \quad \text{whence} \quad \frac{d^2u'}{dh^2}=\frac{d^2u'}{dx^2},$$

and generally

$$\frac{d^mu'}{dh^m}=\frac{d^mu'}{dx^m}.$$

This being premised, when $h=0$, u' is changed into u; and there results

$$U'=\frac{du}{dx}, \quad U''=\frac{d^2u}{dx^2}, \quad U'''=\frac{d^3u}{dx^3}, \text{ \&c.}$$

and $$u'=u+\frac{du}{dx}\cdot\frac{h}{1}+\frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3}+\text{\&c.}$$

This formula is called *Taylor's Theorem*, from the name of the English geometer, by whom it was first discovered.*

---

* I shall add in this place another demonstration of this formula. Having proved as above, that

du'

It contains implicitly the developement of the binomial, for if we suppose $u = x^n$, u' will become $(x+h)^n$, and we shall have

$$(x+h)^n = x^n + n x^{n-1}\frac{h}{1} + n(n-1)\,x^{n-2}\frac{h^2}{1.2}$$
$$+ n(n-1)(n-2)\,x^{n-3}\frac{h^3}{1.2.3} + \&c.$$

22. The formula of Taylor shews, that the various differential coefficients possess the remarkable property of forming, when they are respectively divided by the products

1, 1 . 2, 1 . 2 . 3, &c.

the coefficients of the powers of the increment h, in the complete developement of the difference

$$u' - u = \frac{du}{dx}\cdot\frac{h}{1} + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2} + \&c.$$

$\frac{du'}{dh} = \frac{du'}{dx}$, if we make $u' = A + Bh + Ch^2 + \&c.$
and if we suppose the coefficients A, B, C, &c. not to contain h, they depend only on the variable x, and on the constant quantities which enter into the proposed function; we have therefore

$$\frac{du'}{dh} = B + 2\,Ch + 3\,Dh^2 + \&c.$$

$$\frac{du'}{dx} = \frac{dA}{dx} + \frac{dB}{dx}h + \frac{dC}{dx}h^2 + \&c.$$

and equating each term of these two results, we find

$$B = \frac{dA}{dx}, \quad C = \frac{1}{2}\frac{dB}{dx}, \quad D = \frac{1}{3}\frac{dC}{dx}, \ \&c.$$

but $A = u$, therefore

$$B = \frac{du}{dx}, \quad C = \frac{1}{1.2}\frac{d^2u}{dx^2}, \quad D = \frac{1}{1.2.3}\frac{d^3u}{dx^3} \ \&c.$$

This demonstration was presented to me at a public examination at one of the principal seminaries in Paris.

This developement, when we make $h = dx$, becomes (17)

$$u' - u = \frac{du}{1} + \frac{d^2u}{1.2} + \frac{d^3u}{1.2.3} + \&c.$$

This is a very simple formula, and shows clearly in what manner the difference of u corresponding to any increment dx, is composed of the differentials of various orders relative to the same increment*.

On the Differentiation of Transcendental Functions.

23. Those functions which are not comprehended in the enumeration made in No. 14, are called transcendents. The exponential function $u = a^x$ is the most simple of this sort. When we substitute $x + dx$ instead of x, the difference becomes

$$a^{x+dx} - a^x = a^x(a^{dx} - 1);$$

and in order to express it according to the powers of dx, we make $a = 1 + b$, when it becomes

$$a^{dx} = (1+b)^{dx} = 1 + \frac{dx}{1}b + \frac{dx(dx-1)}{1.2}b^2$$
$$+ \frac{dx(dx-1)(dx-2)}{1.2.3}b^3 + \&c.$$

Whence

$$a^{dx} - 1 = \left\{\frac{dx}{1}b + \frac{dx(dx-1)}{1.2}b^2 + \frac{dx(dx-1)(dx-2)}{1.2.3}b^3 + \&c.\right\},$$

and arranging this according to the powers of dx,

$$a^{dx} - 1 = dx\left(\frac{b}{1} - \frac{b^2}{2} + \frac{b^3}{3} - \&c.\right)$$
$$+ \&c.$$

---

* See note (C.)

replacing b by its value $a - 1$, there results (5)

$$d \, . \, a^x = a^x dx \left(\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{\&c.} \right);$$

and making

$$k = \frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{\&c.}$$

we have $d \, . \, a^x = k a^x dx$.

This is the form of the differential of the proposed function, and we shall soon find a new expression for the constant quantity k.

24. It is evident, that

$$\begin{aligned} d^2 . a^x &= k\,dx\,d\,.\,a^x = k^2 a^x dx^2 \\ d^3 . a^x &= k^3 a^x dx^3 \\ &\cdots\cdots\cdots\cdots \\ d^n . a^x &= k^n a^x dx^n; \end{aligned}$$

and therefore

$$\frac{du}{dx} = k a^x, \quad \frac{d^2 u}{dx^2} = k^2 a^x, \quad \frac{d^3 u}{dx^3} = k^3 a^x, \text{ \&c.}$$

When $x = 0$, the function u and its differential coefficients become

$$U = 1, \quad U' = k, \quad U'' = k^2, \quad U''' = k^3, \text{ \&c.}$$

We have, therefore, (19)

$$a^x = 1 + \frac{kx}{1} + \frac{k^2 x^2}{1 \, . \, 2} + \frac{k^3 x^3}{1 \, . \, 2 \, . \, 3} + \text{\&c.}$$

25. The developement of the function a^x, just found, will be of use in discovering from what quantity the series represented by k derives its origin.

If we suppose $x = 1$, we find

$$a = 1 + \frac{k}{1} + \frac{k^2}{1 \, . \, 2} + \frac{k^3}{1 \, . \, 2 \, . \, 3} + \text{\&c.}$$

this series not being convenient for discovering a by means of k, let us enquire into the value of a, when $k = 1$, and denoting it by e, we shall find

$$e = 1 + \frac{1}{1} + \frac{1}{1 \, . \, 2} + \frac{1}{1 \, . \, 2 \, . \, 3} + \text{\&c.}$$

Adding together the ten first terms, we have

$$e = 2.7182818.$$

This being premised, since this value of e corresponds to that of $k = 1$, it follows, that

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

and similarly,

$$e^k = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \&c.$$

and consequently, $e^k = a$. If we take the logarithms on both sides, we have

$$k \log. e = \log. a, \text{ or } k = \frac{\log. a}{\log. e},$$

we have, therefore,

$$d.a^x = \frac{\log. a}{\log. e} a^x dx.$$

26. We may now arrive at the differential of a logarithmic function. If we call a the base of the system, y the number, and x the logarithm, we have, from the principles of Algebra, the equation $y = a^x$; considering x as a function of y, and taking the differentials of each member, we find $dy = a^x k\, dx$, whence we deduce (9)

$$dx = \frac{dy}{a^x k},$$

or replacing a^x and k by their values y and $\frac{1}{\log. e}$, since a is the base of the proposed system of logarithms, we have

$$d.\log. y = \frac{dy}{y} \log. e.$$

27. The number e frequently occurs in analytical enquiries; it is taken for the base of a system of logarithms, which we shall call Naperian, from the name of Naper their inventor:* and which we shall at present denote by

* These logarithms were known under the very improper name of natural or hyperbolic logarithms.

the characteristic log.'; we have then $\log'.e = 1$, $k = \log.' a$, and the results in the preceding articles become

$$a^x = 1 + \frac{x \log.' a}{1} + \frac{x^2 (\log.' a)^2}{1.2} + \frac{x^3 (\log.' a)^3}{1.2.3} + \&c.$$

$$d.a^x = a^x dx \log.' a \text{ (25)}, \quad d.\log.' y = \frac{dy}{y} \text{ (26)}.$$

In order to pass from the system whose base is e to that whose base is a, we have, (denoting these systems by the characteristics log. and log.')

$$\log. y = \log. e. \log.' y.$$

And since all systems of logarithms are compared with the Naperian system, the number log. e, by which we multiply a Naperian logarithm in order to pass to the corresponding logarithm of another system, is called the modulus of that system.

The *logarithmic differential* being of considerable use, we must keep in mind, that *the differential of a logarithm is equal to the product of the modulus of the system, and the differential of the quantity, divided by the quantity itself.*

28. If from this we would derive the developement of x in terms of y, or of the logarithm in terms of the number and its powers, we should find, that the quantities

$$x, \quad \frac{dx}{dy}, \quad \frac{d^2 x}{dy^2}, \; \&c.$$

become infinite, when $y = 0$, and we may thence conclude, that the logarithm cannot be developed in the form of

$$A + By + Cy^2 + Dy^3 + \&c.$$

It would have been easy to have shewn this, *a priori*, by observing that the function x becomes infinite, when $y = 0$ (Alg. 251), which does not take place in the formula just mentioned, which then reduces itself to $x = A$.

Our result would be different, had we made $y = 1 + u$; for then we should find, taking the Naperian logarithms,

$$x = \log.' (1 + u), \quad \frac{dx}{du} = \frac{1}{1 + u} = (1 + u)^{-1},$$

$$\frac{d^2x}{du^2} = -(1+u)^{-2},\ \frac{d^3x}{du^3} = 2(1+u)^{-3},\ \&c.$$

and making $u = 0$, we find

$$\log.'(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \&c.$$

and for any base a, (denoting the logarithm of e, to that base, by M) we have

$$\log.(1+u) = M\left\{u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} - \&c.\right\}^*.$$

29. This series is not sufficiently convergent to be employed for the calculation of logarithms, except when u is a fraction; but means have been discovered for transforming it into others which are applicable with greater or less advantage to different cases. It was first observed, that by changing u into $-u$, there would result

$$\log.(1-u) = M\left\{-u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} - \&c.\right\}$$

and subtracting this equation from the preceding

$$\log.(1+u) - \log.(1-u) = \log.\left(\frac{1+u}{1-u}\right) = 2M\left\{\frac{u}{1} + \frac{u^3}{3} + \frac{u^5}{5} + \&c.\right\},$$

making $\frac{1+u}{1-u} = 1 + \frac{z}{n}$, which gives $u = \frac{z}{2n+z}$, and observing that

$$\log.\left(1+\frac{z}{n}\right) = \log.\left(\frac{n+z}{n}\right) = \log.(n+z) - \log.(n)$$

---

* It will undoubtedly have been remarked, that the equation $k = \frac{\log. a}{\log. e}$ in No. 25, combined with the equation in No. 23, which is

$$k = \frac{(a-1)}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.$$

leads to

$$\log. a = \log. e\left\{\frac{(a-1)}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.\right\}$$

and putting $a = 1+u$, we shall obtain the developement which is found above.

there arises

$$\log.(n+z)-\log.(n)=2M\left\{\frac{z}{2n+z}+\frac{1}{3}\left(\frac{z}{2n+z}\right)^3+\frac{1}{5}\left(\frac{z}{2n+z}\right)^5+\&c.\right\}$$

whence

$$\log.(n+z)=\log.n+2M\left\{\frac{z}{2n+z}+\frac{1}{3}\left(\frac{z}{2n+z}\right)^3+\frac{1}{5}\left(\frac{z}{2n+z}\right)^5+\&c.\right\}$$

This series, which affords the logarithm of $n+z$, when we are acquainted with that of n, gives, by supposing $n=1$, $z=1$,

$$\log.2=2M\left\{\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\&c.\right\}.$$

since $\log.1=0$. This series is already very convergent, and it becomes still more so for larger numbers. If we take $M=1$, we find $\log.'2=0.6931472$.

The modulus M may be obtained by calculating the logarithm of the same number in the system we wish to adopt, and also in the Naperian system, and by taking the ratio of the two results (27). We may find the modulus of common logarithms very readily, by calculating the Naperian logarithm of 5 by means of that of 4, (which may be deduced from that of 2, since $\log.4=2\log.2$); then knowing $\log.'5$ and $\log.'2$, we have $\log.'10=\log.'5+\log.'2$, and dividing unity which is the common logarithm of 10 by this latter quantity, we obtain the required modulus: we thus find

$$M=0.434294482.$$

This is the number by which we must multiply Naperian logarithms to obtain common logarithms, or those of Briggs.

Reciprocally, to return to Naperian logarithms, we must divide common logarithms by this number, or multiply them by

$$\frac{1}{0.434294482}=2.302585093.$$

30. We will now give some examples of the application of the rules for the differentiation of logarithmic functions; but for the sake of simplicity we shall always suppose them Naperian logarithms, unless the contrary be particularly mentioned: we shall assume hereafter, in all cases, l as the characteristic of these logarithms.

1st. Let $u = \mathrm{l}\left(\frac{x}{\sqrt{a^4+x^2}}\right)$, making $\frac{x}{\sqrt{a^2+x^2}} = z$,

we have $du = \frac{dz}{z}$; but

$$dz = \frac{dx\sqrt{a^4+x^2} - \frac{x^2dx}{\sqrt{a^4+x^2}}}{a^2+x^2} = \frac{a^2dx}{(a^2+x^2)^{\frac{3}{2}}}:$$

therefore, $du = \frac{a^2dx}{x(a^2+x^2)}$.

2d. Let $u = \mathrm{l}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right\}$; make $\sqrt{1+x}+\sqrt{1-x} = y$, and $\sqrt{1+x}-\sqrt{1-x} = z$, which gives

$$u = \mathrm{l}\left(\frac{y}{z}\right) = \mathrm{l}\,y - \mathrm{l}\,z,\quad du = \frac{dy}{y} - \frac{dz}{z};$$

but we find

$$dy = \frac{dx}{2\sqrt{1+x}} - \frac{dx}{2\sqrt{1-x}} = \frac{-dx}{2\sqrt{1-x^2}}\left\{\sqrt{1+x}-\sqrt{1-x}\right\}$$

$$= -\frac{z\,dx}{2\sqrt{1-x^2}},$$

$$dz = \frac{dx}{2\sqrt{1+x}} + \frac{dx}{2\sqrt{1-x}} = \frac{dx}{2\sqrt{1-x^2}}\left\{\sqrt{1+x}-\sqrt{1-x}\right\}$$

$$= \frac{y\,dx}{2\sqrt{1-x^2}},$$

whence we obtain

$$\frac{dy}{y}-\frac{dz}{z}=-\frac{z\,dx}{2y\sqrt{1-x^2}}-\frac{y\,dx}{2z\sqrt{1-x^2}}=\frac{-(y^2+z^2)\,dx}{2yz\sqrt{1-x^2}};$$

and observing that

$$y^2+z^2=4, \text{ and } yz=2x,$$

we find at last

$$du=-\frac{dx}{x\sqrt{1-x^2}}.$$

This example is remarkable, from the reductions which the differential undergoes, and from its simplicity, considering the functions from which it is derived; the following examples, whose results only we shall state, will present no difficulty.

3d. $u=1\left\{x+\sqrt{1+x^2}\right\};\qquad du=\dfrac{dx}{\sqrt{1+x^2}}.$

4th. $u=1\dfrac{1}{\sqrt{-1}}\left\{x\sqrt{-1}+\sqrt{1-x^2}\right\},\ du=\dfrac{dx}{\sqrt{1+x^2}}.$

5th. $u=1\left\{\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right\}^{\frac{1}{2}},\quad du=\dfrac{dx}{\sqrt{1+x^2}}.$

6th. If we had $u=(1\,x)^n$, making $1\,x=z$, we should find

$$(1\,x)^n=z^n,\qquad d.\,z^n=n\,z^{n-1}\,dz;$$

and replacing z and $d\,z$ by their respective values, there would result

$$d.\,(1\,x)^n=n\,(1\,x)^{n-1}\frac{dx}{x}.$$

7th. Let $u=1.\,1\,x$, that is to say, the logarithm of the logarithm of x; putting as above $1\,x=z$, we have

$$u=1\,z,\ du=\frac{dz}{z},\qquad dz=d.\,1\,x=\frac{dx}{x};$$

whence we deduce

$$du = \frac{dx}{x\,\mathrm{l}\,x}.$$

31. The consideration of logarithms facilitates very much the differentiation of exponentials, when they are much complicated.

1st. Let, for example, $u=z^y$, z and y being any two functions of x; taking the logarithms of each member, we have $\mathrm{l}\,u = y\,\mathrm{l}\,z$, and then differentiating, we find

$$\frac{du}{u} = dy\,\mathrm{l}\,z + y\,d.\,\mathrm{l}\,z\ (11, 27), \text{ or}$$

$$\frac{du}{u} = dy\,\mathrm{l}\,z + y\frac{dz}{z}$$

and thence

$$du=u\left(dy\,\mathrm{l}\,z + y\frac{dz}{z}\right), \text{ and } d.\,z^y=z^y\left(dy\,\mathrm{l}\,z + y\frac{dz}{z}\right).$$

2d. Let $u=a^{b^x}$; make $b^x=y$, and we find

$u = a^y$, $du = a^y\,dy\,\mathrm{l}\,a$ (27);

but $dy = d.\,b^x = b^x\,dx\,\mathrm{l}\,b$:

then $du = a^{b^x}\,b^x\,dx\,\mathrm{l}\,a\,\mathrm{l}\,b$.

3d. Let $u=z^{t^s}$, z, t and s being functions of x; make $t^s=y$; then

$$u=z^y,\quad du=z^y\left(dy\,\mathrm{l}\,z + \frac{y\,dz}{z}\right);$$

$$dy=t^s\left(ds\,\mathrm{l}\,t + \frac{s\,dt}{t}\right);$$

and consequently

$$du=z^{t^s}t^s\left(ds\,\mathrm{l}\,t\,\mathrm{l}\,z + \frac{s\,dt\,\mathrm{l}\,z}{t} + \frac{dz}{z}\right).$$

By means of these formulæ, it will be easy to find the differential of any exponential function whatever.

32. Sines, cosines, tangents, and other trigonometrical lines, considered with respect to the arc of the circle on which they depend, are also transcendental functions; they are commonly called *circular functions*.

For the sake of simplicity, we shall always suppose the radius equal to unity.

Substituting $x + dx$, for x, in the function $\sin x$, it becomes

$$\sin(x + dx) = \sin x \cos dx + \cos x \sin dx;$$

from which we find the difference,

$$\sin(x + dx) - \sin x =$$
$$\sin x \cos dx + \cos x \sin dx - \sin x =$$
$$\sin x (\cos dx - 1) + \cos x \sin dx.$$

We might now develope the second member of this equation according to the powers of the increment dx; but we may, without this, obtain the limit of the ratio of the increment of the function to that of the variable in the following manner. Taking the ratio of the increments, we find

$$\frac{\sin(x + dx) - \sin x}{dx} = \sin x \frac{(\cos dx - 1)}{dx} + \cos x \frac{\sin dx}{dx}.$$

If we observe that

$$(\sin dx)^2 = 1 - (\cos dx)^2 = (1 + \cos dx)(1 - \cos dx)$$

and that, consequently,

$$1 - \cos dx = \frac{(\sin dx)^2}{1 + \cos dx},$$

we have

$$\frac{\sin(x + dx) - \sin x}{dx} = -\sin x \frac{\sin dx}{1 + \cos dx} \cdot \frac{\sin dx}{dx} + \cos x \frac{\sin dx}{dx}$$
$$= \left(-\sin x \frac{\sin dx}{1 + \cos dx} + \cos x\right) \frac{\sin dx}{dx}.$$

We may pass to the limits of these quantities by finding what the two factors of the second member become

when the increment dx vanishes (8). In this case, $\sin dx=0$, $\cos dx=1$, and the first factor is reduced to $\cos x$.

The factor $\frac{\sin dx}{dx}$ constantly tends towards unity: for, from the expression $\tan A=\frac{\sin A}{\cos A}$, we deduce $\frac{\sin A}{\tan A}=\cos A$; and since $\cos A=1$, when $A=0$, unity will be the limit of the ratio of the sine and the tangent, when the arc vanishes; but since the arc is less than the tangent, and greater than the sine, it follows, *a fortiori*, that its ratio to the sine will constantly approximate to unity.

We find then, from these remarks,

$$\frac{d.\sin x}{dx}=\cos x, \text{ or } d.\sin x=dx\cos x.$$

33. Having obtained this differential, the others may easily be deduced from it; for we have

1st. $\operatorname{Cos} x=\sin\left(\frac{\pi}{2}-x\right)$, $d.\cos x=d.\sin\left(\frac{\pi}{2}-x\right)$;

but from the preceding article,

$$d.\sin\left(\frac{\pi}{2}-x\right)=d\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)$$

$$=-dx\cos\left(\frac{\pi}{2}-x\right),$$

and since $\cos\left(\frac{\pi}{2}-x\right)=\sin x$, we have

$$d.\cos x=-dx\sin x.$$

2d. Since $\operatorname{vers} x=1-\cos x$, we have

$$d.\operatorname{vers} x=-d.\cos x=dx\sin x.$$

3d. $\operatorname{Tan} x=\frac{\sin x}{\cos x}$

$$d.\tan x=\frac{\cos x\, d.\sin x-\sin x\, d.\cos x}{(\cos x)^2} \quad (12)$$

$$=\frac{\left\{(\cos x)^2+(\sin x)^2\right\}dx}{(\cos x)^2};$$

but $(\cos x)^2 + (\sin x)^2 = 1$; therefore

$$d.\tan x = \frac{dx}{(\cos x)^2}.$$

4th. $\text{Cot } x = \frac{1}{\tan x}$,

$$d.\cot x = -\frac{d.\tan x}{(\tan x)^2} = \frac{-dx}{(\tan x)^2 (\cos x)^2} = -\frac{dx}{(\sin x)^2},$$

putting for $\tan x$ its value.

5th. $\text{Sec } x = \frac{1}{\cos x}$,

$$d.\sec x = -\frac{d.\cos x}{(\cos x)^2} = \frac{dx \sin x}{(\cos x)^2} = dx \tan x \sec x,$$

since $\frac{\sin x}{\cos x} = \tan x$, and $\frac{1}{\cos x} = \sec x$.

6th. $\text{Cosec } x = \frac{1}{\sin x}$,

$$d.\operatorname{cosec} x = -\frac{d.\sin x}{(\sin x)^2} = -\frac{dx \cos x}{(\sin x)^2} = -dx \cot x \operatorname{cosec} x.$$

34. By the aid of these formulæ, we may find the differential of any expression involving sines, cosines, tangents, &c. It will be necessary, however, first to consider these quantities as simple and uncompounded; and after the differential of the expression is found upon this supposition, to substitute for their differentials the results above written. We shall give but one example.

Let $u = (\cos x)^{\sin x}$. Make $\cos x = z$, $\sin x = y$; then $u = z^y$, and

$$du = d.z^y = z^y \left(dy \,1 z + \frac{y\,dz}{z.} \right) (31)$$

$$= dx (\cos x)^{\sin x} \left\{ \cos x \,1 \cos x - \frac{(\sin x)^2}{\cos x} \right\}.$$

35. Having discussed the sine, cosine, &c. regarded as functions of the arc, it will now be proper to consider in

its turn the arc, as a function of the sine, cosine, &c. successively, and to determine its differential under these different points of view. For this purpose, let x be the function proposed, and u the variable, on which that function depends. 1st. The equation $d.\ \sin x = dx \cos x$, gives, (in consequence of the equations $\sin x = u$, and $\cos x = \sqrt{1-u^2}$)

$$du = dx\sqrt{1-u^2}, \text{ and consequently } dx = \frac{du}{\sqrt{1-u^2}}.$$

This is the value of the differential of the arc expressed in terms of the $\sin x$, and its differential.

If we would express the differential of the arc by means of its cosine, we must begin from the equation

$$d.\ \cos x = -dx \sin x,$$

which gives, making $\cos x = u$,

$$du = dx\sqrt{1-u^2}, \text{ or } dx = \frac{-du}{\sqrt{1-u^2}}.$$

In order to pass from this to the versed sine, make $u = 1 - y$, since $\cos x = 1 - \text{vers } x$, and we consequently shall have $du = dy$, and $dx = \dfrac{dy}{\sqrt{2y-y^2}}$.

2d. Let $\tan x = u$; the equation $d.\ \tan x = \dfrac{dx}{(\cos.\ x)^2}$ gives

$$du = \frac{dx}{(\cos.\ x)^2}, \text{ and } d\ \ = du(\cos x)^2. \quad \text{But since}$$

$\dfrac{\sin x}{\cos. x} = \tan x$, we have $\sin x = \cos x \tan x$, and

$$(\sin x)^2 = (\cos x)^2 (\tan x)^2;$$

and substituting $1-(\cos x)^2$, for $(\sin x)^2$, it becomes
$1 = (\cos x)^2 + (\cos x)^2 (\tan x)^2 = (\cos x)^2 \ (1 + \tan x)^2$;
we have, therefore,

$$(\cos x)^2 = \frac{1}{1+(\tan x)^2} = \frac{1}{1+u^2},$$

and substituting this value in that of dx, there results

$$dx = \frac{du}{1+u^2};$$

whence we conclude, that *the differential of the arc is equal to that of the tangent divided by the square of the secant*; for $\sqrt{1+u^2}$ expresses the secant when the tangent is represented by u.

We shall conclude this article by the following example:

Let x be an arc, having for its sine the function $2u\sqrt{1-u^2}$,

$$\text{make } 2u\sqrt{1-u^2} = z;$$

$$\text{then } dx = \frac{dz}{\sqrt{1-z^2}};$$

But

$$dz = \frac{2\,du\,(1-2u^2)}{\sqrt{1-u^2}}, \text{ and } \sqrt{1-z^2} = 1-2u^2;$$

therefore

$$dx = \frac{2\,du}{\sqrt{1-u^2}}.$$

36. By the help of the differential expressions just obtained, we may derive the developement of the principal circular functions.

1st. For $\sin x$, we have

$$\frac{du}{dx} = \cos x,\ \frac{d^2u}{dx^2} = -\sin x,\ \frac{d^3u}{dx^3} = -\cos x,\ \frac{d^4u}{dx^4} = \sin x, \text{ \&c.}$$

and making $x=0$, it follows, from No. 19, that

$$U=0,\quad U'=1,\quad U''=0,\quad U'''=-1,\quad U''''=0, \text{ \&c.}$$

whence we conclude, that

$$\sin x = \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \text{\&c.}$$

2d. We find for $\cos x$

$$\frac{du}{dx} = -\sin x,\ \frac{d^2u}{dx^2} = -\cos x,\ \frac{d^3u}{dx^3} = \sin x,$$

$$\frac{d^4u}{dx^4} = \cos x, \text{ \&c.}$$

and making $x = 0$, we shall get

$$U=1,\quad U'=0,\quad U''=-1,\quad U'''=0,\quad U''''=1,\ \&c.$$

which gives

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$$

These two formulæ, whose law is both evident and simple, present the most expeditious method of calculating the sine and cosine, corresponding to any given arc, particularly when this arc is of small magnitude. We shall find analogous formulæ for the tangent, and for other trigonometrical lines; but their law is not so simple as that of the preceding; and they are much less convenient in their application than those relations which are given by the expressions for the tangent and secant, in terms of the sine and cosine. For this reason we shall not stop to consider them.*

37. If we represent by y the arc of a circle, whose sine is x, we have (35)

$$dy = \frac{dx}{\sqrt{1-x^2}},$$

an equation which will lead to the developement of the arc, according to the powers of the sine; for, from this equation we deduce

$$\frac{dy}{dx} = (1-x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = x(1-x^2)^{-\frac{3}{2}}$$

$$\frac{d^3y}{dx^3} = (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}}$$

$$\frac{d^4y}{dx^4} = 3.3\,x(1-x^2)^{-\frac{5}{2}} + 3.5\,x^3(1-x^2)^{-\frac{7}{2}}$$

$$\frac{d^5y}{dx^5} = 3.3(1-x^2)^{-\frac{5}{2}} + 2.5.9\,x^2(1-x^2)^{-\frac{7}{2}} + 3.5.7\,x^4(1-x^2)^{-}$$

&c.

* See Note (D).

Making $x=0$, and observing that, upon this supposition, the arc y vanishes, we find

$$y = x + \frac{x^3}{1.2.3} + \frac{3.3\,x^5}{1.2.3.4.5} + \&c.$$

Let us now inquire into the value of the arc expressed in terms of its tangent; we have (35)

$$dy = \frac{dx}{1+x^2}, \text{ whence}$$

$$\frac{dy}{dx} = (1-x^2)^{-1}$$

$$\frac{d^2y}{dx^2} = -2x\,(1+x^2)^{-2}$$

$$\frac{d^3y}{dx^3} = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}$$

$$\frac{d^4y}{dx^4} = 24x(1+x^2)^{-3} - 48x^3(1+x^2)^{-4}$$

$$\frac{d^5y}{dx^5} = 24(1+x^2)^{-3} - 288x^2(1+x^2)^{-4} + 384x^4(1+x^2)^{-5}$$

&c.

and making $x=0$, we find

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c.$$

The law here manifests itself in the first few terms; this is not the case with respect to the preceding series; but since the successive differentials of y become more and more complicated, on account of their denominators, the method above employed is not the most proper to lead to the developement required. The integral calculus will furnish more convenient methods.

The latter of these developements furnishes a remarkable expression for the arc $\frac{\pi}{4}$, whose tangent is equal to unity; for, if we make $x=1$, it gives

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$$

This series is not sufficiently convergent to be employed; but we may calculate the same arc, by dividing it into several parts; and the tangent of each being less than unity, we shall have converging series for each. The English Geometer, Machin, found that the arc $\frac{\pi}{4}$ is equal to four times the arc, whose tangent is $\frac{1}{5}$, minus the arc, whose tangent is $\frac{1}{239}$, which may be easily proved as follows: Let $\tan a = \frac{1}{5}$,

$$\tan 2a = \frac{2 \tan a}{1 - (\tan a)^2} = \frac{5}{12},$$

$$\tan 4a = \frac{2 \tan 2a}{1 - (\tan 2a)^2} = \frac{120}{119}.$$

This last number is but a little greater than unity, or the tangent of $\frac{\pi}{4}$, which shows that $4a > \frac{\pi}{4}$; making

$4a = A$, $\frac{\pi}{4} = B$,

the difference

$4a - \frac{\pi}{4}$, or $A - B$, has for its tangent

$$\tan(A - B) = \tan b = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{1}{239},$$

and since $B = A - (A - B)$, it follows that $\frac{\pi}{4} = 4a - b$;

But taking successively $x = \frac{1}{5}$, $x = \frac{1}{239}$, we find

$$b = \frac{1}{239} - \frac{1}{3.(239)^3} + \frac{1}{5.(239)^5} - \frac{1}{7.(239)^7} + \&c.$$

$$a = \frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.5^7} + \&c.$$

whence

$$\frac{\pi}{4} = \begin{cases} 4\left(\frac{1}{1.5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.5^7} + \&c.\right) \\ -\left(\frac{1}{1.239} - \frac{1}{3.(239)^3} + \frac{1}{5.(239)^5} - \&c.\right) \end{cases}$$

On the Differentiation of Equations of two Variables.

Hitherto we have only differentiated equations, in which the variables are separated; that is to say, in which the function itself occurs on one side, and the variable on the other; such are all equations of the form $X=Y$, Y being a function of y, and X a function of x. But the greater number of equations which occur in analytical enquiries do not present themselves under this form: the variable and the function are frequently mixed or combined together.

When we have an equation $V=0$, between x and y, such an equation determines x in terms of y, or y in terms of x; so that one of these quantities is a function of the other.

If we conceive y to be determined in terms of x, and if we substitute the value of y in the quantity V, this necessarily reduces itself to an expression, consisting of functions of x alone; but it will be composed of terms which destroy each other, independently of any particular value of x, since x must continue indeterminate. It follows, therefore, that when x receives any increment whatever, h, the change which y undergoes, must be such that the function V shall remain equal to zero, as it was before. If, then, we denote by V' what the expression V apparently becomes, we must have $V'=0$, whence we conclude, that

$$V'-V=0, \text{ and } \frac{V'-V}{h}=0;$$

whatever may be the increment h; and consequently, if the expression $\frac{V'-V}{h}$ is susceptible of a limit P, we ought to have $P=0$.*

* In order to conceive this clearly, it is sufficient to consider in general, that if, in the expression V, we substitute $x+h$,

 instead

But since V is composed of x, and of y considered as a function of x; this limit may be obtained by differentiating V, taking care to make y and x vary, according to the rules of Nos. 10, 11, 12, 13, 15; and, observing that $P dx = dV$, we may conclude from what has been said, that the equation $V = 0$, necessarily leads to the equation

$$dV = 0,$$

the first determining the value of y, and the second that of dy.

---

instead of x, and $y+k$, instead of y, the result may be developed in the form

$$V+Mh+Nk+Ph^2+Qhk+Rk^2+Sh^3+\&c. = 0,$$

M, N, P, Q, R, S, &c. being quantities independent of h and k.

This equation, on account of the given one, $V=0$, reduces itself to

$$Mh+Nk+Ph^2+Qhk+Rk^2+Sh^3+\&c. = 0,$$

and assigns a relation between h and k.

If we now make $k=\pi h$, it acquires a factor h in all its terms; which being suppressed, the equation becomes

$$M+N\pi+Ph+Q\pi h+R\pi^2 h+Sh^2+\&c. = 0.$$

Though the ratio π changes as h diminishes, it does not vanish with that quantity; and we shall have, for determining π on the hypothesis of $h=0$, the equation

$$M+N\pi = 0.$$

The value of π, which results from this, will be the limit of all the values which that quantity can assume in consequence of the change of h; it is, therefore, evident, that if we denote this limit by p, the equation

$$M+Np = 0$$

will be rigorously exact.

It appears from the process just gone through, that the expression $M+Np$, is the differential coefficient of the function V, taken on the hypothesis of y being a function of x.

The following example will illustrate this. Let there be the equation

$$y^2 - 2\,m\,x\,y + x^2 - a^2 = 0.$$

The expression V is, in this case, $y^2 - 2\,m\,x\,y + x^2 - a^2$; and if we differentiate it, upon the supposition that y is a function of x, and make the result equal to zero, we shall find

$$2y\,dy - 2\,m\,x\,dy - 2\,m\,y\,dx + 2\,x\,dx = 0,$$

$$\text{or, } y\,dy - m\,x\,dy - m\,y\,dx + x\,dx = 0 \qquad (1),$$

if we suppress the common factor 2; and making $dy = p\,dx$, we have

$$(y - m\,x)\,p - m\,y + x = 0,$$

whence

$$p = \frac{m\,y - x}{y - m\,x}.$$

To obtain p in terms of x alone, we must subsitute in this expression the value of y, deduced from the proposed equation, which is

$$y = m\,x \pm \sqrt{a^2x^2 - m^2 + x^2};$$

this gives

$$p = \frac{-x + m^2x \pm m\sqrt{a^2 - x^2 + m^2x^2}}{\pm\sqrt{a^2 - x^2 + m^2x^2}} =$$

$$= m \pm \frac{-x + m^2\,x}{\sqrt{a^2 - x^2 + m^2x^2}},$$

the same result that would have been deduced from the original equation, which would have become

$$y = m\,x \pm \sqrt{a^2 - x^2 + m^2x^2},$$

had the variables first been separated.

39. The equation

$$(y - m\,x)\,p - m\,y + x = 0,$$

being differentiated, considering y and p as functions of x, leads to the equation

$$(dy - m\,dx)\,p + (y - m\,x)\,dp - m\,dy + dx = 0,$$

and if we make

$$dy = p\,dx, \qquad dp = q\,dx,$$

we find

$$(p-m)p + (y-mx)q - mp + 1 = 0,$$

an equation which expresses the relation which the differential coefficient of the second order q, or $\frac{d^2y}{dx^2}$ (17), has to that of the first order p, or $\frac{dy}{dx}$, and the variables x and y.

By continuing to differentiate in the same manner, we may form the equation on which the differential coefficient of the third order depends; and so on for others.

40. If we take notice that $q = \frac{d^2y}{dx^2}$, and $d^2y = d\,(dy)$, we shall perceive that the equation

$$(p-m)p + (y-mx)q - mp + 1 = 0,$$

is immediately deducible from the equation

$$y\,dy - mx\,dy - my\,dx + x\,dx = 0 \qquad (1),$$

by differentiating it, (supposing dy to vary as being a function of x), and then dividing the whole by dx^2. In fact, by the first operation, we have

$$dy^2 + y\,d^2y - 2m\,dx\,dy - mx\,d^2y + dx^2 = 0 \quad (2);$$

and by the second

$$\frac{dy^2}{dx^2} - 2m\frac{dy}{dx} + 1 + (y - mx)\frac{d^2y}{dx^2} = 0,$$

which equation, if we change $\frac{dy}{dx}$ into p, and $\frac{d^2y}{dx^2}$ into q, agrees with that before found for the determination of q.

In general, when we make the quantities p, q, &c. vary, as functions of x, we do nothing more than take the differentials of the equivalent expressions $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c.

which differentials are respectively represented by $\frac{d^2 y}{d x}$, $\frac{d^3 y}{d x^2}$, &c. It is, in short, the same thing as considering the quantities $d y$, $d^2 y$, &c. as functions of x.

The equation (1) is the *first differential* of the proposed equation; the equation (2) is the *second differential* of it, &c. and from the preceding remark it appears, that *the successive differentials of any proposed* PRIMITIVE *equation may be deduced one from another by successive differentiation, regarding* y, d y, d^2 y, *&c. as functions of* x.

We may derive the equations which determine the differential coefficients, either by observing that these coefficients are represented by

$$\frac{d y}{d x}, \ \frac{d^2 y}{d x^2}, \ \&c.$$

or by making $d y = p\, d x, \qquad d^2 y = q\, d x^2$, &c.

By these last substitutions, the differentials disappear, and there remain in the results only the functions p, q, &c. absolutely independent of the value of the increment $d x$.

41. The equation

$$y^2 - 2\, m x y + x^2 - a^2 = 0, \qquad (1).$$

being of the second degree, affords two values of y, in consequence of which the equation

$$(y - m x)\, d y - (m y - x)\, d x = 0 \qquad (1),$$

or

$$\frac{d y}{d x} = \frac{m y - x}{y - m x},$$

gives for the differential coefficient $\frac{d y}{d x}$, two values corresponding to those of the function y.

If, instead of resolving the proposed equation to determine the value of y, we eliminate that variable between the two equations

$$y^2 - 2mxy + x^2 - a^2 = 0,$$

$$(y - mx)\,dy - (my - x)\,dx = 0 \quad (1),$$

from the second of these we find,

$$y = \frac{x(m\,dy - dx)}{dy - m\,dx};$$

and substituting this in the first, and making the proper reductions, we find

$$(x^2 - a^2 - m^2 x^2)\,dy^2 - (2m\,x^2 - 2m\,a^2 - 2m^3 x^2)\,dx\,dy,$$
$$+ (x^2 - m^2 x^2 - a^2 m^2)\,dx^2 = 0.$$

The values of dy deduced from this equation are the same as those which would be deduced by differentiating those of y: dividing this equation by dx^2, we may immediately deduce the values of the differential coefficient. We thus find

$$(x^2 - a^2 - m^2 x^2)\frac{dy^2}{dx^2} - (2m\,x^2 - 2m\,a^2 - 2m^3 x^2)\frac{dy}{dx},$$
$$+ (x^2 - m^2 x^2 - a^2 m^2) = 0;$$

and disengaging the second power of the differential coefficient from its multiplier, we have

$$\frac{dy^2}{dx^2} - 2m\frac{dy}{dx} + \frac{x^2 - m^2 x^2 - a^2 m^2}{x^2 - m^2 x^2 - a^2} = 0.$$

42. It is easy to apply the preceding method to much more complicated examples, or to those in which the variables rise to a higher degree.

Take, for instance, the equation

$$y^3 - 3axy + x^3 = 0;$$

from which, by differentiation, we obtain

$$3y^2\,dy - 3ax\,dy - 3ay\,dx + 3x^2\,dx = 0,$$

or suppressing the common factor 3,

$$y^2 dy - ax\,dy - ay\,dx + x^2\,dx = 0, \quad (1),$$

and consequently

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

The function y, in this example, being given by means of an equation of the third degree, ought to have three values; and by substituting them successively in the expression for $\frac{dy}{dx}$, we should obtain an equal number of values for this differential coefficient. We may observe, in general, that this coefficient will have as many different values, as the function y has in the original equation; and the same holds good with regard to the differential itself.

If we eliminate y between the two equations

$$y^3 - 3axy + x^3 = 0,$$

$$y^2 dy - ax dy - ay dx + x^2 dx = 0. \quad (1),$$

the result would be an equation of the third degree, with respect to dy; and it would contain, as its roots, the three values of which that differential is susceptible.

Having found the expression for dy, or that of $\frac{dy}{dx}$, we may obtain those of $d^2 y$, or $\frac{d^2 x}{dx^2}$, by differentiating the first differential of the proposed equation

$$y^2 dy - ax dy - ay dx + x^2 dx = 0, \quad (1).$$

with respect to dy, y and x, according to the rule established in No. 40.

Performing these operations, we have

$$y^2 d^2 y - ax d^2 y + 2y dy^2 - a dy dx - a dx dy + 2x dx^2 = 0;$$

which, by reduction, becomes

$$(y^2 - ax) d^2 y + 2y dy^2 - 2a dx dy + 2x dx^2 = 0. \quad (2).$$

This is the second differential of the proposed equation; and if we combine it with the first differential, we can eliminate dy, and the result will give the expression of $d^2 y$, in terms of x, dx, and y. From this the function y may be eliminated, if required, by means of the proposed equation.

Dividing the equation (2) by dx^2, it takes the form

$$(y^2 - a x) \frac{d^2 y}{d x^2} + 2 y \frac{d y^2}{d x^2} - 2 a \frac{d y}{d x} + 2 x = 0,$$

and only contains the differential coefficients $\frac{d^2 y}{d x^2}$ and $\frac{d y}{d x}$.

Putting, instead of $\frac{d y}{d x}$, its value $\frac{a y - x^2}{y^2 - a x}$, deduced from equation (1), we have

$$(y^2 - ax)\frac{d^2 y}{d x^2} + 2 y \left(\frac{a y - x^2}{y^2 - ax}\right)^2 - 2 a \left(\frac{a y - x^2}{y^2 - a x}\right) + 2 x = 0;$$

and, reducing a common denominator

$$(y^2 - a x)^3 \frac{d^2 y}{d x^2} + 2 x y^4 - 6 a x^2 y^2 + 2 x^4 y + 2 a^3 x y = 0;$$

but the quantity

$$2 x y^4 - 6 a x^2 y^2 + 2 x^4 y,$$

is the same with

$$2x y (y^3 - 3 a x y + x^3),$$

which is equal to zero, by the original equation, and consequently we have

$$(y^2 - a x)^3 \frac{d^2 y}{d x^2} + 2 a^3 x y = 0,$$

$$\text{or } \frac{d^2 y}{d x^2} = - \frac{2 a^3 x y}{(y^2 - a x)^3}.$$

By differentiating equation (2), relative to $d^2 y$, dy, y and x, we might derive the third differential of the proposed equation, and by eliminating $d^2 y$ and dy, by means of the equations (1) and (2), we might deduce the value of $d^3 y$; dividing this result by $d x^3$, we should have the expression for the coefficient $\frac{d^3 y}{d x^3}$. By continuing this process, we should arrive at the differential coefficients of any order whatever.

43. The observation in No. 7, respecting the constants which disappear by the differentiation of functions, is

equally applicable to equations. If we had, for example, the equation $y^2=ax+b$, its differential $2y\,dy=a\,dx$, being independent of b, would equally belong to all those equations, which result from giving to b all possible values in the proposed equation.

But we might also, in the present case, arrive at an equation independent of a, although the differentiation has not made this constant disappear; for this purpose it is sufficient to eliminate a between the two equations

$$y^2=ax+b, \text{ and } 2y\,dy=a\,dx,$$

and we should find

$$y^2\,dx=2xy\,dx+b\,dx.$$

Although this latter equation is not the direct differential of the proposed one; it is, however, derived from it in such a manner, that, being divided by dx, it expresses the relation which ought to subsist between the variable x, the function y, and the differential coefficient $\frac{dy}{dx}$, whatever be the value of a.

If the constant which we eliminate is above the first degree in the given equation, the result at which we arrive will contain higher powers than the first of dy and dx. As an example, let us take the equation

$$y^2-2ay+x^2=a^2.$$

By differentiating, we have

$$y\,dy-a\,dy+x\,dx=0;$$

whence

$$a=\frac{y\,dy+x\,dx}{dy},$$

and substituting this in the proposed equation, after having arranged the result, according to powers of dy, and having divided the whole by dx^2, we shall obtain

$$(x^2-2y^2)\frac{dy^2}{dx^2}-4xy\frac{dy}{dx}-x^2=0:$$

Such is the relation which exists between the variable x, the function y, and its differential coefficient $\frac{dy}{dx}$, independently of any particular value of the constant a.

By solving the equation

$$y^2 - 2ay + x^2 = a^2,$$

with respect to a, we deduce

$$a = -y \pm \sqrt{2y^2 + x^2},$$

and the quantity a being now disengaged from the variables x and y, differentiation alone will be sufficient to make it disappear. Thus we arrive at

$$-dy \pm \frac{2ydy + xdx}{\sqrt{2y^2 + x^2}} = 0.$$

By clearing this equation from the radical, we may convince ourselves that it is the same as that which results from elimination.

44. Any number of constants may be made to disappear by differentiating as many times as there are constants in the equation. Let

$$y^2 = m(a^2 - x^2);$$

we have

$$ydy = -mxdx:$$

differentiating again, we find,

$$yd^2y + dy^2 = -mdx^2,$$

and substituting for m its value $\frac{-ydy}{xdx}$, derived from the preceding equation, it becomes

$$y\frac{dy}{dx} - x\frac{dy^2}{dx^2} - xy\frac{d^2y}{dx^2} = 0,$$

a result independent of the constants a and m.

45. Differentiation, combined with elimination, affords a means of making irrational functions disappear from an equation: take, for example,

$$P^n = Q,$$

P and Q being any functions of x and y; taking the differential of this equation, there arises

$$nP^{n-1}dP = dQ, \text{ whence } nP^n dP = PdQ,$$

and substituting for P^n its value, we obtain

$$nQdP = PdQ,$$

in which equation the quantity P is freed from the exponent n.

We should have arrived at the same result by taking the logarithms of each side of the proposed equation, by which means we obtain successively

$$n\,\mathrm{l}\,P = \mathrm{l}\,Q, \quad \text{and } n\frac{dP}{P} = \frac{dQ}{Q} \text{ (27)},$$

and consequently $nQdP = PdQ$.

This remark will assist us in developing, according to the powers of x, the function

$$(a + bx + cx^2 + dx^3 + \&c.)^n,$$

whatever may be the exponent n. For this purpose, let

$$(a + bx + cx^2 + \&c.)^n =$$
$$A + Bx + Cx^2 + Dx^3 + \&c.$$

and taking the logarithms, we have

$$n\,\mathrm{l}(a + bx + cx^2 + dx^3 + ex^4 + \&c.)$$
$$= \mathrm{l}(A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.)$$

By differentiation, we obtain

$$\frac{n(b + 2cx + 3dx^2 + 4ex^3 + \&c.)dx}{a + bx + cx^2 + dx^3 + ex^4 + \&c.}$$
$$= \frac{(B + 2Cx + 3Dx^2 + 4Ex^3 + \&c.)dx}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.},$$

which, by omitting the common factor dx, taking away the denominator, and arranging the terms according to the powers of x, becomes

$$\left.\begin{array}{l} nbA + 2ncAx + 3ndAx^2 + 4neAx^3 + \&c. \\ \quad + nbBx + 2ncBx^2 + 3ndBx^3 + \&c. \\ \qquad + nbCx^2 + 2ncCx^3 + \&c. \\ \qquad\quad + nbDx^3 + \&c. \end{array}\right\} =$$

$$\left.\begin{array}{l} aB + 2aCx + 3aDx^2 + 4aEx^3 + \&c. \\ \quad + bBx + 2bCx^2 + 3bDx^3 + \&c. \\ \qquad + cBx^2 + 2cCx^3 + \&c. \\ \qquad\quad + dBx^3 + \&c. \end{array}\right\};$$

But since x ought to remain indeterminate, the two members of this equation must be identical, by means of their coefficients; that is to say, the coefficients of each power of x must be respectively equal in each member. This consideration already employed in No. 193. of the Elements of Algebra, furnishes the following equations:

$$nbA = aB$$
$$2ncA + nbB = 2aC + bB$$
$$3ndA + 2ncB + nbC = 3aD + 2bC + cB$$
&c.

whence the values of the coefficients B, C, D, &c. may be deduced: the coefficient A apparently remains indeterminate; it may, however, be found by making $x = 0$, in the equation

$$(a + bx + cx^2 + \&c.)^n = A + Bx + Cx^2 + \&c.$$

which in this case is reduced to

$$a^n = A.$$

Substituting this expression in the preceding equations, we have

$$B = \frac{n}{1} a^{n-1} b$$

$$C = n a^{n-1} c + \frac{n(n-1)}{1.2} a^{n-2} b^2$$

$$D = n a^{n-1} d + \frac{n(n-1)}{1.1} a^{n-2} bc + \frac{n(n-1)(n-2)}{1.2.3} a^{n-3} b^3,$$

&c.

whence, $(a + bx + cx^2 + dx^3 + \&c.)^n =$

$$a^n + \frac{n}{1} a^{n-1} b x + \left\{ n a^{n-1} c + \frac{n(n-1)}{1.2} a^{n-2} b^2 \right\} x^2$$

$$+ \left\{ n a^{n-1} d + \frac{n(n-1)}{1.1} a^{n-2} b c + \frac{n(n-1)(n-2)}{1.2.3} a^{n-3} b^3 \right\} x^3$$

$+$ &c.

46. The transcendents which occur in an equation, may also be made to disappear by combining it with its differentials. One of the most simple of these functions is

$$\mathrm{l}\,(a + bx + cx^2 + \&c.).$$

If we represent its developement by

$$A + Bx + Cx^2 + Dx^3 + \&c.$$

and differentiate the equation

$$\mathrm{l}\,(a + bx + cx^2 + \&c.) = A + Bx + Cx^2 + \&c.$$

we shall obtain

$$\frac{b + 2cx + 3dx^2 + \&c.}{a + bx + cx^2 + dx^3 + \&c.} = B + 2Cx + 3Dx^2 + \&c.$$

and we may determine the coefficients A, B, C, &c. in the usual manner.

As another example, take

$$\sin(a + bx + cx^2 + \&c.) = A + Bx + Cx^2 + \&c.$$

and for the sake of brevity, putting

$$a + bx + cx^2 + dx^3 + \&c. = u,$$

$$A + Bx + Cx^2 + Dx^3 + \&c. = y,$$

we have $y = \sin u$; and, by differentiation, it becomes $dy = du \cos u$. We may eliminate $\cos u$ by means of the equation, $\cos u = \sqrt{1 - (\sin u)^2}$, which gives $\cos u = \sqrt{1 - y^2}$, and we should then have $dy = du\sqrt{1 - y^2}$; but the radical must be made to disappear from this equation. To avoid this inconvenience, we must differentiate the equation $dy = du \cos u$, a second time; and observing that u is a function of x, as well as of y, we have

$$d^2y = d^2u \cos u - d u^2 \sin u,$$

and putting for sin u and cos u, their values y and $\frac{dy}{du}$, we have

$$d^2y = \frac{dy}{du} d^2u - y\, d u^2, \text{ or } d u\, d^2 y - d y\, d^2 u + y\, d u^2 = 0.$$

Nothing now remains, but to substitute for y, $d y$, d^2y, $d u$, d^2u, and $d u^2$, their respective values; but the equation

$$y = A + B x + C x^2 + D x^3 + \&c.$$

gives

$$d y = (B + 2 C x + 3 D x^2 + \&c.)\, d x.$$
$$d^2 y = (2 C + 2 . 3 D x + \&c.)\, d x^2.$$

To avoid the trouble of very complicated calculations, let the function be reduced to sin $(a + b x + c x^2)$, by making d, e, &c. $= 0$. In this particular case,

$$d u = (b + 2 c x)\, d x,$$
$$d^2u = 2 c\, d x^2,$$
$$d u^3 = (b^3 + 6 b^2 c x + 12 b c^2 x^2 + 8 c^3 x^3)\, d x^3.$$

By substituting these values, the equation

$$d u\, d^2 y - d y\, d^2u + y\, d u^2 = 0,$$

becomes divisible by $d x^3$, and arranging it according to the powers of x, it takes the following form:

$$\left.\begin{array}{r} 2 b C + 6 b D x + 12 b E x^2 + \&c. \\ +4 c C x + 12 c D x^2 + \&c. \\ + b^3 A + 6 b^2 c A x + 12 b c^2 A x^2 + \&c. \\ + b^3 C x^2 + \&c. \\ - 2 c B - 4 c C x - 6 c D x^2 - \&c. \end{array}\right\} = 0.$$

and making the coefficient of each power of x equal to zero, we have equations for determining C, D, E, &c. With regard to the values of A and B, we must recur to the equations

$$y = \sin u, \text{ and } \frac{dy}{du} = \cos u.$$

When $x = 0$, we have

$$u = a,\ y = A,\ d u = b\, d x,\ d y = B\, d x;$$

and from these values we get

$$A = \sin a, \qquad B = b \cos a.^*$$

On the Investigation of the Maxima and Minima of Functions of one Variable.

47. The investigation of the greatest and least values of which a given function admits, forms one of the most important analytical applications of the Differential Calculus. It rests on the following principles.

When the variable on which any proposed function depends, passes successively through all degrees of magnitude, the different values of this function may form, at first, an increasing, and then a decreasing series. In this case one of these values must be greater than any of the others. If, on the contrary, the values of the proposed function form, at first, a decreasing, and afterwards an increasing series, there must necessarily occur one which is less than any of the others. That value at which the increase of the function ends, is called the *Maximum*, and that at which the diminution ends, and the increase begins, is called the *Minimum*.

Take, for example, the function $y = b - (x - a)^2$. In this equation, when $x = 0$, we have $y = b - a^2$, and the quantity $(x - a)^2$ decreasing as x increases, y is also increased until we have $x = a$, when $y = b$, which is a *maximum* value; but beyond this term, although x be increased, y diminishes, and becomes nothing, when $(x - a)^2 = b$. The progress of the proposed function is easily traced, and it may readily be shewn, that the greatest value of y corresponds to $x = a$; by substituting successively $a + \delta$ and $a - \delta$, instead of x, we find, in both cases, $y = b - \delta^2$, or always less than b.

---

* See Note E.

Again, let $y=b+(x-a)^2$. In this example, if x be equal to nothing, then $y=b+a^2$; afterwards, whilst x increases, the quantity $(x-a)^2$ diminishes, as well as y, until $x=a$; after this $(x-a)^2$ increases, and also y, whose *minimum* value consequently corresponds to the supposition $x=a$. This may be easily verified by substituting $a+\delta$ and $a-\delta$, instead of x: since we find in both cases $y=b+\delta^2$, which is always greater than b.

Every function which increases or decreases continually, whilst the variable on which it depends increases, admits neither of a *maximum* nor a *minimum*, since each succeeding value is always greater or less than the preceding.

The essential character of a maximum *consists in its being greater than both the values which immediately precede and follow it*; *that of a* minimum, *on the contrary, consists in its being less than both these values.*

It is necessary to say *immediately*, because it frequently happens, that a function admits of values which surpass its *maximum*, or which are less than its *minimum*, or that it admits of several *maxima*, or several *minima*, of unequal magnitudes: and this is not very difficult to conceive; for if, for example, the function, after having increased and then diminished, should again increase indefinitely, it would at length exceed the *maximum*, which it had before attained.

If this second increase, instead of being indefinite, should cease at a certain point, a new *maximum* would arise, which might be different from the former. It is easy to perceive what would take place were these changes repeated, and varied, in their respective extent.

We will now proceed to state the method of discovering the *maxima* and *minima* of functions of one variable.

48. Let y be any function whatever of x, in which this variable has attained that value which makes it a *maximum*, or a *minimum*: from what has been already observed, it

follows, that if we examine the values of y corresponding to $x-h$, and $x+h$, we ought, (however small the quantity h may be), to obtain results less than the *maximum*, or greater than the *minimum*. Denoting by ${}_{,}y$ the value of y, which corresponds to $x-h$, and by y' that which corresponds to $x+h$, we shall have, by Taylor's theorem (21)

$$ {}_{,}y = y - \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} - \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c. $$

$$ y' = y + \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c. $$

The powers of a quantity less than unity, becoming so much the smaller as the exponent is increased, it is easy to perceive that it is always possible to assume h so small, that the term $\frac{dy}{dx}h$ may exceed the sum of all those which follow it; and since this term enters into the values of ${}_{,}y$ and y', with different signs, it follows that one of these quantities must be greater than y, and the other less; consequently the proposed function cannot be a *maximum* or a *minimum*, unless $\frac{dy}{dx}$ be equal to nothing.* But if this coefficient vanishes, we must have, in that case,

---

* If there is any doubt concerning this assertion, it will be sufficient, for the purpose of showing its accuracy, to observe, that a series of the form

$$ A\,h + B\,h^2 + C\,h^3 + \&c. $$

may be put into this form:

$$ h\left\{ A + B\,h + Ch^2 + \&c. \right\} $$

and that the part

$$ B\,h + C\,h^2 + D\,h^3 + \&c. $$

vanishes when $h=0$; it may consequently become less than the quantity A, whose value remains the same, whatever be the value of h.

$$,y = y + \frac{d^2 y}{d x^2} \frac{h^2}{1.2} - \frac{d^3 y}{d x^3} \frac{h^3}{1.2.3} + \&c.$$

$$y' = y + \frac{d^2 y}{d x^2} \frac{h^2}{1.2} + \frac{d^3 y}{d x^3} \frac{h^3}{1.2.3} + \&c.$$

$,y$ and y' are both greater, or both less, than y, according as the value of the coëfficient $\frac{d^2 y}{d x^2}$ is positive or negative: in the first case y is a *minimum*, and in the second a *maximum*. From this it follows, *that the value of* x, *which is necessary to make a function* y *a* maximum, *or a* minimum, (for they are both given by the same equation), *is that which when substituted in the differential coefficient* $\frac{dy}{dx}$, *will make it equal to nothing.*

In the example $y = b - (x-a)^2$, considered above, we have $\frac{dy}{dx} = -2(x-a)$; and equating this to zero, we have $x = a$. To discover whether this value relates to a *maximum* or a *minimum*, we must examine the value of $\frac{d^2y}{dx^2}$; and since this is reduced to -2, a negative quantity, we may conclude that the value $x = a$ gives a *maximum* value of y.

Treating the function

$$y = b + (x - a)^2,$$

in the same manner, we find $x = a$; but in this case $\frac{d^2 y}{d x^2}$ will be a positive quantity: this value of x, therefore, must, in the case before us, correspond to a *minimum*.

49. We must not, however, conclude, because $\frac{dy}{dx} = 0$, in the case of a *maximum* or a *minimum*, that the one o the other must necessarily take place, whenever this condition is fulfilled. In fact, if the value of x, which makes

$\frac{dy}{dx}$ equal to zero, causes, at the same time, $\frac{d^2y}{dx^2}$ to vanish, and not $\frac{dy^3}{dx^3}$, since, in that case, we should have

$$,y = y - \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4} - \&c.$$

$$y' = y + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4} + \&c.$$

and since, by assigning a suitable value to h, the term $\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}$ may be made to surpass all those which follow it, there will no longer exist between the quantities $,y, y, y'$, that arrangement of magnitudes which agrees with a *maximum* or a *minimum*; the middle term being greater than one of the extremes, and less than the other: as we may observe in the function $y = b + (x-a)^3$.

But if the same value of x causes $\frac{d^3y}{dx^3}$ to vanish, we have

$$,y = y + \frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4} - \&c.$$

$$y' = y + \frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4} + \&c.$$

in which the conditions relative to the *maximum* and *minimum* would be again fulfilled; and we may discover from the sign of $\frac{d^4y}{dx^4}$ which of the two takes place. In this manner we shall find that the value $x = a$ gives a *maximum* for the function $y = b - (a-x)^4$, and a *minimum* for $y = b + (a-x)^4$.

Without carrying these considerations any farther, we may readily perceive, that a *maximum* or a *minimum* can only take place when the first of the differential coefficients, which does not vanish, is of an even order, this coefficient

being negative in the case of a *maximum*, and positive in the case of a *minimum*.

As we shall have occasion to return to this subject, when we consider the theory of curves, a few applications will be sufficient for the present.

50. Suppose that it was required *to divide a quantity into two such parts, that the* m*th power of the one multiplied by the* n*th power of the other, may be greater than any other similar product formed of the parts of the same quantity.*

If x be one of the parts of the quantity a, the other will be $a-x$, and the product, whose *maximum* we seek, being represented by y, we shall have $y=x^m(a-x)^n$, whence we deduce

$$\frac{dy}{dx}=m\,x^{m-1}(a-x)^n-n\,x^m(a-x)^{n-1}$$

$$=[\,m\,a-m\,x-n\,x\,]\,x^{m-1}(a-x)^{n-1};$$

and making each of these factors equal to zero, we shall find

$$x=\frac{m\,a}{m+n},\quad x=0,\quad x=a.$$

The first of these values corresponds to a *maximum*; for when we substitute it in the general expression for $\frac{d^2y}{dx^2}$, it gives the negative quantity

$$-\frac{m^{m-1}\,n^{n-1}\,a^{m+n-2}}{(m+n)^{m+n-2}}:$$

the two others correspond to *minima*, when m and n are even, as we may readily see, if we examine the differential coefficients; or more simply, if we make $x=\pm h$ and $x=a\pm h$. We shall constantly find a positive result in both cases, whatever be the sign of the quantity h: this shews that the proposed function, after having decreased to zero, does not then become negative, but immediately begins to increase.

51. Let us consider the function denoted by y in the equation

$$y^2 - 2mxy + x^2 - a^2 = 0,$$

whose differential is

$$(y - mx)\,dy - (my - x)\,dx = 0, \quad (38)$$

whence

$$\frac{dy}{dx} = \frac{my - x}{y - mx},$$

from which we have

$$my - x = 0.$$

In order to obtain the value of x, we must combine this equation with the one proposed; by this means we get

$$y = \frac{x}{m}, \quad \frac{x^2}{m^2} - x^2 - a^2 = 0;$$

from which there results

$$x = \frac{ma}{\sqrt{1 - m^2}}, \quad y = \frac{a}{\sqrt{1 - m^2}}.$$

We must now examine the value of the differential coefficient $\frac{d^2y}{dx^2}$. The second differential of the proposed equation gives the following one:

$$(y - mx)\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} - 2m\frac{dy}{dx} + 1 = 0,$$

and if we make $\frac{dy}{dx} = 0$, it becomes

$$(y - mx)\frac{d^2y}{dx^2} + 1 = 0,$$

whence we deduce, by putting for y its value, in terms of x,

$$\frac{d^2y}{dx^2} = \frac{-m}{x(1 - m^2)};$$

we must now substitute the value of x, and we have

$$\frac{d^2y}{dx^2} = -\frac{1}{a\sqrt{1-m^2}}:$$

this result being negative, shows that the value of y, which has just been determined, is a *maximum*.

Of the Values of Differential Coefficients in certain Circumstances, and of Expressions which become $\frac{0}{0}$.

52. If we enquire into the value of the *maximum* or the *minimum* of the function $a\,y = \sqrt{a^2x^2 - x^4}$, we easily deduce $a\frac{dy}{dx} = \frac{a^2x - 2x^3}{\sqrt{a^2x^2 - x^4}}$; and making $x = 0$, it becomes

$a\,\frac{dy}{dx} = \frac{0}{0}$.

With a little attention, however, we may perceive that the numerator and denominator of the fraction $\frac{a^2x - x^3}{\sqrt{a^2x^2 - x^4}}$ vanish at the same time, only because they are multiplied by the common factor x. If we free them from this factor, we shall have $a\,\frac{dy}{dx} = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$, and consequently $\frac{dy}{dx} = \pm 1$, when $x = 0$.

Generally, when we make $x = a$, in an expression of the form $\frac{P(x-a)^m}{Q(x-a)^n}$, it becomes $\frac{0}{0}$; nevertheless its real value is either nothing, or finite, or infinite, according as $m > n$, $m = n$, or $m < n$; for by suppressing the factors common to the numerator and denominator, we have $\frac{P(x-a)^{m-n}}{Q}$ in the first case; $\frac{P}{Q}$ in the second, and $\frac{P}{Q(x-a)^{n-m}}$ in the third; it being understood, that the quantities P and Q neither become evanescent nor infinite by the supposition of $x = a$.

Whenever, therefore, an expression presents itself under the form $\frac{0}{0}$, we must, in order to discover its real value, disengage it from the factors which are common to its numerator and denominator; differentiation will furnish us with the means by which this may be effected.

The differential of the expression $P(x-a)$, in which P denotes any function of x which does not involve the factor $x-a$, being

$$(x-a)\,dP+P\,dx,$$

will not vanish when $x=a$.

If we differentiate twice in succession the function $P(x-a)^2$, we should find

$$(x-a)^2\,dP+2\,(x-a)\,P\,dx,$$
$$(x-a)^2\,d^2P+4\,(x-a)\,dP\,dx+1.2\,P\,dx^2;$$

and since P does not contain $x-a$, the second differential is reduced to its last term. By pursuing this method, we may easily prove that all the differentials of an expression of the form of $P(x-a)^m$ as far as the order $m-1$ inclusively, vanish on the supposition of $x=a$, whenever m is a whole number, and that the mth differential will be reduced to $1.2.3\ldots m\,P\,dx^m$: the factor $(x-a)^m$ would disappear therefore, on this hypothesis, after m differentiations.

It is not necessary that we should know the value of the exponent m; neither is it necessary that the factor $(x-a)^m$ should be apparent, in order to know when the expression $P(x-a)^m$ is freed from it; it is sufficient to try after each differentiation, whether the result vanishes or not, when a is put for x: in the latter case the operation is completed, and the result we have found represents the quantity $1.2\ldots m\,P\,dx^m$. For example, take the function $x^3-ax^2-a^2x+a^3$, which vanishes by the supposition of $x=a$; its first differential vanishes on the same hypothesis, but its second differential, which is $(6x-2a)\,dx^2$ does not vanish. It is, therefore, freed from the factor

$(x-a)^2$; and since two differentiations were necessary for that purpose, we conclude, that it is of the form $P(x-a)^2$, which may be easily ascertained by other means; for

$$x^3-ax^2-a^2x+a^3=(x+a)(x-a)^2.$$

This being premised, if in the case of $m=n$ we differentiate both the numerator and denominator of the fraction $\frac{P(x-a)^m}{Q(x-a)^n}$, m times successively, they will be disengaged from the factor $x-a$; for we have, when $x=a$

$$\frac{d^m.P(x-a)^m}{d^m Q(x-a)^m}=\frac{1.2.3\ldots m P d x^m}{1.2.3\ldots m Q d x^m}=\frac{P}{Q}.$$

If it is the numerator which first affords a result which does not vanish, it is a proof that the factor $x-a$ is raised to a higher power in the denominator, and consequently that the proposed fraction is infinite; if, on the contrary, the denominator first affords a result, which does not vanish, the proposed fraction is nothing. The rule may, therefore, be thus enunciated. *To obtain the true value of a function which becomes* $\frac{0}{0}$, *when we assign to* x *some particular value, we must differentiate the numerator and the denominator, until we find for one or for the other a result which does not vanish: the proposed function will be infinite in the first case, and zero in the second; and if we find, at the same time, two results which do not vanish, it will have a finite value.*

A few examples will render this sufficiently clear.

53. 1st. The formula $\frac{x^n-1}{x-1}$ which expresses the sum of the n first terms of the geometrical series $1, x, x^2, x^3$, &c. becomes $\frac{0}{0}$, when $x=1$; the sum, however, of the series, in this case, has a determinate value, and is equal to n, as we shall also find from the preceding rule. In fact, by differentiating the numerator and denominator of the expression $\frac{x^n-1}{x-1}$, we find $\frac{n x^{n-1} d x}{d x}$, and putting unity for x, it becomes equal to n.

2d. The real value of $\frac{a x^2 - 2 a c x + a c^2}{b x^2 - 2 b c x + b c^2}$, in the case of $x = c$, can only be obtained after two differentiations; for the first gives $\frac{a x - a c}{b x - b c}$, a result which becomes $\frac{0}{0}$, when $x = c$; but by differentiating again, it becomes $\frac{a}{b}$.

3d. If we enquire into the value of $\frac{x^3 - a x^2 - a^2 x + a^3}{x^2 - a^2}$, when $x = a$, we shall find, after one differentiation of the numerator and the denominator, that the former alone becomes equal to nothing, when we put a in the place of x; from which we learn, that the value of the proposed function is zero. The contrary would be the case, for the function

$$\frac{a x - x^2}{a^4 - 2 a^3 x + 2 a x^3 - x^4}.$$

4th. Although we do not immediately see how it is possible to give the form $\frac{P (x - a)^m}{Q (x - a)^n}$ to the transcendental function $\frac{a^x - b^x}{x}$, which becomes $\frac{0}{0}$, when $x = 0$, we may nevertheless apply the rule to it, and after having differentiated its numerator and denominator, we find $a^x 1 a - b^x 1 b$: and putting 0 for x, we have $1 a - 1 b$, which is its true value.

This result might be immediately obtained by substituting the developement of the functions a^x and b^x: for, there thence arises

$$\frac{a^x - b^x}{x} = (1 a - 1 b) + \left\{ (1 a)^2 - (1 b)^2 \right\} \frac{x}{1 . 2} + \&c.$$

and the supposition of $x = 0$ reduces the second member of the equation to its first term. We may remark in this operation, that the factor x has disappeared by division.

5th. The function $\frac{1-\sin x+\cos x}{\sin x+\cos x-1}$ is reduced to $\frac{0}{0}$, when the arc $x=\frac{\pi}{2}$; but by applying the rule to it, we shall find, that its true value is in that case 1.

6th. The reader may exercise himself on the functions

$$\frac{a-x-a\,\mathrm{l}\,a+a\,\mathrm{l}\,x}{a-\sqrt{2ax-x^2}}, \quad \frac{x^x-x}{1-x+\mathrm{l}\,x};$$

the former becomes $\frac{0}{0}$, when $x=a$, and the latter when $x=1$: their real values are respectively -1 and -2.

54. The rule in No. 52. would not be applicable to those cases, where the factors are raised to fractional powers; for since the successive differentiations only abstract units from the exponent m of the factor $x-a$, they can never exhaust that exponent when it is a fraction: it will only become negative when the number of differentiations surpasses the greatest whole number which it contains (13). If we had, for example, $\frac{(x^2-a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}$, although the true value of this fraction, when $x=a$, is $(2a)^{\frac{3}{2}}$, we should never arrive at it by differentiation: we should find, successively,

$$\frac{3x(x^2-a^2)^{\frac{1}{2}}}{\frac{3}{2}(x-a)^{\frac{1}{2}}},$$

$$\frac{3(x^2-a^2)^{\frac{1}{2}}+3x^2(x^2-a^2)^{-\frac{1}{2}}}{\frac{1}{2}\cdot\frac{3}{2}(x-a)^{-\frac{1}{2}}}, \text{ \&c.}$$

the first of these results becomes $\frac{0}{0}$, when $x=a$, and the same supposition renders infinite the numerators and denominators of all those which follow. If we make the negative exponents disappear by transferring into the denominator those which appear in the numerator, and *vice versâ*, the new expressions which result from this change will all reduce themselves to $\frac{0}{0}$.

55. This difficulty results from the differential not being of the form $p\,d\,x$, in the case in which a particular value of x causes an irrational part of the function to disappear.

If we had, for example, $y = b + \sqrt{x-a}$, and if we wished, when $x=a$, to find the consecutive value of y, we must put $a+d\,x$, in the place of x, and there arises

$$y' = a + \sqrt{d\,x},$$

and the difference will be

$$y' - y = \sqrt{d\,x} = d\,x^{\frac{1}{2}}.$$

It is reduced to the single term $d\,x^{\frac{1}{2}}$, which is consequently, the differential relative to this case: from this we deduce

$$\frac{y'-y}{d\,x} = \frac{d\,x^{\frac{1}{2}}}{d\,x} = \frac{1}{d\,x^{\frac{1}{2}}},$$

an expression whose denominator alone vanishes when $dx=0$, and from which it results, that the differential coefficient $\frac{d\,y}{d\,x}$ is infinite for the particular value of $x=a$.

In the sequel the consideration of curves will explain more clearly this species of paradox.*

56. The following method is general, exceedingly simple, and comprehends also the rule in No. 52. It was only reserved for this place, because the preceding considerations seemed better calculated to explain the nature and difficulties of the subject.

Let $\frac{X}{X'}$ be a function whose numerator and denominator become evanescent, when $x=a$; substituting $a+h$ for x, the functions X and X' may be developed in *ascending* series of the form

$$A\,h^{\alpha} + B\,h^{\beta} + \text{\&c.} \qquad A'\,h^{\alpha'} + B'\,h^{\beta'} + \text{\&c.}$$

that is to say, in which the exponents α, β, &c. continue

* See Note (F).

increasing, and are positive; since these series ought to vanish on the hypothesis of $h=0$, which corresponds to that of $x=a$: we shall have, therefore,

$$\frac{Ah^{\alpha}+Bh^{\beta}+\&c.}{A'h^{\alpha'}+B'h^{\beta'}+\&c.}$$

instead of the proposed fraction. If, in this result, we suppose $h=0$, we ought to have the value which the function $\frac{X}{X'}$ assumes, when we change x into a; and although it appears at first, that it would, in that case, reduce itself to $\frac{0}{0}$, we shall presently see that it has always a determinate value.

Distinguishing the three cases of $\alpha > \alpha'$, $\alpha = \alpha'$, and $\alpha < \alpha'$, we may, in the two first cases, write the expression thus:

$$\frac{Ah^{\alpha-\alpha'}+Bh^{\beta-\alpha'}+\&c.}{A'+B'h^{\beta'-\alpha'}+\&c.}$$

Under this form it is easy to perceive, that as long as α is greater than α', the supposition of $h=0$ will make the fraction equal to nothing, and that it will be reduced to $\frac{A}{A'}$, when $\alpha=\alpha'$. In the third case, on the contrary, when $\alpha<\alpha'$, we have

$$\frac{A+Bh^{\beta-\alpha}+\&c.}{A'h^{\alpha'-\alpha}+B'h^{\beta'-\alpha}+\&c.}$$

and this result becomes infinite, by the supposition of $h=0$. In all cases, the true value depends on the first term alone of each series.

The following rule extends to all functions which can present themselves under the indeterminate form $\frac{0}{0}$. *Take the first term of each of the series which express the developement of the numerator and denominator, when* $x=a+h$; *reduce the resulting fraction to its most simple form, and then make* $h=0$:

the result thus obtained will be the value of the proposed fraction, when x=a.

The fraction $\frac{(x-a)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}$, whose value we could not find by differentiation, when $x=a$ (54), becomes by this method

$$\frac{(2ah+h^2)^{\frac{3}{2}}}{h^{\frac{3}{2}}}=(2a+h)^{\frac{3}{2}},$$

when x is changed into $x+h$; making $h=0$, we obtain its true value $(2a)^{\frac{3}{2}}$.

The same method will be sometimes more convenient than differentiation, even when that process is applicable. For example, it requires four successive differentiations of the numerator and denominator of the fraction

$$\frac{x^3-4ax^2+7a^2x-2a^3-2a^2\sqrt{2ax-a^2}}{x^2-2ax-a^2+2a\sqrt{2ax-x^2}},$$

to arrive at its true value, in the case of $x=a$.

Writing $a+h$, instead of x, as the rule directs, it becomes

$$\frac{2a^3+2a^2h-ah^2+h^3-2a^2\sqrt{a^2+2ah}}{-2a^2+h^2+2a\sqrt{a^2-h^2}};$$

and reducing the radicals into series, we shall have

$$\sqrt{a^2+2ah}=a+h-\frac{h^2}{2a}+\frac{h^3}{2a^2}-\frac{5h^4}{8a^3}+\&c.$$

$$\sqrt{a^2-h^2}=a-\frac{h^2}{2a}-\frac{h^4}{8a^3}-\&c.$$

The substitution of these two series in the preceding fraction will give $-5a$ for the true value required.

57. A function may present itself under several indeterminate forms, apparently different from that of $\frac{0}{0}$: it is proper to be acquainted with these forms, which will be

found, upon examination, to be identical with the one preceding.

1st. The numerator and denominator of the fraction $\frac{X}{X'}$, may become infinite at the same time; but this fraction being written thus: $\frac{\frac{1}{X'}}{\frac{1}{X}}$, is reduced to the form $\frac{0}{0}$, when X and X' are infinite.

2d. We may sometimes meet with a product composed of two factors, one infinite, and the other nothing. Let $P\,Q$ be a product of this kind, in which the supposition of $x=a$ gives $P=0$, $Q=\frac{b}{0}$; we may write it thus: $P\,Q = \frac{P}{\frac{1}{Q}}$, and since $\frac{1}{Q}=0$, we have

$$P\,Q=\frac{0}{0}\ *.$$

---

* The method of No. 56. presents some difficulties, when it is required to apply it to differential coefficients, given by an equation, in which the variables x and y are mixed. We must have recourse to particular artifices for deducing from the proposed primitive equation, a value of y, developed and arranged according to the powers of h. (See the larger Treatise on the Differential and Integral Calculus.)

When, however, the equation is free from radicals, we may arrive at the true value of $\frac{dy}{dx}$, by the considerations indicated in the note at page 41.

In fact, the equation $M+Np=0$, giving $p=-\frac{M}{N}$, leads to $p=\frac{0}{0}$, when the quantities M and N vanish together; but in this case, the complete developement, from which the equation is deduced, is

$Mh+$

58. If we wish to obtain the value of the function $\frac{1\,x}{x^n}$, when x is infinite, or, what amounts to the same thing, to obtain its limit, we shall not be able to obtain it by any of the methods we have just employed, on account of the impossibility of reducing x to a series of the required form; we must, therefore, have recourse to some considerations peculiar to the nature of the proposed function $1\,x$.

Changing x into n, and a into x, in the developement of a^x (24), we find

$$x^n = 1 + \frac{n\,1\,x}{1} + \frac{n^2(1\,x)^2}{1.2} + \frac{n^3(1\,x)^3}{1.2.3} + \&c.$$

whence we conclude that

$$\frac{1\,x}{x^n} = \frac{1\,x}{1 + \frac{n\,1\,x}{1} + \frac{n^2(1\,x)^2}{1.2} + \frac{n^3(1\,x)^3}{1.2.3} + \&c.}$$

$$= \frac{1}{\frac{1}{1\,x} + n + \frac{n^2\,1\,x}{1.2} + \frac{n^3(1\,x)^2}{1.2.3} + \&c.}$$

$$Mh + N\pi h + Ph^2 + \mathfrak{Q}\pi h^2 + R\pi^2 h^2 + Sh^3 + \&c. = 0;$$

which, in this case, becomes

$$Ph^2 + \mathfrak{Q}\pi h^2 + R\pi^2 h^2 + Sh^3 + \&c. = 0,$$

and is divisible by h^2: by taking the limits, making $h=0$, and changing π into p, we have

$$P + \mathfrak{Q}p + Rp^2 = 0;$$

and, in this case, we find two values of p.

If the values of x and y, which make the quantities M and N disappear, likewise annihilate the quantities P, $\mathfrak{Q}$, R, we must have recourse to those terms, in which the increment h rises to the third power, in order to obtain p, which will then have three values. We shall afterwards have occasion to observe, in what manner the quantities P, $\mathfrak{Q}$, R, &c. which we may calculate, *a priori*, by making the substitution indicated in the note, page 41, may also be formed by differentiation.

a quantity which approximates towards zero, in proportion as x augments, unless n is incomparably smaller than $\frac{1}{\mathrm{l}\,x}$ (57).

59. An equation $V=0$, which has n equal roots, is of the form

$$V=P(x-a)^n=0,$$

the factor P containing the unequal roots; and it follows, from what has been said, in No. 52, that all the differential coefficients $\frac{dV}{dx}$, $\frac{d^2V}{dx^2}$, as far as $\frac{d^{n-1}V}{dx^{n-1}}$ inclusively, vanish by the supposition of $x=a$, since they all contain the factor $x-a$. The equations

$$V=0,\ \frac{dV}{dx}=0,\ \frac{d^2V}{dx^2}=0,\ \ldots\ \frac{d^{n-1}V}{dx^{n-1}}=0,$$

will, therefore, all hold good at the same time; and if we seek the factor common to the first and second equations, which are respectively

$$P(x-a)^n=0, \qquad \frac{dP}{dx}(x-a)^n+nP(x-a)^{n-1}=0,$$

it is evident that we shall find $(x-a)^{n-1}$.

We may easily recognise the equations $\frac{dV}{dx}=0$, $\frac{d^2V}{dx^2}=0$, &c. as precisely the same with those denoted by (*A*), (*B*), &c. in No. 205. of the Elements of Algebra.

These considerations may be easily applied to the case in which the proposed equation contains several sets of equal roots; that is to say, when it is of the form

$$X(x-a)^n(x-b)^p=0;$$

for by differentiating the first member, according to the rule of No. 11, we find

$$\left.\begin{array}{r}(x-a)^n (x-b)^p \dfrac{d X}{d x} + nX (x-a)^{n-1} (x-b)^p \\ + p X (x-a)^n (x-b)^{p-1}\end{array}\right\}$$

a quantity which vanishes when $x = a$, and $x = b$; and which has, with the given equation, the common divisor

$$(x-a)^{n-1} (x-b)^{p-1}.$$

We may operate in the same manner, whatever be the number of factors $(x-a)^n$, $(x-b)^p$, $(x-c)^q$, &c. and we shall always find, *that the divisor common to the equations* $V=0$, *and* $\frac{dV}{dx}=0$, *will contain the equal factors, each raised to a power less by unity, than the same factors in the proposed equation* $V=0$.

---

*On the Application of the Differential Calculus to the Theory of Curves.**

60. It was in the course of enquiries relative to curve lines that Geometers first arrived at the Differential Calculus, which has since been exhibited under so many different points of view; but whatever may be the origin we assign to this calculus, it will always depend on an *analytical fact* antecedent to any hypothesis, as the phænomenon of the fall of heavy bodies to the surface of the earth, is antecedent to all explanations that have been given of it; and this fact is precisely that property which all functions possess, of admitting a limit in the ratio between their increments and that of the variable on which they depend. This limit, which is different for different functions; but constantly the same for the same function, and which is always independent of the absolute values of the increments themselves, characterises, in a peculiar manner, the *course* of the function in the different stages through which it may pass. In fact, the smaller the limits of the independent

---

* See Note (G.)

variable, the more nearly the successive values of the function approximate to each other; the more does the function also approximate to coincidence with the law of continuity; and the more nearly does the ratio of its changes to that of the independent variable approximate to the limit assigned by the calculus. By the law of continuity is meant that which is observed in the description of lines by motion, and according to which the consecutive points of the same line, succeed each other without any interval. The method of considering magnitude in analysis does not appear to admit of this law, since we always suppose an interval between two consecutive values of the same quantity; but the smaller this interval is, the more nearly we approach to the law of continuity, with which the limit accurately agrees: it is also in virtue of this law that the increments, although evanescent, still preserve the ratio to which they have gradually approached, before they vanish.

The preceding statement appears to involve the true and philosophical explanation of the nature and properties of the Differential and Integral Calculus, when viewed in its application to questions connected with curve lines, and the theory of motion. The difficulty, in both cases, arises from the existence of a continuity, in the changes of lines and of velocities; and the consideration of limits (or any other equivalent to it), furnishes the means of establishing this continuity in the Calculus.

61. Geometrical considerations show very clearly that the ratio of the increments of a function and its variable, is, generally, susceptible of limits.

Every function of one variable may be represented by the ordinate of a curve, whose abscissa is the variable itself (Trig. 77)*; and the ratio of the ordinate of a curve to its

* "Our author here refers to his Traité Elémentaire de "Trigonométrie Rectiligne et Spherique, et d'Application "d'Algébre à la Geométrie, which forms a part of his Elementary System of Analysis."

subtangent, corresponds to the differential coefficient of that function. In fact, if in any curve whatever, $C\,D$, fig. 1, we draw through two points M and M', a secant $M\,M'$, which is prolonged until it meets the axis of the abscissæ $A\,B$, in the point S; and if we also draw the two ordinates $P\,M$, $P'\,M'$, and the right line $M\,Q$, parallel to $A\,B$, we have, from the similar triangles $M'\,Q\,M$ and $M\,P\,S$, the ratio $\frac{M'\,Q}{M\,Q}$, equal to the ratio $\frac{P\,M}{P\,S}$: but if we conceive the point M' to approach continually to the point M, the point S will also approach towards the point T; and consequently the line $P\,S$ will constantly tend to become equal to the subtangent $P\,T$; the ratio $\frac{P\,M}{P\,S}$ will, therefore, approximate to the ratio $\frac{P\,M}{P\,T}$, which will be its limit; and also that of the ratio of the increments $M\,Q$, and $M'\,Q$, which the abscissa and ordinate simultaneously receive. FIG. 1.

From this it follows, that when the function which represents the ordinate is known, its differential coefficient will give the expression for the ratio $\frac{P\,M}{P\,T}$, and, reciprocally, if this ratio is found by other means, it will furnish the differential coefficient of the function corresponding to the ordinate.*

---

* Although there can scarcely exist a doubt of the equality of two quantities which are the limit of the same variable quantity, yet it has been usual to demonstrate its truth in the following manner:

Let A and B be the two first quantities, and V the third. If $A = B - D$, and if V be always less than A, however near it may be to that magnitude, the difference between it and B will always exceed D. Nor can it be said that V is contained between A and B; for it would then differ from B or from A, by a quantity at least equal to $\frac{1}{2}\,D$: thus, in all cases, V cannot

62. When we assign successive values to the abscissæ, the ordinates which correspond to these values will determine points in the curve, which may be regarded as being situated at the angles of a polygon inscribed in that curve.

FIG. 2. If we take, for example, on the axis of the abscissæ the points P, P', P'', fig. 2, distant from each other by the constant quantity h, we shall have

$$AP=x, \quad AP'=x+h, \quad AP''=x+2h, \text{ \&c.}$$

and if we draw the ordinates PM, $P'M'$, $P''M''$, and join the points M, M', M'', &c. by the chords of the intercepted arcs, we shall form the polygon M, M', M'', &c. which will differ so much the less from the proposed curve, the nearer the points M, M', M'' are to each other. But at the same time, the number of sides will be increased, since the distance PP' will be contained a proportionally greater number of times in the given abscissa AB. The curve CD will be evidently the limit of all these polygons; and consequently whatever properties can be proved to belong to this limit, must belong likewise to the proposed curve.*

V cannot approach at the same time, as near as we please, to the two quantities A and B, which is contrary to the definition of limits.

* Leibnitz always considered the Differential Calculus under a point of view very nearly similar.

" Sentio autem et hanc et alias (methodos) hactenùs adhibi-
" tas omnes deduci posse ex generali quodam meo dimetiendo-
" rum curvilineorum principio, *quod figura curvilinea censenda*
" *sit æquipollere polygono infinitorum laterum*; unde sequitur,
" quicquid de tali polygono demonstrari potest, sive ità, ut
" nullus habeatur ad numerum laterum respectus, sive ità, ut
" tantò magis verificetur, quanto major sumitur laterum nume-
" rus, ità, ut error tandem fiat quovis dato minor; id de curvà
" posse pronuntiari." (*Acta Eruditorum ann.* 1684, *page* 585.)

This explanation of the principles of the Differential Calculus is very luminous, and differs from that above given in one circumstance

This being premised, if we draw $M Q$ and $M' Q'$ parallel to the axis $A B$, $M' Q$ will be the difference of two consecutive ordinates, $P M$ and $P' M'$, and $M'' Q'$ that of the ordinates $P' M'$ and $P'' M''$. Producing the right line $M M'$ to N', we shall form the equal triangles $M M' Q$, and $M' N' Q'$, which will give $M' Q = N'' Q$; from this there results

$$M' N' = N' Q' - M'' Q', \text{ or } M'' N'' = M'' Q' - N'' Q';$$

and consequently $M'' Q' - M' Q = \mp M'' N''$, according as the curve is concave or convex towards the axis of the abscissæ: $M'' N''$ will, therefore, be the difference of the lines $M' Q$ and $M'' Q'$.

The Differential Calculus will give us expressions for these different lines; for we have (21).

$$P M = y$$

$$P' M' = y + \frac{dy}{dx}\frac{h}{1} + \frac{d^2 y}{dx^2}\frac{h^2}{1.2} + \&c.$$

$$P'' M'' = y + \frac{dy}{dx}\frac{2h}{1} + \frac{d^2 y}{dx^2}\frac{4h^2}{1.2} + \&c.$$

$$P' M' - P M = M' Q = \frac{dy}{dx}\frac{h}{1} + \frac{d^2 y}{dx^2}\frac{h^2}{1.2} + \&c.$$

$$P'' M'' - P' M' = M'' Q' = \frac{dy}{dx}\frac{h}{1} + \frac{d^2 y}{dx^2}\frac{3h^2}{1.2} + \&c.$$

$$M'' Q' - M' Q = \mp M'' N'' = \frac{d^2 y}{dx^2} h^2 + \&c.$$

from whence it follows, that if we take $h = dx$, the value of $M' Q$ will approach nearer and nearer to the first differential dy; and that of $M'' N''$ to the second differential

---

cumstance only, that the *limit* is here considered as a polygon of an infinite number of infinitely small sides.

d^2y. By considering a fourth point in the polygon, we should, in a similar manner, find a line corresponding to the third differential.

63. The lines PM, $M'Q$, $M''N'$, have, in relation to the calculus of limits, a remarkable subordination, indicated by the exponents with which the increment h is affected in the first term of their respective expressions, which exponent is the same as that of the order of the differential to which they correspond. We observe, in fact, that the ratio of $M'Q$ to PM, continually diminishes, and at last vanishes, when $h=0$, which is likewise the case with the ratio of $M''N''$ to $M'Q$; but if we compare the first of these with the square of the second, the ratio will have an assignable limit, which will be that of $\frac{d^2y}{dx^2}$ to $\frac{dy^2}{dx^2}$ (56).*

64. We may observe, from what precedes, that the differential coefficient of the first order $\frac{dy}{dx}$, which expresses the ratio, $\frac{PM}{PT}$, fig. 1, gives also the value of the trigonometrical tangent of the angle MTP, which the right line, touching the curve in the point M, makes with AB, the axis of the abscissæ.

FIG. 1.

Further, if we observe, that when the ordinate is positive, the difference $M''Q' - M'Q$, fig. 2, is negative or positive, according as the curve is concave or convex towards the axis of the abscissæ; and as this must be the case,

FIG. 2.

---

* This furnishes a very simple explanation of the various orders of infinitesimals admitted by Leibnitz. He considered the first differential as infinitely small, with respect to the ordinate; the second differential as infinitely small, with respect to the first; and so on successively. Upon this principle, he neglected differentials of all orders, when compared with those preceding, which is, in fact, what must always be virtually done whenever we would pass to the consideration of the limits.

however small we suppose the distances of the points P, P', P'', or the value of h, to be, we conclude that the first term $\frac{d^2 y}{d x^2} h^2$, in the expression for $M'' Q' - M' Q$, which may be rendered greater than the sum of all those which succeed it, must consequently have the same sign as the difference $M'' Q' - M' Q$ itself; but the quantity h^2 being essentially positive, it follows, from what we have said, that $\frac{d^2 y}{d x^2}$ is negative or positive, according as the curve is concave or convex to the axis of the abscissæ.

The inspection of the curves $c\,m$, placed below the axis of the abscissæ, shows that the signs of $\frac{d^2 y}{d x^2}$ ought to be taken in an inverse order, when the ordinate is negative; and that consequently, *a curve is convex or concave towards the axis of the abscissæ, according as the ordinate and its second differential coefficient are of the same or different signs.*

65. Since the angle $M\,T\,P$ becomes known by means of the differential coefficient $\frac{d y}{d x}$, we shall find no difficulty in constructing the tangent $M\,T$, fig. 1, but we generally make use of the subtangent $P\,T$, which may be readily calculated, by observing that FIG. 1.

$$\frac{P\,M}{P\,T} = \frac{d y}{d x} \text{ gives } P\,T = \frac{P\,M\,d x}{d y} = \frac{y\,d x}{d y}.$$

The triangle $P\,M\,T$, which has a right angle at T gives the *tangent*

$$M\,T = \sqrt{\overline{P\,M}^2 + \overline{P\,T}^2} = y\sqrt{1 + \frac{d x^2}{d y^2}}.$$

The consideration of the similar triangles $P\,M\,T$ and $P\,M\,R$, gives the *subnormal*

$$P\,R = P\,M\,\frac{P\,M}{P\,T} = \frac{y\,d y}{d x}.$$

The triangle PMR, which has a right angle at P, gives the *normal*

$$MR = \sqrt{\overline{PM}^2 + \overline{PR}^2} = y\sqrt{1 + \frac{dy^2}{dx^2}}.$$

66. The following are a few examples of the application of these formulæ.

The general equation of lines of the second order being

$$y^2 = mx + nx^2, \quad \text{(Trig. 148.)},$$

we have

$$\frac{dy}{dx} = \frac{m + 2nx}{2y} = \frac{m + 2nx}{2\sqrt{mx + nx^2}},$$

from which we deduce

$$PT = \frac{y\,dx}{dy} = \frac{2(mx + nx^2)}{m + 2nx}$$

$$MT = y\sqrt{1 + \frac{dx^2}{dy^2}} = \sqrt{mx + nx^2 + 4\left(\frac{mx + nx^2}{m + 2nx}\right)^2}$$

$$PR = \frac{y\,dy}{dx} = \frac{m + 2nx}{2}$$

$$MR = y\sqrt{1 + \frac{dy^2}{dx^2}} = \sqrt{mx + nx^2 + \tfrac{1}{4}(m + 2nx)^2}.$$

When $n = 0$, the curve becomes a parabola (Trig. 114.), in which case we have

$$PT = 2x, \qquad MT = \sqrt{mx + 4x^2}$$

$$PR = \frac{m}{2} \qquad MR = \sqrt{mx + \tfrac{1}{4}m^2}.$$

We may deduce from these values, the results and constructions indicated in the application of Algebra to Geometry, for lines of the second order.

In the curve represented by the equation

$$x^3 - 3axy + y^3 = 0,$$

we have

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax},$$

and we shall find

$$PT = \frac{y^3 - axy}{ay - x^2} = \frac{2axy - x^3}{ay - x^2},$$

which value may be easily constructed when we have assigned the value of x, and determined that of y. (Trig. 64.)

67. It is often more convenient, and besides more elegant, to consider the tangent and the normal, by means of their equation (Trig. 148). To obtain that of the former, let us examine generally what relations ought to exist so that any two lines may touch each other. Considering these lines as at first having two points M and M', fig. 1, in common, it is evident that their equations ought to give the same values of the ordinate PM, and the difference $M'Q$, corresponding to the abscissa AP, and its increment PP'. If then x and y denote the co-ordinates of the point M of the proposed curve, and if we denote by x' and y' those of any point in the line which cuts it in M and M', we shall have for these two points FIG. 1.

$$y' = y, \qquad \frac{dy'}{dx'}h + \text{\&c.} = \frac{dy}{dx}h + \text{\&c.} \quad (62.)$$

The second equation is divisible by h, and when we take the limit, and suppose $h = 0$, it is reduced to

$$\frac{dy'}{dx'} = \frac{dy}{dx};$$

But on this hypothesis, the two points of intersection are united in one, which becomes a point of contact for the proposed lines; since they have now only this one point in common. It follows from this, that when two lines touch each other, we have, for their point of contact,

$$y' = y, \ \frac{dy'}{dx'} = \frac{dy}{dx}.$$

When one of these is a right line, whose equation is of the form $y' = Ax' + B$, (Trig. 83), and which gives $\frac{dy'}{dx'} = A$,

we have, for the contact of this line with the curve proposed,

$$y = Ax + B, \qquad A = \frac{dy}{dx};$$

whence we conclude, that

$$B = y - x\frac{dy}{dx}, \text{ and } y' = \frac{dy}{dx}x' + y - x\frac{dy}{dx},$$

or

$$y' - y = \frac{dy}{dx}(x' - x);$$

from this equation we deduce that of the normal, which is perpendicular to the tangent, and which passes through the point M: this will be

$$y' - y = -\frac{dx}{dy}(x' - x) \qquad \text{(Trig. 86.)}$$

For the circle, whose equation is

$$y^2 + x^2 = a^2,$$

we have

$$\frac{dy}{dx} = -\frac{x}{y},$$

and the equation of its tangent will therefore be

$$y' - y = -\frac{x}{y}(x' - x), \text{ or } yy' - y^2 = -xx' + x^2;$$

or

$$yy' + xx' = a^2;$$

since $x^2 + y^2 = a^2$.

The equation of its normal becomes

$$y' - y = \frac{y}{x}(x' - x),$$

which is reduced to

$$y' = \frac{y}{x}x';$$

this proves that all the normals of the circle pass through its centre, which is here the origin of the co-ordinates (Trig. 83.), and this ought to be the case, since the normals to the circle are nothing more than its radii.

The tangent of the curve given by the equation

$$x^3-3\,a\,x\,y+y^3=0,$$

has for its equation

$$y'-y=\frac{a\,y-x^2}{y^2-a\,x}\,(x'-x);$$

whence

$$y^2\,y'-a\,x\,y'-y^3+a\,x\,y=a\,y\,x'-x^2\,x'-a\,x\,y+x^3;$$

and if we put for y^3 its value, and reduce the equation, we obtain

$$(y^2-a\,x)\,y'+(x^2-a\,y)\,x'=a\,x\,y.$$

68. If it were proposed to draw from a point without the curve, whose abscissa is α, and whose ordinate is β, a tangent to the curve; it is evident that we must substitute α for x', and β for y' in the equation of the tangent, which will then become

$$\beta-y=\frac{d\,y}{d\,x}\,(\alpha-x),$$

and will serve, when combined with the equation of the curve, to determine the co-ordinates x and y of the point of contact.

Let us take, for our first example, the circle, the equation of whose tangent is

$$y\,y'+x\,x'=a^2\quad(67),$$

we shall have

$$\beta\,y+\alpha\,x=a^2.$$

This equation, combined with that of the circle, will determine the co-ordinates x and y of the points of contact; or, which amounts to the same thing, it will determine the points where the circle meets the right line, whose equation is

$$\beta\,y+\alpha\,x=a^2.\qquad\text{(Trig. 110.)}$$

In the curve corresponding to the equation

$$x^3-3\,a\,x\,y+y^3=0,$$

the point of contact will be found by determining the intersection of this curve, with the line of the second order, which results from the equation

$$\beta\,(y^2-a\,x)+\alpha\,(x^2-a\,y)=a\,x\,y.$$

69. In order to draw a line to touch a given curve, and which shall also be parallel to a given straight line, or which shall make with the axis of the abscissæ, an angle, whose tangent is represented by a, it is sufficient to make $\frac{dy}{dx} = a$ (Trig. 85); combining this equation with that of the proposed curve, we may determine the values of x and y at the point of contact.

If the proposed curve be the common parabola, we have

$$y^2 = mx, \qquad \frac{dy}{dx} = \frac{m}{2y} = a,$$

which gives

$$y = \frac{m}{2a}, \text{ and } x = \frac{m}{4a^2}.$$

70. In all that precedes, we have supposed the co-ordinates x and y to be perpendicular to each other. But it is easy to perceive, that when they are inclined at any given angle, the ratio of $M'Q$ to MQ, will still have for its limit that of PM to PT: the equation of the tangent will also preserve the same form. With respect to MT, MR, and PR, we may find expressions for them by means of the triangle MPT, MTR, and MPR, in which we always know either an angle and two sides, or a side and two angles.

By examining the positions which the tangent of any given curve assumes, when the point of contact is more and more removed from the origin of the co-ordinates, we may discover whether this curve has, like the hyperbola, any right lines for its asymptotes (Trig. 154.), whose position we may also determine.

FIG. 3. We observe, that in the curve MX, fig. 3, which has an asymptote RS, whilst the point M moves further and further from the origin of the abscissæ, the tangent MT continually approaches to the asymptote; and the points T and D approach the points R and E; so that AR and

AE are the limits which the values of AT and AD cannot exceed, nor even attain; but to which they may approximate as near as we chuse. From this it follows, that in order to discover whether a curve has any asymptotes, we must find whether the expressions for AT and AD, relative to this curve, are susceptible of limits: if that be the case, and these limits be determined, we shall know the two points R and E through which the line RS must be drawn, which will be the asymptote required.

The expressions for AT and AD may be deduced from that of PT; the first by observing that $AT = AP - PT$; the second by means of the similar triangles ADT and MPT: they may also be deduced from the equation of the tangent, by making successively $y' = 0$ and $x' = 0$, (Trig. 83.): we shall find

$$AT = x - y\,\frac{dx}{dy},\ AD = y - x\,\frac{dy}{dx}.$$

72. If we apply these principles to the equation

$$y^2 = mx + nx^2,$$

we shall find

$$AT = x - \frac{2y^2}{m + 2nx} = \frac{-mx}{n + 2nx}$$

$$AD = y - \frac{mx + 2nx^2}{2y} = \frac{mx}{2\sqrt{mx + nx^2}}.$$

The second members of these equations, being put under the form

$$-\frac{m}{\frac{m}{x} + 2n},\quad \frac{m}{2\sqrt{\frac{m}{x} + n}},$$

their respective limits, when x becomes infinite, are seen to be

$$-\frac{m}{2n} = AR, \text{ and } \frac{m}{2\sqrt{n}} = AE.$$

If $n = 0$, the expressions for AT and AD become infinite at the same time with x; and the curve has no asymp-

totes; nor will it have any, when x is negative; because, in that case, its equation will not admit of an infinite value of x.

In the curve represented by the equation

$$x^3 - 3axy + y^3 = 0,$$

we have

$$AT = \frac{axy}{x^2 - ay},\ AD = \frac{axy}{y^2 - ax}:$$

in order to find the limit to which these expressions approach, whilst y increases, we must substitute for y the limit to which it tends, and must consequently know the value of y in terms of x; we may, however, in the present instance, supersede the necessity of obtaining this value, by a very simple artifice. If we make $x = ty$, the proposed equation becomes divisible by y^2; and we have $y = \frac{3at}{1+t^3}$; it is readily seen, that the supposition of $t = -1$, will make y infinite, and will give $x = -y$. Changing x into $-y$ in the expressions for AT and AD, and then taking the limits, we shall have

$$AR = -a = AE;$$

FIG. 4. and drawing through the points R and E, fig. 4, determined by means of the preceding values, the line RE, it will be the asymptote of the branches AY and AZ.

73. If, whilst one of the quantities AR or AE remains finite, the other should become infinite, it is evident that the asymptote will be parallel to the axis, on which this latter is measured.

In order, therefore, to determine all the asymptotes, which any proposed curve ought to have, we must successively make x and y infinite, and afterwards substitute, in the expressions for AT and AD, each of the different results which these two hypotheses afford. When AT and AD are always infinite at the same time, we may conclude that the proposed curve has no asymptote.

It may happen, that both these quantities are evanescent: in this case the curve will have for an asymptote a right line passing through the origin of the co-ordinates; but as we only know one point of this line, we must find its direction: for this purpose, we must take the limit of the expression $\frac{dy}{dx}$, which represents the tangent of the angle MTP (64), for any point of the curve, and we shall thus get the tangent of the angle SRB.

74. It is generally considered as a proposition nearly self-evident, that *a small arc of a curve may be taken for its chord*; or, what is the same thing, *that the ratio of an arc to its chord has unity for its limit.* This proposition, which is very important, ought nevertheless to be demonstrated, which may be done in the manner following:

The right-angled triangle $MM'Q$, fig. 5, gives FIG. 5.

$$MM' = \sqrt{\overline{MQ}^2 + \overline{M'Q}^2};$$

and we have also (62.),

$$MQ = h,\ M'Q = \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.$$

This developement may be put into the form

$$(p + Ph)h,$$

by making

$$\frac{dy}{dx} = p, \text{ and } \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c. = Ph^2:$$

thus we obtain

$$MM' = \sqrt{h^2 + (p+Ph)^2h^2} = h\sqrt{1 + (p+Ph)^2}.$$

Drawing the tangent MN, we shall find

$$NQ = MQ \text{ tang } NMQ = \frac{dy}{dx}h = ph \text{ (64)}.$$

$$MN = \sqrt{h^2 + p^2h^2} = h\sqrt{1+p^2}$$

$$M'N = NQ - M'Q = -\frac{d^2y}{dx^2}\frac{h^2}{1.2} - \&c. = -Ph^2;$$

and from this we obtain

$$\frac{MN+M'N}{MM'} = \frac{h\sqrt{1+p^2}-Ph}{h\sqrt{1+(p+Ph)^2}} = \frac{\sqrt{1+p^2}-Ph}{\sqrt{1+(p+Ph)^2}},$$

which ratio has, for its limit,

$$\frac{\sqrt{1+p^2}}{\sqrt{1+p^2}} = 1.$$

But the arc $M\,O\,M'$ is always comprised between the chord $M\,M'$ and the bent line $M\,N+M'\,N$; we may, therefore, conclude, *a fortiori*, that the ratio $\frac{M\,O\,M'}{M\,M'}$ has unity for its limit.

75. It is evident that the arc of a curve is a function of its abscissa; and in order to obtain the differential coefficients of that function, we must find the limit of the ratio $\frac{M\,O\,M'}{P\,P'}$; now we have

$$\frac{M\,O\,M'}{P\,P'} = \frac{M\,M'}{P\,P'} \times \frac{M\,O\,M'}{M\,M'}.$$

Substituting the value of the first ratio $\frac{M\,M'}{P\,P'}$ on the second side of the equation, and making $h=0$, in order to obtain the limit, the second ratio will become unity, and we shall thus get (8.) $\sqrt{1+p^2}$. If, therefore, we call the arc $C\,M$, z, we shall have

$$\frac{dz}{dx} = \sqrt{1+\frac{dy^2}{dx^2}}, \text{ or } dz = \sqrt{dx^2+dy^2}.$$

The circle whose equation is

$$x^2+y^2=a^2,$$

giving

$$x\,dx+y\,dy = 0, \text{ or } dy = -\frac{x\,dx}{y},$$

there results

$$dz = \sqrt{dx^2 + \frac{x^2 dx^2}{y^2}} = \frac{dx}{y}\ \sqrt{x^2+y^2}$$

$$= \frac{a\,dx}{y} = \frac{a\,dx}{\sqrt{a^2-x^2}},$$

which result is conformable to that of No. 35, when we suppose $a=1$.

76. The differential of the area of the *segment* $ACMP$ of any curve, may be obtained, by observing that the ratio of the rectangles $PP'QM$ and $PP'M'N$, fig. 6, which have the same base, is equal to $\frac{P'M'}{PM}$, and that its limit is consequently unity. It follows from this, that the curvilinear trapezium $PP'MM'$, which is always comprised between the two rectangles just mentioned, and which represents the increment which the segment $ACMP$ receives, when the abscissa increases by PP', must approximate constantly to equality with the rectangle $PP'QM$, or that the ratio FIG. 6.

$$\frac{PP'M'M}{PP' \times PM} = \frac{PP'M'M}{PP'} \times \frac{1}{PM}$$

must have unity for its limit. Calling s the function of x, corresponding to the area $ACMP$, we shall have for the limit (8).

$$\frac{PP'MM'}{PP'} = \frac{ds}{dx},$$

and

$$\frac{ds}{dx} \times \frac{1}{y} = 1, \text{ or } ds = y\,dx.$$

In the circle

$$ds = dx\sqrt{a^2-x^2};$$

thus, though we cannot assign the algebraic expression of a circular segment, yet we may arrive at that of its differential, from the consideration of limits.

On the Determination of the Nature and Positions of remarkable Points in Curve Lines.

Those points of a curve which are distinguished by some remarkable circumstance, are sometimes called *singular points*. The Differential Calculus furnishes very simple methods of discovering their existence, and of determining their position.

When the differential coefficient $\frac{dy}{dx}$, which expresses the tangent of the angle MTP (64), becomes evanescent, it follows, that the right line which touches the curve at the point M, is parallel to the line of the abscissæ; and also, that if it change its sign after this point, the tangent is then inclined towards a different side of the ordinate to what it was in the former case. An examination of the two
Fig. 7 & 8. figures 7 and 8, will show that, in this case, the ordinate, after having attained a certain magnitude, begins to diminish (fig. 7); or otherwise, that after having diminished to a certain point, it begins to increase (fig. 8).

The first circumstance evidently corresponds to a *maximum* value of the ordinate, and the second to a *minimum* value. When either of these takes place, we have equally $\frac{dy}{dx} = 0$, as we have also shewn from analytical considerations (48).

78. Geometrical considerations also prove, that this character is not confined to those points alone, where the *maximum* or *minimum* takes place; but that it may also hold good in other circumstances. Although the tangent
Fig. 9. at the point M, of fig. 9, is parallel to the line of the abscissæ, this point does not, therefore, correspond to a *maximum*, since the ordinate, beyond it, still continues to increase; but we must remark here, that the concavity of

the curve, which was at first turned towards the axis of the abscissæ, is afterwards turned the opposite way. This circumstance is what is called an *inflexion*, and the point M is called *a point of inflexion*, or of *contrary flexure*. It may be recognized by the change of sign which the coefficient $\frac{d^2 y}{d x^2}$ undergoes, before and after the point M (64). It may also be determined, by seeking the position of the curve with respect to its tangent, before and after this point.

The equation of the tangent being in general

$$(y'-y) = \frac{d y}{d x}(x'-x) \quad (67),$$

we shall have, making $x'=x+h$,

$$y'-y = \frac{d y}{d x} h,$$

or

$$y = y' + \frac{d y}{d x} h,$$

for the expression of $P' N'$, fig. 10, which is the ordinate of the tangent, corresponding to the point P', whose abscissa is $x+h$; but since y is a function of x, we have (21), for $P' M'$ this series FIG. 10.

$$y + \frac{d y}{d x}\frac{h}{1} + \frac{d^2 y}{d x^2}\frac{h^2}{1.2} + \frac{d^3 y}{d x^3}\frac{h^3}{1.2.3} + \text{\&c.}$$

whence we deduce

$$P'M' - P'N' = \frac{d^2 y}{d x^2}\frac{h^2}{1.2} + \frac{d^3 y}{d x^3}\frac{h^3}{1.2.3} + \text{\&c.}$$

Again, taking a point $P_{,}$, upon the axis of the abscissæ, behind the point P, and whose abscissa is $x-h$, we should, in like manner, find

$$P_{,}M_{,} - P_{,}N_{,} = \frac{d^2 y}{d x^2}\frac{h^2}{1.2} - \frac{d^3 y}{d x^3}\frac{h^3}{1.2.3} + \text{\&c.}$$

Now it is evident, that if $\frac{d^2 y}{d x^2} = 0$, the two differences

$$P' M' - P' N' = \quad \frac{d^3 y}{d x^3} \frac{h^3}{1.2.3} + \text{\&c.}$$

$$P_, M_, - P_, N_, = - \frac{d^3 y}{d x^3} \frac{h^3}{1.2.3} + \text{\&c.}$$

will have contrary signs, at least when h is taken so small, that the first term of the series shall be greater than the sum of all the rest (48); and thus the proposed curve, after having been situated in the former part of its course, below the tangent, will now pass above it, and *vice versâ.*

There will, therefore, take place, an inflexion at M, and not a *maximum* or a *minimum*; if $\frac{d^2 y}{d x^2}$ vanish, at that point, together with $\frac{d y}{d x}$, and in general, if the first of the differential coefficients, which does not vanish, is of an odd order. Such is the geometrical meaning of the analytical characters indicated in Art. 49.

79. An inflexion may take place at a point where the tangent is not parallel to the line of the abscissæ; and where, consequently, $\frac{d y}{d x}$ does not vanish; but what always characterises this point is, the change of sign of $\frac{d^2 y}{d x^2}$ with respect to y.

It is evident, that every integral quantity can only change its sign, by passing through zero; but a fraction may also change its sign in its passage through infinity, as happens, for instance, to $\frac{a}{x}$, which successively becomes

$$\frac{a}{b}, \quad \frac{a}{0}, \quad - \frac{a}{b},$$

when x is made $= b$, $x = 0$, $x = -b$: we may, therefore, conclude, by what has been said, that *at a point of contrary*

flexure, $\frac{d^2y}{dx^2}$ *is nothing, or infinite*; but we cannot reverse this proposition.

80. When $\frac{dy}{dx}$, instead of vanishing, becomes infinite, the ordinate becomes a tangent, as at E, fig. 11: this circumstance indicates a *limit* of the curve, in the direction of the abscissæ; that is to say, a *maximum* or a *minimum* of the abscissa, provided no inflexion of the curve take place at that point, where the tangent is perpendicular to the line of the abscissæ. FIG. 11.

81. We have seen (55), that $\frac{dy}{dx}$ becomes infinite, when some radical quantity disappears from the expression of the function y. It must be noticed, that at that moment the quantity by which this function changes by the variation of x, must, contrary to the ordinary rule, have more than one value corresponding to a single value of y; for, unless this were the case, we should not again get the number which the degree of the function entitles us to, and which must always remain the same, the equality of several of these values being, necessarily, only momentary.

This circumstance takes place at the point E, where it is evident that, for the abscissa $A\,c$, consecutive to $A\,C$, the curve has two ordinates, and consequently the same ordinate $C\,E$ has two differences, the one $c\,e' - C\,E$, and the other $C\,E - c\,e$.

The same thing equally happens at a point, as G, where two branches of the curve cut each other; that particular ordinate $F\,G$ has also, for one and the same increment, $F\,f$ of the abscissa, two differences; the one $f\,g' - F\,G$, and the other $F\,G - f\,g$; but in other parts of the curve, one and the same ordinate has, for each increment of the abscissa, but one single difference.

The points where several branches intersect, as G, or

the junction of two branches, $G D E$ and $G D' E$, as E, are called *multiple points*. They are recognized by one ordinate having, for the same abscissa, more than one differential, which causes the differential coefficient to become infinite at those points. It may also exhibit itself under the form $\frac{0}{0}$, which always happens when its expression contains, at the same time, both the variables x and y.

At each of these points, the curve has more than one tangent. At G, for example, it has two distinct ones; at E it has also two, but united in one, which is the limit of those of the superior branch $G D' E$, and the inferior $G D E$.

82. The multiple points sometimes assume two particular forms, to which have been given the names *cusps*, or *points of reflexion*, because the branches of the curve which meet there, extend no farther, and the curve is, as
FIG. 12. & 13. it were, bent backwards. That of fig. 12, where the convexities of the branches are opposed, is a *cusp of the first species*, and that of fig. 13, where their concavities are turned the same way, is a *cusp of the second species*.

These points have only one tangent, but which must be looked upon as double, like that of the point E in the figure of the preceding article; and they are distinguished from other multiple points, by the course of the curve before and after them, with respect to its tangent; and which may be recognized by the sign of the differential coefficient $\frac{d^2y}{dx^2}$ (64), when we take successively for x, values greater and less than the abscissa corresponding to the multiple point under examination.

93. All which has been said may be reduced to a rule as simple in its enunciation, as it is unequivocal in its application: *the determination of the abscissa, to which a singular point corresponds, is obtained by inquiring, when the differential coefficients, of whatever order, become nothing or infinite, or* $\frac{0}{0}$:

the species of the point is assigned 1*st*, *by examining what number of branches of the curve passes through this point, and whether or no they are extended on both sides of it ; 2dly, by determining the position of their tangent ; and 3dly, the direction in which their concavities or convexities are turned.*

84. In the family of curves, represented by the very simple equation

$$y = b + c\,(x-a)^m,$$

we shall find examples of almost all the particulars above enumerated ; and their discussion is very proper to illustrate the rule there delivered.

The expression

$$\frac{d^n y}{d x^n} = m\,(m-1) \ldots\ldots (m-n+1)\,c\,(x-a)^{m-n},$$

vanishing by the supposition of $x = a$, in every case where $m > n$, and becoming infinite when $m < n$, it follows, that the abscissa a corresponds to a singular point.

The exponent m may be positive or negative, greater or less, than unity. We shall at first suppose it positive and > 1.

If it be an even number, or a fraction with an even numerator, we find, 1st, the same value of y, whether we take $x < a$, or $x > a$; the curve, therefore, pursues its course above the abscissæ, which precede and follow a ; and there passes only one branch through the point under examination : 2dly, from the expression

$$\frac{d y}{d x} = m\,c\,(x-a)^{m-1},$$

which vanishes when $x = a$, we see that the tangent at this point is parallel to the line of the abscissæ.

3dly. If we make x alternately $< a$ and $> a$, in the value of

$$\frac{d^2 y}{d x^2} = m\,(m-1)\,c\,\,(x-a)^{m-2},$$

and if we notice that, on the supposition we set out with, the exponent $m-2$ is also an even number, or a fraction with an even numerator, we shall perceive that this differential coefficient retains the same sign in both cases with the ordinate y itself; and of course the curve turns its concavity the same way, on both sides of the point we are considering. Its course, beyond this point, is therefore, of one of the two kinds represented in fig. 14; the first, if c be negative, the second, in the other case.

FIG. 14.

If m be an odd number, or a fraction whose numerator and denominator are both odd, the ordinate corresponding to each abscissa, has only one value; and by taking $x < a$, and $x > a$, we find for y two real values: the curve is, therefore, continued on both sides of the point we are examining, and it has but one branch passing through this point. The tangent is as before parallel to the line of the abscissæ; but the exponent $m-2$, being now an odd number, or a fraction with an odd numerator and denominator, the coefficient $\frac{d^2 y}{d x^2}$ will change its sign when x is made successively $< a$ and $> a$. The curve, in consequence, has not its concavity turned the same way on both sides of the point under consideration: this point is, therefore, one of contrary flexure, as in fig. 15.

FIG. 15.

Lastly, if the exponent m be a fractional number, whose denominator is even, the quantity $(x-a)^m$ being susceptible of the signs $\pm$, the ordinate y will have, for every abscissa, two real values, when x is greater than a, and only imaginary ones when $x < a$; two branches, therefore, of the curve passes through the point under consideration; but which extend only on one side of it. The tangent is still parallel to the axis of the abscissæ. The coefficient $\frac{d^2 y}{d x^2}$ has two values with opposite signs, while those of the ordinate have the same. Hence it follows, that one branch turns its concavity towards the axis of the abscissæ, and

the other its convexity, as is shewn in fig. 16, which produces a cusp of the first species. FIG. 16.

3dly. If the exponent $m < 1$, since we should then have

$$\frac{dy}{dx} = \frac{mc}{(x-a)^{1-m}},$$

the value $x=a$ will render this differential coefficient infinite; and the line which touches the curve at the point where $x=a$, will be perpendicular to the axis of the abscissæ. We shall find also by considerations similar to the foregoing, that the point C is a limit of the curve in the direction of the axis of the abscissæ, when m is a fraction, whose numerator is odd, and denominator even: that this point is a cusp when the numerator is even, and a point of contrary flexure, when the numerator and denominator are both odd.*

The ordinate y would become infinite, and be changed into an asymptote, if m were negative.

85. The curve represented by the equation

$$(y-x^2)^2 = x^5$$

offers an example of a cusp of the second kind (82). In this curve

$$y = x^2 \pm x^{\frac{5}{2}}.$$

In order to know whether it has any singular point, we must enquire whether any of the differential coefficients of the function y become nothing or infinite. We first obtain

$$\frac{dy}{dx} = 2x \pm \frac{5}{2}x^{\frac{3}{2}}, \qquad \frac{d^2y}{dx^2} = 2 \pm \frac{5}{2} \cdot \frac{3}{2}x^{\frac{1}{2}};$$

the first of these results vanishes when $x=0$; the second reduces itself to 2, and we see also that the third differential coefficient

* The cusp of fig. 17, may, strictly speaking, be taken for a *maximum*, and that of fig. 18. for a *minimum*. FIG. 17. & 18.

$$\frac{d^3 y}{d x^3} = \pm \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} x^{-\frac{1}{2}},$$

becomes infinite in that case; the point corresponding to $x=0$ is, therefore, *a singular point*. It is evident that this point is a limit of the curve, which does not extend on the side of the negative abscissæ, since the term $x^{\frac{5}{2}}$ then becomes imaginary. The values of the coefficient $\frac{d^2 y}{d x^2}$ are both positive when x is very small, and are of the same sign as those of y; both branches of the curve, therefore, turn their
FIG. 13. convexities towards the axis of the abscissæ $A\,B$, fig. 13; they touch at A, for they have $A\,B$ for a common tangent, $\frac{d y}{d x}$ vanishing at this point. It results from all these characters taken together, that the form of the curve at this point is such as the figure represents.

The preceding examples relating only to those singular points, where the branches of the curve touch one another, afford instances of a single tangent only. The curve corresponding to the equation $a y = \sqrt{a^2 x^2 - x^4}$ of Art. 52, presents, at the point where $x=0$, two branches which cut one another: but we shall not dwell on this example, because farther on, we shall discuss another, in which the same circumstance takes place.

86. Curves are sometimes accompanied by insulated points, which have the character of multiple points; but may be distinguished from them by this, that in the case of the former, the coefficient $\frac{d y}{d x}$ assumes an imaginary value.

Take the equation

$$a y^2 - x^3 + b x^2 = 0;$$

from which may be obtained

$$y = \sqrt{\frac{x^2(x-b)}{a}},$$

$$\frac{dy}{dx} = \frac{x(3x-2b)}{2\sqrt{ax^2(x-b)}}.$$

The differential coefficient, when $x = 0$, becomes $\frac{0}{0}$; but its true value may be had by suppressing the factor x, common to the numerator and denominator: thus we obtain

$$\frac{dy}{dx} = \frac{3x-2b}{2\sqrt{a(x-b)}};$$

whence, making $x=0$, there results

$$\frac{dy}{dx} = -\frac{2b}{2\sqrt{-ab}},$$

an imaginary expression.

Upon the same supposition the proposed equation gives $y=0$; but this ordinate, which is imaginary, when x is negative, becomes so again, until $x=b$. Thus the point A, fig. 20, although comprised in the equation, is absolutely detached from the curve. FIG. 20.

Points of this kind are called *conjugate points*: they result from certain finite portions of the curve vanishing, owing to the particular value of some constant in the equation. The curve represented by the equation

$$ay^2 - x^3 + (b-c)x^2 + bcx = 0,$$

which gives

$$y = \pm\sqrt{\frac{x(x-b)(x+c)}{a}},$$

offers an example of these changes. Its course is at first, as represented in fig. 19; the supposition $c=0$ reduces the part AF to the single point A, fig. 20, as we have seen above. When $b=0$ (c not vanishing), it assumes the figure 21, and if $b=0$, at the same time that $c=0$, the figure 22. FIG. 19, 20, 21, & 22.

Curves have also occasionally singular points, which are

not visible: they are such as result from an even number of inflexions, uniting into one. (See for these points, and for those of *undulation**, from which they take their origin, the Traité du Calcul Differentiel et du Calcul Integral, 4to. Art. 190.)

Example of the Analysis of a Curve.

87. We divide lines into different orders, according to the degree of their equations. The right line constitutes the first order, since it is represented by the general equation of the first degree, involving two indeterminate quantities. The lines of the second, and of the third order, are those whose equations are of the second or third degree; and so on for the others. Newton, considering that the first order included only the right line, and that curves did not exhibit themselves before the second, divided these latter into classes, and called lines of the second order curves of the first class; those of the third order curves of the second class, and so on for the higher orders.

Lines of the same order are subdivided into species from a consideration of the principal circumstances which characterise their course.

Were it possible to resolve equations of every degree, there would be no difficulty in tracing the course of a curve represented by any algebraic equation whatever. In fact, if we suppose that this equation, being resolved with respect to one of the indeterminate quantities, y, for instance, which it involves, should furnish the different roots X', X'', X''', &c. which will be necessarily functions of x, and constant quantities; the question will be then reduced to the particular examination of the courses of the lines produced by the equations,

---

* Points de serpentement.

$$y=X', \quad y=X'', \quad y=X''', \text{\&c.}$$

when we give to x every possible value, both positive and negative, which the functions X', X'', X''', &c. will admit of, without becoming imaginary. These lines will constitute so many branches of the curve which is represented by the proposed equation.

The extent of each branch will be determined by the extent of the limits, between which are comprised the different solutions of which the particular equation by which it is represented, is susceptible. If amongst the quantities X', X'', &c. there be found any which become infinite, or in which we may suppose x to be infinite, there will correspond to them branches, whose course will be infinite, since they will recede to an infinite distance from one, and sometimes also from both of the axes to which the curve is referred.

A branch of a curve never terminates, unless the expression for its ordinate become imaginary, though it does not, therefore, follow, that the course of the curve is interrupted; it only happening, that in this case, two branches are united, and are reciprocally continuations of each other. We may be easily convinced of this fact, by observing that the number of imaginary values of y is necessarily even, and that each pair of them consisted of real and equal roots, before they become imaginary. In fact, the proposed equation being always decomposable into factors of the first and second degree, if we represent one of these latter by $y^2 - 2Py + Q = 0$, we shall find, that its roots $P \pm \sqrt{P^2 - Q}$, are not imaginary, unless Q be greater than P^2, than which it was originally less; and that there must be a point where the functions of x, which are designated by the letters P and Q, are such as to give $Q = P^2$, which will annihilate the radical quantity, and give to y two equal values.

88. Let us take the equation

$$y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = 0.$$

This equation, which is resolvible, both with respect to x and to y, gives, in the first case,

$$y = \pm \sqrt{48\,a^2 \pm \sqrt{2304\,a^4 - 100\,a^2x^2 + x^4}}.$$

By discussing each of the values of y, in the same manner as those of the general equation of the second degree, involving two indeterminate quantities (Trig. 107, and succeeding Nos.), we may discover the extent and limits of the branches of which the proposed curve is formed; and determine the points where they meet the axis (Trig. 81), and where they intersect each other, or unite into one; but the application of the Differential Calculus materially abridges these investigations, and has the advantage of shewing in what manner they may be effected, even when the equation of the curve proposed is of a degree too elevated to enable us to obtain the general expression for one of the variables in terms of the other.

89. To determine the limits of the curve in the direction of the ordinates, or to discover whether y is susceptible of a *maximum* or a *minimum*, we must examine in what case the differential coefficient

$$\frac{dy}{dx} = \frac{x^3 - 50\,a^2x}{y^3 - 48\,a^2y}$$

becomes equal to nothing; we shall then have

$$x^3 - 50\,a^2x = 0;$$

whence

$$x=0, \qquad x = \pm\, 5\,a\sqrt{2}.$$

The first value of x, substituted in the proposed equation, gives

$$y=0 \text{ and } y = \pm\, 4\,a\sqrt{6}.$$

The two values of y, which are equal to $\pm\, 4\,a\sqrt{6}$,
FIG. 23. determine the points D and D', fig. 23; the one situated above, and the other below, the axis of the abscissæ, and which are also *maximum* values. We may easily convince ourselves of this, by finding the value of $\frac{d^2y}{dx^2}$, correspond-

ing to this hypothesis, or by shewing from the expression for y, that the values of the ordinates which immediately succeed, and follow it, are both less than $4a\sqrt{6}$.

90. The concurrence of the two values $x=0$, and $y=0$, indicates the point A, and makes, at the same time, $\frac{dy}{dx}=\frac{0}{0}$. To discover the value and import of this last expression, which in general characterises a multiple point, we must have recourse to the process in No. 56; but this may also be effected by finding the differential coefficient of the second order. For this purpose, we observe, that the first differential of the proposed equation is

$$(y^3-48a^2y)\,dy+(50a^2x-x^3)\,dx=0,$$

and the second differential

$$\left.\begin{array}{r}(y^3-48a^2y)\,d^2y+(3y^2-48a^2)\,dy^2\\ +\,(50a^2-3x^2)\,dx^2\end{array}\right\}=0;$$

and that, in the case in which x and y are evanescent, this reduces itself to

$$-48a^2dy^2+50a^2dx^2=0,$$

which consequently gives, for this case alone, the values of the coefficient $\frac{dy}{dx}$, which we were not able to deduce from the first differential: we thus get

$$\frac{dy}{dx}=\pm\sqrt{\frac{50}{48}}=\pm\frac{5}{4}\frac{\sqrt{2}}{\sqrt{3}}.$$

It follows, from these values, that the curve has, at the point A, two tangents, which make, with the axis of the abscissæ, angles, whose trigonometrical tangents are respectively

$$\frac{5}{4}\sqrt{\frac{2}{3}},\quad:\quad-\frac{5}{4}\sqrt{\frac{2}{3}},$$

and which consequently admit of a very easy construction.*

* We shall succeed in general, as above, in finding the true value of $\frac{dy}{dx}$, in the case in which it becomes $\frac{0}{0}$, by examining the

91. There yet remain, to be examined, the two roots,

$$x = \pm 5a\sqrt{2}.$$

By substituting them in the proposed equation, they make y imaginary, and consequently give neither a *maximum* nor a *minimum*.

91. To obtain the limits of the curve in the direction of the abscissæ; or, what amounts to the same thing, to find the *maximum* and *minimum* of x (80), we must make the denominator of the fraction which expresses $\frac{dy}{dx}$, equal to nothing, which will furnish the equation $y^3 - 48a^2y = 0$, whence $y = 0$, and $y = \pm\sqrt{48a^2}$. The first value gives $100a^2x^2 - x^4 = 0$, from which we deduce $x = 0$, and $x = \pm 10a$. The root $x = 0$, again indicates the multiple point placed at the origin A; but the two others correspond to the points I and I', where the curve meets the axis AB of the abscissæ, and which have not yet been remarked.

The two last values $y = \pm\sqrt{48a^2} = \pm 4a\sqrt{3}$, lead us to $x = \pm 6a$, and $x = \pm 8a$: the one of these results enables us to recognise the point F, and those corresponding to it in the other branches; the other determines the point H with those corresponding to it likewise. We may also observe, that at the points F and I the abscissa is a *maximum*, and at the point H, a *minimum*; since the curve turns its concavity in the first case, and its convexity in the second, towards the axis AC of the ordinates.

92. To complete the determination of the principal circumstances which distinguish the proposed curve, it yet remains to inquire into the nature of its principal branches, and its different points of inflexion; for know-

---

the successive differentials of the proposed equation, and by continuing up to that order whose exponent is equal to the number of values which $\frac{dy}{dx}$ ought to have.

ing its multiple points, we know already that it has no cusp or point of reflexion. We shall begin with discussing the nature and number of its infinite branches. We may easily assure ourselves, that the two values of y, mentioned in No. 88, become infinite at the same time with x; but without recurring to these values, if we make $y = t\,x$, the proposed equation will be divisible by x^2, and will thus become

$$t^4 x^2 - 96\,a^2 t^2 + 100\,a^2 - x^2 = 0;$$

whence we deduce

$$x^2 = \frac{100\,a^2 - 96\,a^2 t^2}{1 - t^4},$$

a result which gives $x = \pm$ an infinite quantity, when $t = 1$; in which case also $y = x$.

We shall also have (71)

$$x - y\,\frac{d\,x}{d\,y} = \frac{x^4 - 50\,a^2 x^2 - y^4 + 48\,a^2 y^2}{x^3 - 50\,a^2 x},$$

$$y - x\,\frac{d\,y}{d\,x} = \frac{y^4 - 48\,a^2 y^2 - x^4 + 50\,a^2 x^2}{y^3 - 48\,a^2 y}.$$

These expressions, when we substitute the value of x^4, become

$$\frac{50\,a^2 x^2 - 48\,a^2 y^2}{x^3 - 50\,a^2 x}, \qquad \frac{48\,a^2 y^2 - 50\,a^2 x^2}{y^3 - 48\,a^2 y},$$

which diminish continually, whilst x and y increase, and are actually evanescent, when we suppose $y = x$. We thus see (73), that the asymptotes of the proposed curve are two right lines drawn through the origin A; and as the expression for $\frac{d\,y}{d\,x}$ has unity for its limit, it follows that they must make an angle of 45° with the axis of the abscissæ. We have not drawn them, lest the figure should be too complicated.

93. We now proceed to find the inflexions or points of contrary flexure.

We have

$$\frac{d^2 y}{d x^2} = \frac{(3x^2 - 50a^2) - (3y^2 - 48a^2)\frac{dy^2}{dx^2}}{y^3 - 48a^2 y}:$$

this expression becomes $\frac{0}{0}$, when x and y are evanescent, a case in which $\frac{dy^2}{dx^2} = \frac{50}{48}$; and to determine its true value it will be necessary to find the third differential of the proposed equation. Making, in the result, x and y equal to zero, we shall have simply $-144\, a^2\, dy\, d^2 y = 0$, which gives $\frac{d^2 y}{d x^2} = 0$, and proves that the point A is, in fact, a point of contrary flexure.

To discover whether the proposed curve has any others, we must make the numerator of the expression for $\frac{d^2 y}{d x^2}$ equal to zero, and there will result the equation

$$3x^2 - 50a^2 - (3y^2 - 48a^2)\frac{dy^2}{dx^2} = 0;$$

putting for $\frac{d^2 y}{d x^2}$, its value, and making the denominator disappear, we shall have

$$(3x^2 - 50a^2)(y^3 - 48a^2 y)^2$$
$$-(3y^2 - 48a^2)(x^3 - 50a^2 x)^2 = 0:$$

we may give to this equation the following form:

$$y^2(y^2 - 48a^2)^2 (3y^2 - 50a^2)$$
$$- x^2(x^2 - 50a^2)^2 (3y^2 - 48a^2) = 0.$$

If we afterwards observe, that the proposed equation is reducible to the form which follows,

$$(y^2 - 48a^2)^2 - (x^2 - 50a^2)^2 + 196a^4 = 0,$$

and if we deduce from this the value of $(y^2 - 48a^2)^2$, for the purpose of substituting it in the equation preceding, we shall find, after proper reductions,

$$(x^2 - 50a^2)^2 (25y^2 - 24x^2)$$
$$+ 98a^2 y^2 (3x^2 - 50a^2) = 0:$$

this last equation, combined with the one proposed, will serve to determine the abscissæ and ordinates of the point of contrary flexure K, and those which correspond to it in the other branches; we shall be easily able to deduce from it the value of y^2; and by substituting for it in the equation of the proposed curve, we shall have a result which involves x only.

By making $\frac{d^2 y}{d x^2}$ infinite, or the denominator $y^3 - 48\, a^2 y$, in the expression for it equal to zero, we shall find

$$y = 0 \text{ and } y = \pm \sqrt{48\, a^2}:$$

these results inform us of nothing new;, they belong to the point A, which has already been remarked, and to the points F, H, and I, which are not points of inflexion, but merely the limits of the curve in the direction of the abscissæ.

If we consider collectively all that precedes, we see that the form of the proposed curve is successively determined by the circumstances presented by the points A, D, F, I, H, K, and the infinite branches X and X'.

Of Osculating Curves.

94. It is by considering the relation which a curve bears to its tangent, that the method of determining the various circumstances of its course has been learnt. Geometers, however, have not restrained themselves to this comparison of curves with right lines, from which they immediately separate themselves: they have proposed to themselves the investigation of those curves (among those of the most simple kind, as the parabola, the circle, &c.) which, within a given small space, approach nearest to any given curve.

The tangent of a curve being the limit of all the right lines which meet a curve in two points, we are led, by

analogy, to seek in general among all lines of a given species, the limit of those which cut the curve in any given number of points.

We know, for instance, that to determine a circle requires three points; we may suppose now, that they are taken in the proposed curve, and inquire, what circle we shall obtain on the supposition that these three points become coincident. This circle, called the *osculating circle*, will be the limit of all the others, in the same manner as the tangent is that of all the secants.

The latter line is determined by the two constants which enter into its equation (Trig. 83); and the circle, by the three constants, which express the abscissa and ordinate of its centre, and the length of its radius. (Trig. 90).

It is plain, that when any two curves DX, EY, have
FIG. 24. three common points, M, M', M'', fig. 24. they will necessarily have three common ordinates; or, which comes to the same, there are two sides of the polygon $M\,M'\,M''$,
FIG. 2. &c. fig. 2, which are at once inscribed in both the curves; and the lines PM, $M'Q$, and $M''N''$ (62), have the same values in each. Denoting always, therefore, by x, y, the co-ordinates of the particular point M, of the proposed
FIG. 24. curve DX, fig. 24; and by x', y', those of any point whatever of the curve EY, we shall have, by No. 62, for the points M, M', M'',

$$\left.\begin{array}{c} y'=y \\ \dfrac{dy'}{dx'}h+\text{ \&c.}=\dfrac{dy}{dx}h+\text{\&c.} \\ \dfrac{d^2y'}{dx'^2}h^2+\text{\&c.}=\dfrac{d^2y}{dx^2}h^2+\text{\&c.} \end{array}\right\}\text{or}\left\{\begin{array}{c} y'=y \\ \dfrac{dy'}{dx'}+\text{ \&c.}=\dfrac{dy}{dx}+\text{ \&c.} \\ \dfrac{d^2y'}{dx'^2}+\text{ \&c.}=\dfrac{d^2y}{dx^2}+\text{\&c.} \end{array}\right.$$

on the supposition that x is changed to x', in the expressions of

y', $\dfrac{dy'}{dx'}$, $\dfrac{d^2y'}{dx'^2}$, &c. deduced from the equation of the

curve EY. Now if we pass to the limit, by making $h=0$,

the three intersections will unite in one point of contact, in which we find the following conditions must hold:

$$y' = y$$

$$\frac{dy'}{dx'} = \frac{dy}{dx}$$

$$\frac{d^2y'}{dx'^2} = \frac{d^2y}{dx^2}.$$

If the curve EY be the circle represented by the equation

$$(x'-\alpha)^2+(y'-\beta)^2=\gamma^2 \quad \text{(Trig. 90)},$$

differentiating twice successively, we get

$$(x'-\alpha)+(y'-\beta)\frac{dy'}{dx'}=0$$

$$1+\frac{dy'^2}{dx'^2}+(y'-\beta)\frac{d^2y'}{dx'^2}=0,$$

and supposing that x' is changed to x, in these equations, they must then give the same values for y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, as in the proposed curve; that is, they must be satisfied by the substitution of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in them. Making this last substitution, they become

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2$$

$$(x-\alpha)+(y-\beta)\frac{dy}{dx}=0$$

$$1+\frac{dy^2}{dx^2}+(y-\beta)\frac{d^2y}{dx^2}=0;$$

but since the quantities derived from the proposed curve are already determined, by the condition of their corresponding to the particular point M, it follows that α, β, and γ, must have values assigned to them, proper for verifying these equations.

If we determine, from the two last of them, the values

of $y-\beta$, and of $x-\alpha$; and then substitute them in the first, we have

$$y-\beta = -\frac{dx^2+dy^2}{d^2y}$$

$$x-\alpha = \frac{dy}{dx}\left(\frac{dx^2+dy^2}{d^2y}\right)$$

$$\gamma = \pm\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y}.$$

95. The circle whose magnitude and position we have determined, varies for every point in the curve, since the quantities α, β, and γ, on which these depend, are functions of x and y. It possesses remarkable properties, discoverable either by geometrical or analytical considerations. We shall begin by explaining the former.

Fig. 25. Let $M\,M'\,M''\,M'''$, &c. fig. 25, be the polygon inscribed in the proposed curve. The circle which passes through the three points M, M', M'', has its center situated in the intersection of the right lines $N\,O$ and $N'\,O'$, erected perpendicularly at the middle points of the line $M\,M'$ and $M'\,M''$. If with the points M', M'', we combine a fourth point M''', we shall, by these three points, determine a new circle, whose centre will be in O', at the intersection of the perpendiculars $N'\,O'$ and $N''\,O''$, drawn from the middle points of $M'M''$ and $M''\,M'''$. Conceiving now the same operation continued throughout all the extent of the polygon $M\,M'\,M'\,M'$, &c. the center of all the successive circles will form, when joined, a polygon, such, that all its sides, when produced, will meet those of the first at right angles.

When we consider the limits, that is, when we substitute curves for polygons, the points M, M', M'' becoming, coincident, the right line NO becomes a normal to the curve, which is the limit of the polygon $M\,M'\,M''\,M'''$, &c. and a tangent to that which is the limit of the polygon $O\,O'\,O''\,O'''$, &c. and the circle which passes through the points M, M', M'',

becomes the osculating circle. We must substitute then the figure 26, instead of 25, so as to replace the polygons by the curves $D\,X$ and $F\,Z$, the second being the *locus* of the centres of all the osculating circles of the first, which have the tangent $M\,O$ for their radius. Fig. 26.

96. In order to exhibit the analysis of the preceding properties, we assume the three equations of No. 94; and clearing the two last of the differentials, which enter, as divisors, into them, we have

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2 \ldots\ldots\ldots\ldots (1)$$
$$(x-\alpha)\,dx+(y-\beta)\,dy=0 \ldots\ldots (2)$$
$$dx^2+dy^2+(y-\beta)\,d^2y=0 \ldots\ldots\ldots (3).$$

Now, 1st. Since the second equation gives

$$\beta-y=-\frac{dx}{dy}\,(\alpha-x)$$

it is (67) that of the normal drawn from the point whose co-ordinates are α, β, that is, from the point O of the curve $F\,Z$, to the point M of the proposed curve $D\,X$.

2dly. Differentiating the two first equations, not only with respect to x, y, but also to the quantities α, β, γ, (inasmuch as these last are functions of the others [95]), we get

$$(x-\alpha)\,dx+(y-\beta)\,dy-(x-\alpha)\,d\alpha-(y-\beta)\,d\beta=\gamma\,d\gamma$$
$$dx^2+dy^2+(y-\beta)\,d^2y-d\alpha\,dx-d\beta\,dy=0.$$

Now the equations (2) and (3) reduce these to

$$-(x-\alpha)\,d\alpha-(y-\beta)\,d\beta=\gamma\,d\gamma \ldots\ldots\ldots (4)$$
$$-d\alpha\,dx-d\beta\,dy=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

the latter of which gives $\dfrac{d\beta}{d\alpha}=-\dfrac{dx}{dy}$, an expression which changes the equation

$$\beta-y=-\frac{dx}{dy}\,(\alpha-x)$$

into

$$y-\beta=\frac{d\beta}{d\alpha}\,(x-\alpha),$$

and which shews therefore (67), that the normal MO is a tangent to the curve whose co-ordinates are α, β, that is, to the curve FZ.

3dly. If we eliminate $x-\alpha$, $y-\beta$, $\frac{dy}{dx}$, between the equations (1), (2), (4) and (5), we shall have

$$d\gamma^2 = d\alpha^2 + d\beta^2, \text{ or } \frac{d\gamma}{d\alpha} = \sqrt{1+\frac{d\beta^2}{d\alpha^2}},$$

which gives the differential coefficient of γ, with respect to the variable α: now (75), this expression is also that of the differential coefficient of the arc of the curve, whose co-ordinates are α, β; and it follows, from this identity, that the radius of the osculating circle varies by the same differences as the arc of the curve FZ (22), a property which merits the greatest attention.

In fact, the radius of the osculating circle at the point M, being a tangent to the curve FZ, has its direction necessarily the same with that of a thread, wrapped round the convexity of this curve, and then unwound, as far as the point O. We may observe, if we trace this *developement* in its progress from O to O', that the thread increases in length by a part equal to $O\,O'$, the arc of the curve FZ; and since, by what we have before said, the difference of the radii $O\,M$ and $O'\,M'$, is also equal to the same arc $O\,O'$, it follows that the extremity M of the thread must be still found in M', a point in the proposed curve, which it has never quitted during the progress of the developement from one of these points to the other. We may, therefore, regard the curve DX as generated by the developement of FZ.

This process has much analogy with the description of a circle. The curve FZ performs the part of a centre, and the radius MO, instead of being constant, varies at

every point. $F Z$ is called the *evolute*,* the curve $D X$ its *involute*, and the radius of the osculating circle, the *radius of curvature*.†

In general, the osculating circle at once touches and cuts the curve, in the manner of a tangent at a point of inflexion (78). If the radius of the osculating circle increase from M to M', it is evident that the arc $M M'$ of the curve must lie above $G H$, the osculating circle of the point M, while the part $M D$ lies below it. Moreover, since we may always conceive the points M and M' so near each other, that the radii $M O$ and $M' O'$ may differ by any quantity, however small; and since, if we describe the circle $G' M H'$ with the radius $M o = M' O'$, the arc $D M'$ will be entirely *below* the curve, we shall easily perceive that no other circle can pass between a curve and its osculating circle; for every circle, whose radius is less than $M O$, will pass entirely within the arc $G M H$, while every circle, whose radius is greater than $M o$, will lie entirely without the arc $G' M H'$.

The osculating circle being, therefore, that which of all circles touching the proposed curve at the point M, approaches nearest to it, on either side of the point of contact, is consequently that which differs the least from the curve at the point under consideration. The curvature of a circle is uniformly the same in every point of it; but in arcs of a given length, that of a smaller circle is greater than that of a larger, so that the curvatures of these arcs are in the inverse ratio of the radii of the circles to which they belong.

* La développée—la développante—rayon de la développée.

† It is by this latter consideration that Huyghens determined the osculating circle, which he first noticed, and the preceding formulæ might be deduced from it; but this view of the subject, separating, as it does, the investigation of the osculating circle from the general theory of curves, of which it ought to form a part, is too limited for the present state of science.

We may, therefore, by the radius of the osculating circle, estimate the curvature of the curve at any point. This is the reason for calling the radius of that circle the radius of curvature: and it appears *that the curvature of any curve is in the inverse ratio of the radius of curvature.*

The evolute may also be considered as the limit of the intersections of the normals of the proposed curve, taken two and two consecutively, since the point K, the intersection of the two radii MO and $M'O'$, perpendicular to the curve DX, at M and M' approaches so much the nearer to the curve FZ, as the points M and M' are nearer to each other.

97. We may likewise prove by the assistance of analysis, that between the curve proposed, and its circle of curvature, no other circle whatever can pass; and this property leads us immediately to the others.

In general, when two curves, whose ordinates and abscissæ are designated by x and y, x' and y', have a common point, and in which consequently $x'=x$, $y'=y$, if we take the difference of the series

$$y+\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{1.2.3.}+\&c.$$

$$y'+\frac{dy'}{dx'}\frac{h}{1}+\frac{d^2y'}{dx'^2}\frac{h^2}{1.2.}+\frac{d^3y'}{dx'^3}\frac{h^3}{1.2.3.}+\&c.$$

which express the ordinates of the points corresponding to the abscissa $x+h$, we shall find, generally,

$$\left(\frac{dy}{dx}-\frac{dy'}{dx'}\right)\frac{h}{1}+\left(\frac{d^2y}{dx^2}-\frac{d^2y'}{dx'^2}\right)\frac{h^2}{1.2.}$$
$$+\left(\frac{d^3y}{dx^3}-\frac{d^3y'}{dx'^3}\right)\frac{h^3}{1.2.3}+\&c.$$

for the expression for the distance of these curves, in the direction of the ordinate; but if at the particular point which we are now considering, we have $\frac{dy'}{dx'}=\frac{dy}{dx}$, this distance will be then reduced to

$$\left(\frac{d^2 y}{d x^2} - \frac{d^2 y'}{d x'^2}\right)\frac{h^2}{1.\ 2.} + \left(\frac{d^3 y}{d x^3} - \frac{d^3 y'}{d x'^3}\right)\frac{h^3}{1.\ 2.\ 3.} + \&c.$$

The ratio which this developement bears to the preceding, becoming smaller in proportion as h increases (56), it results from it, that the distance which it expresses between the two curves, will terminate by being less than that which is expressed by the former; and that consequently no curve whatever, for which we have not $\frac{d y'}{d x'} = \frac{d y}{d x}$, can pass between those which satisfy this condition. It is on this account, that between a curve and its tangent, we can draw no straight line passing through their point of contact.

The proximity would become still greater, if we also had $\frac{d^2 y'}{d x'^2} = \frac{d^2 y}{d x^2}$. A curve which only satisfies the two conditions

$$y' = y, \quad \frac{d y'}{d x'} = \frac{d y}{d x},$$

that is to say, which has only a simple contact, cannot approach, at those parts which are indefinitely near the common point, so near to the second curve as the first approaches, and cannot possibly be drawn so as to pass between them. This is the case of a circle which is merely a tangent, when compared with the circle of curvature.

Since the first term of the expression for the difference of the ordinates involves h^3, which changes its sign when we substitute $-h$ in the place of $+h$, we readily see, that the circumstances which we have remarked, of the tangent at points of inflexion (78), will likewise take place in the circle of curvature, excepting those cases only in which we have

$$\frac{d^3 y'}{d x'^3} = \frac{d^3 y}{d x^3}.$$

It is not necessary to extend these considerations farther, in order to be convinced, that curves may have with

each other degrees of contact more or less immediate. By pursuing the course indicated in No. 94, we should find, that if two curves had four points in common, and if we would determine one of them in such manner, that these points might coincide, we must then have, at the same time,

$$y' = y, \quad \frac{dy'}{dx'} = \frac{dy}{dx}, \quad \frac{d^2y'}{dx'^2} = \frac{d^2y}{dx^2}, \quad \frac{d^3y'}{dx'^3} = \frac{d^3y}{dx^3}:$$

this contact would differ from the preceding in this circumstance, that a curve which has with either of the proposed curves, a contact of the above-mentioned species only, cannot be drawn between this and the other.

By employing the preceding conditions, in the determination of the constants which particularize the equation whose variables are x and y, we shall discover, that this equation must necessarily involve four constants.

98. We divide contacts into different orders, according to the number of points of intersection which are supposed to be united in them; or, what amounts to the same thing, according to the number of terms which are supposed to be equal in the developements of the ordinates relative to a consecutive point. The contact of the highest order which can take place between the *tangent* curve, and the one proposed, depends on the number of constants which the equation of the former involves, and is also called the contact of *osculation*.

Thus the tangent, between which and a given curve, a simple contact only can take place, is an *osculating* line of the first order: the circle whose equation involves three constants, may have either a simple contact of the first order, or a contact of the second; but this last, being the most elevated, is termed that of osculation, and distinguishes the *circle* of *curvature* from all those circles which are merely tangents.

99. We shall not detain ourselves long with the application of the formulæ

$$\gamma = \pm \frac{(d x^2 + d y^2)^{\frac{3}{2}}}{d x\, d^2 y},$$

$$x - \alpha = \frac{d y\,(d x^2 + d y^2)}{d x\, d^2 y},$$

$$y - \beta = -\frac{d x^2 + d y^2}{d^2 y}.$$

since they can present no difficulty when we are well acquainted with the structure of the Differential Calculus.

The value of γ being susceptible of the double sign $\pm$, it may be asked, which of the two we ought to employ; for it is very evident, that in general for one point of the curve there is but one radius of curvature; and since this radius has not, except at some particular points, the same direction with the ordinate or abscissa, it cannot properly be said to have any peculiar sign, with respect to those lines. The determination, therefore, of that by which we commonly affect it, must depend upon a convention previously established on the direction of the curvature, with respect to the normal. If we agree to consider the radius of curvature as positive, for those curves whose concavities are turned towards the axis of the abscissæ, as the value of $\frac{d^2 y}{d x^2}$ is, in these cases, negative (64), we must affect the expression for γ with the sign $-$; and the same assumption will also make the radius of curvature negative, when the concavities of the curves are turned from the axis of the abscissæ, since it changes its sign at the same time with $\frac{d^2 y}{d x^2}$. In conformity to this convention, we shall always, in the application of the formulæ, assume

$$\gamma = -\frac{(d x^2 + d y^2)^{\frac{3}{2}}}{d x\, d^2 y}.$$

The general equation of lines of the second order,

$$y^2 = m x + n x^2,$$

leading to

$$dy=\frac{(m+2nx)dx}{2y},$$

$$d^2y=\frac{2nydx^2-(m+2nx)dx\,dy}{2y^2}=\frac{[4ny^2-(m+2nx)^2]dx^2}{4y^3},$$

there will thence result

$$\gamma=-\frac{[4y^2+(m+2nx)^2]^{\frac{3}{2}}}{8ny^2-2(m+2nx)^2}.$$

If we substitute the value of y^2, in this expression, we shall have

$$\gamma=\frac{[4(mx+nx^2)+(m+2nx)^2]^{\frac{3}{2}}}{2m^2}.$$

This is the general expression for the radius of curvature, in lines of the second order: we shall deduce its particular value for each species of these lines, by giving to m and n the values which respectively correspond to them. (Trig. 147.)

This result is reduced to $\frac{m}{2}$ in all cases, when $x=0$; the curvature, therefore, of the proposed lines at their vertex, is the same as that of a circle described with a radius equal to the semi-parameter. (Trig. 132.)

By comparing the value of γ with that which we have found in No. 66. for the normal, we shall see that $\gamma=\frac{\overline{MR}^3}{\frac{1}{4}m^2}$, or that the *radius of curvature in lines of the second order is equal to the cube of the normal, divided by the square of the semi-parameter.*

In the parabola, in which $n=0$, we have simply

$$\gamma=\frac{(m^2+4mx)^{\frac{3}{2}}}{2m^2}.$$

We may apply, in a similar manner, the general expressions for $x-\alpha$ and $y-\beta$; and substituting for y its value,

we should have two equations in terms of x, α, and β; from which, by eliminating x, we may deduce the equation for the evolute, in terms of α and β alone. We shall go through this process for the parabola only. We have, in this case,

$$dy = \frac{m\,dx}{2y}, \qquad d^2y = -\frac{m^2\,dx^2}{4y^3},$$

and there results

$$y - \beta = \frac{4y^3}{m^2}\left(\frac{4y^2+m^2}{4y^2}\right) = \frac{4y^3}{m^2} + y$$

$$x - \alpha = -\frac{m}{2y}\,\frac{4y^3}{m^2}\left(\frac{4y^2+m^2}{4y^2}\right) = -\frac{4y^2+m^2}{2m};$$

from which we get

$$-\beta = \frac{4y^3}{m^2}, \qquad x - \alpha = -\frac{2y^2}{m} - \frac{1}{2}m;$$

substituting, in each of these equations, for y its value $m^{\frac{1}{2}}x^{\frac{1}{2}}$, there will arise

$$-\beta = \frac{4x^{\frac{3}{2}}}{m^{\frac{1}{2}}}, \qquad x - \alpha = -2x - \frac{1}{2}m;$$

determining the value of x, in the second result, in order to substitute it in the first, we shall obtain

$$x = \frac{1}{3}\left(\alpha - \frac{1}{2}m\right), \qquad \beta^2 = \frac{16}{27m}\left(\alpha - \frac{1}{2}m\right)^3:$$

the last of these equations belongs to the evolute of the parabola. If we change $\alpha - \frac{1}{2}m$ into α', or transfer the origin of the abscissæ to D, fig. 27, we shall be able to give it this very simple form, $\beta^2 = \dfrac{16\,\alpha'^3}{27\,m}$, which shews that the curve DF is a parabola of the third order*, composed of the two Fig. 27.

* The equation $y^2 = m\,x$, being generalized thus: $y^p = m\,x^n$, represents a family of curves, of which the common parabola is only

branches DF and Df, the first of which generates by its developement, the branch AX of the common parabola XAx, and the second produces the branch Ax.

100. It is necessary to observe, that in order to describe the parabola XAx, by the developement of the curve FDf, the string which is wrapped round one or other of the branches DF and Df, ought to have at the point D, in the prolongation of the tangent BD, a length AD, equal to the radius of curvature at the point A; that is to say, equal to half the parameter of the given curve; every other point I, taken upon this string, would generate a different curve. If the point I should fall upon the point D, the radius of curvature of the curve described in that case, would be equal to nothing at its origin, and consequently the curve would have, at this point, an infinite curvature (96).

Since the arc DF is equal to the difference between the radius of curvature MF, corresponding to the point M, and the radius AD, which belongs to the origin of the abscissæ, we easily see that the curve FDf is *rectifiable*; that is to say, that we can assign a right line which is equal to it in length.

This remark is general; for since we can always deduce an expression for the radius of curvature of algebraic curves, the evolutes of these curves are all rectifiable.

On Transcendental Curves.

101. We have hitherto only considered algebraic curves; we now propose to make the reader acquainted with some of the most remarkable among the transcendental ones.

---

only a particular case: we also name them *parabolas*, but distinguish them by the exponent of the degree of their respective equations.

Those curves are so called, whose equation cannot be obtained in algebraic terms. The *Logarithmic* curve will first engage our attention, in which the ordinates are the logarithms of the abscissæ.

In the curve $y = \mathrm{l}\, x$, and when x is taken $= 1$, $y = 0$, which shews that it meets the axis at the point E, fig. 28, where the abscissa $A\,E$ is equal to unity. The branch $E\,X$, which corresponds to a positive abscissa greater than unity, is infinite; since the logarithms of these abscissæ increase perpetually. Throughout the portion $A\,E$, where the abscissæ are fractions, the ordinates are negative, and increase as these fractions diminish, so as that the branch $E\,x$ shall have, for its asymptote, the negative part $A\,c$ of the axis of the ordinates. Lastly, the curve does not extend on the side of the negative abscissæ, since the logarithms of these are imaginary. (See the *Traité du Calcul Différentiel et du Calcul Intégral*). FIG. 28.

If we differentiate the equation $y = \mathrm{l}\, x$, it becomes

$$\frac{d\,y}{d\,x} = \frac{M}{x} \quad (27);$$

we see, by this, that the tangent of the curve is perpendicular to the line of the abscissæ, when $x = 0$; and that it only becomes parallel to it, when x is infinite (77). The general expression for the sub-tangent (65), gives $P\,T = \frac{x\,y}{M}$; but the elimination of y introduces the logarithm of x; so that this expression is transcendental. If, however, we find the value of the sub-tangent $O\,D$, upon the axis $A\,C$, we get $O\,D = \frac{x\,d\,y}{d\,x} = M$, a very remarkable result, proving, as it does, that the sub-tangent $O\,D$ is constant and equal to the modulus, for all points of the curve. We should find too, that the tangent, the normal, and the sub-normal, taken with respect to the axis $A\,B$, are transcendents, the ordinate y, entering into their expression; but that they

become algebraic, when considered with respect to the axis AC.

We proceed to consider the radius of curvature. Now we have

$$1+\frac{dy^2}{dx^2}=\frac{x^2+M^2}{x^2},\qquad \frac{d^2y}{dx^2}=-\frac{M}{x^2},$$

whence (94) $\gamma=\dfrac{(x^2+M^2)^{\frac{3}{2}}}{Mx}$,

$$y-\beta=\frac{(x^2+M^2)}{M},\qquad x-\alpha=-\frac{(x^2+M^2)}{x}.$$

We shall not stay to consider the evolute, since it is necessarily a transcendent; we shall only observe, that the differential equation of this curve is easily obtained, by eliminating x, dx, dy, from the equation $dy=M\dfrac{dx}{x}$, with the help of the values of $y-\beta$, $x-\alpha$, and their differentials.

Logarithmic curves differ from each other, on account of the modulus relative to the system of logarithms, which they represent. The equation $x=a^y$, gives, if we take the Naperian logarithms, $\mathrm{l}\,x=y\,\mathrm{l}\,a$, whence we find

$$y=\frac{\mathrm{l}\,x}{\mathrm{l}\,a},$$

an equation of which the second member is nothing else than the logarithm of x, calculated for a modulus equal to $\dfrac{1}{\mathrm{l}\,a}$. This equation, therefore, belongs to a logarithmic curve.

102. The *Cycloid*, or the curve described by a point in the circumference of a circle, while the circle itself rolls upon a right line given in position, is another transcendental curve; the relation between its ordinates and abscissæ depends on the arcs of the *generating circle*, and may be expressed in the following manner:

The origin of the motion of the circle being arbitrary, we will assume the point A for it, fig. 29, where the *describing point* was situated in the right line $A\,B$, which the generating circle $Q\,M\,G$ rolls along. Since the circle in rolling applies every point in its circumference to the line $A\,B$, it is plain that in any situation, as $Q\,M\,G$, the distance $A\,Q$ is equal to the arc $M\,Q$, contained between the point M, which at first touched the line $A\,B$ in A, and the point Q, which is in contact with it in its present position. FIG. 28.

If, on $A\,B$, we erect, at the point Q, the perpendicular $Q\,O$, passing through the center of the generating circle, and draw $M\,N$ parallel to $A\,B$: $M\,N$ will be the sine of the arc $M\,Q$, and $N\,Q$ the versed sine (Trig. 5).

Let $Q\,O=a$, $\quad P\,A=x$, $\quad P\,M=Q\,N=y$,
and we shall have

$$M\,N=\sqrt{2\,a\,y=y^2}, \qquad x=A\,Q-P\,Q=\text{arc } M\,Q-M\,N,$$

$$\text{or } x=\text{arc}\left\{\text{vers}=y\right\}-\sqrt{2\,a\,y-y^2}:$$

which is the primitive equation of the cycloid.*

---

* See Note (H.)

If we wished to calculate the length of the arc $M\,2$ from its sine, by trigonometrical tables, in order to construct the curve, we must first refer the sine $M\,N$ (radius a) to the radius 1, and we shall get $\frac{M\,N}{a}$, or $\frac{1}{a}\sqrt{2\,a\,y-y^2}$. Denoting then by t, the length of the arc corresponding to this latter sine, the arc $M\,2$ will necessarily bear the same proportion to the quadrant of the circle of which it is a part, that t does to the quadrant of the tables: whence it follows, that

$$\text{arc } M\,2=at,$$

$$x=at-\sqrt{2\,a\,y-y^2},$$

and $t=$ arc, whose sine $=\frac{1}{a}\sqrt{2\,a\,y-y^2}$.

The

The arc MQ (whose versed sine is y), has also MN, or $\sqrt{2ay-y^2}$ for its sine; and if we differentiate the foregoing equation, the circular arc will disappear; for by the formula (75), in which a represents the radius, and x the sine, if we substitute $\sqrt{2ay-y^2}$ for x, we shall have

$$d.\,\text{arc } MQ = \frac{a\,dy}{\sqrt{2ay-y^2}};$$

Whence also

$$dx = \frac{a\,dy}{\sqrt{2ay-y^2}} - \frac{a\,dy-y\,dy}{\sqrt{2ay-y^2}},$$

consequently

$$dx = \frac{y\,dy}{\sqrt{2ay-y^2}}:$$

which is the differential equation of the cycloid.

Nothing now is easier than to obtain expressions for the sub-tangent, the tangent, the sub-normal, and the normal, in the cycloid. We find, by the general formula of No. 65,

$$PT = \frac{y^2}{\sqrt{2ay-y^2}}, \qquad MT = \frac{y\sqrt{2ay}}{\sqrt{2ay-y^2}},$$

$$PR = \sqrt{2ay-y^2}, \qquad MR = \sqrt{2ay}.$$

We may construct these values in a very simple manner; for it is easy to observe, that MP or y being considered as the abscissa QN in the generating circle QMG, the value given above for PR, is precisely that of the ordinate MN of this circle, and consequently the normal coincides with the chord of the arc MQ, as may be also seen from

---

The expression for x, putting for t its value, is thus written:

$$x = a.\,\text{arc}\left(\sin = \frac{1}{a}\sqrt{2ay-y^2}\right) - \sqrt{2ay-y^2};$$

and by differentiating, by the rule in (32), we come to the same result as before.

the expression for $M R$. It follows from hence, that the chord $M G$ produced is a tangent. If we conceive the circle $Q M G$ to slide upon the point Q, so as to arrive at any other position $q m g$, the lines $m q$ and $m g$ will continue, notwithstanding this change, parallel to $M Q$, $M G$. It is, therefore, sufficient for constructing the tangent and normal at any given point M, to refer this point to the fixed circle $q m g$, which may be done by drawing $M m$ parallel to $A B$, and then to draw $M T$ parallel to $m g$, and $M Q$ parallel to $m q$.

103. Let us next consider the radius of curvature. If we differentiate the equation

$$d x = \frac{y d y}{\sqrt{2 a y - y^2}},$$

we obtain ($d x$ being constant)

$$0 = (y d^2 y + d y^2) \sqrt{2 a y - y^2} - \frac{y d y (a d y - y d y)}{\sqrt{2 a y - y^2}};$$

which, reduced and divided by y, becomes

$$0 = (2 a y - y^2) d^2 y + a d y^2,$$

whence we deduce

$$d^2 y = - \frac{a d y^2}{2 a y - y^2}:$$

substituting now this value, and that of $d y$ in the expression for the radius of curvature (94), we find, after the necessary reductions

$$\gamma = 2^{\frac{3}{2}} (a y)^{\frac{1}{2}} = 2 \sqrt{2 a y}.$$

This result shews, that the radius of curvature $M M'$ is double the normal $M Q$; and that it can never exceed, therefore, twice the diameter of the generating circle, which diameter is at once the ordinate and the normal to the cycloid at the point I, at which the point of contact Q, has traversed one half the circumference.

The expressions for $x-\alpha$ and $y-\beta$, give, moreover

$$y-\beta=2y, \qquad x-\alpha=-2\sqrt{2ay-y^2};$$

whence we conclude, that

$$y=-\beta, \qquad x=\alpha-2\sqrt{-2a\beta-\beta^2}.$$

If we substitute these values in the primitive equation of the cycloid, and make the necessary reduction, we find

$$\alpha=\text{arc}\left\{\text{vers}=-\beta\right\}+\sqrt{-2a\beta-\beta^2}.$$

a result which has a great analogy with that equation. The radical $\sqrt{-2a\beta-\beta^2}$ becomes similar to $\sqrt{2ay-y^2}$, when we make $\beta=-2a+\beta'$, which comes to the same as taking, instead of the ordinate EM', which is always negative, the ordinate $P'M'$, referred to an axis $A'B'$, situated below AB, at a distance $A'I=2a$. By this transformation it becomes

$$\alpha=\text{arc}\left\{\text{vers}=2a-\beta'\right\}+\sqrt{2a\beta'-\beta'^2}:$$

but we must observe, that the two arcs, whose versed sines are together equal to the diameter, are the supplements of each other; and denoting the semi-circumference by π, we may, therefore, write the above as follows:

$$\alpha=\pi-\text{arc}\left\{\text{vers}=\beta'\right\}+\sqrt{2a\beta'-\beta'^2}.$$

Taking then $\alpha=\pi-\alpha'$; that is, substituting for the abscissa AE, another abscissa $A'P'=AI-AE$, we shall find

$$\alpha'=\text{arc}\left\{\text{vers}=\beta'\right\}-\sqrt{2a\beta'-\beta'^2},$$

the equation of a cycloid, whose origin is at the point A', and which is described upon the axis $A'B'$, by the same generating circle as the proposed, but rolling in the direction $A'B'$, opposite to AB.

The same consequence may be also obtained from the determination of the radius of curvature. Producing GQ to meet $A'B'$ in Q', and drawing $Q'M'$, we shall have the triangles GMQ, $QM'Q'$, equal to each other. The angle $QM'Q'$ is, therefore, a right angle; and if a circle be described upon QQ', as a diameter, it will pass through

M, and will be equal to the generating circle. This being premised, since the arc $M' Q'$ is the supplement of $M' Q$, which is itself equal to $M Q$, we shall have

$$\begin{aligned}\text{arc } M' Q' &= Q M G - \text{arc } M Q \\ &= A I - A Q = Q I = A' Q',\end{aligned}$$

which proves very clearly, that the evolute $A' M' A$ is a cycloid described by the circle $Q M' Q'$, rolling on $A' B'$, from A' towards B'.

The reader will have remarked, doubtless, from what has been said before, that the cycloid is rectifiable, since it is its own evolute, and the expression, for its radius of curvature is algebraic; and we thence deduce a curious result, that the length of the arc $A' A$, or its equal $A K$, which compose the half of the branch described by the generating circle, is precisely that of $A' K$, or double the diameter of the circle.

The cycloid is not terminated at L, where the circle has described its whole circumference on $A L$; for there is nothing to limit the extent of its motion. We ought particularly to remark, that in the description of curves, all the different parts which result from the same construction, or from the same motion, belong to the same curve. Thus the circle $Q M G$, by continuing to roll on the right line $A B$, beyond the point L, describes a series of portions, similar to $A K L$; and we must conceive as many to have been described on the left side of the point A, since the circle may have arrived at this point in the course of a motion which has already continued an infinite length of time. The equation of the curve leads naturally to these remarks; for there is nothing to prevent our supposing the arc $Q M$ to increase or diminish, by as many circumferences as we please. We see too, that y can never surpass $2 a$. Hence it follows, that the cycloid conceived as existing in its full extent, may be cut by the same right line in an infinite number of points.

The differential coefficient of the second order $\frac{d^2y}{dx^2}$, being $=-\frac{a}{y^2}$, is always negative, since y^2 is always positive; but when $y=0$, it becomes infinite, as well as $\frac{dy}{dx}$, when $y=0$, which happens whenever the arc $M\,Q$ is either 0, or some multiple of the circumference. The points A, L, &c. therefore, where the different branches of the cycloid touch each other, are cusps of the first species, at which the tangent is perpendicular to the axis of the abscissæ (83).

104. The spirals compose another class of transcendental curves, remarkable from their form, and their properties. That which Conon of Syracuse imagined, and whose principal properties were discovered by Archimedes, is generated as follows:

FIG. 30. While the radius $A\,O$, fig. 30, revolves round the center A of the circle $O\,G\,Q$, a moveable point, setting out from that center, uniformly describes the line $A\,O$, with such a velocity as to arrive at O, when the line has completed a revolution. It follows, therefore, that for any point M of the spiral $A\,M\,O\,M'\,X$, the ratio of $A\,M$ to $A\,N$ is the same as that of the arc $O\,N$ to the circumference $O\,G\,O$: but as there is nothing to prevent the *describing point* from continuing its motion beyond O, upon the radius produced, and as this radius may itself make an indefinite number of revolutions, the curve $A\,M\,O$ will, therefore, extend itself, making continually more and more turns round A, in such a manner, that the ratio of the distances of its several points, from A to the radius of the circle, shall be the same with that of the arc described by O, since the beginning of the motion to the whole circumference. At M', for instance, where the radius $A\,N$ has made one revolution *plus* the arc $O\,N$, we have

$$\frac{A\,M'}{A\,N}=\frac{O\,G\,O+O\,N}{O\,G\,O}.$$

If then we make

$$O\,N=t, \qquad A\,M=u,$$

and if, taking for unity the radius $A\,N$, we represent the circumference $O\,G\,O$ by $2\,\pi$, we shall have $u=\frac{t}{2\,\pi}$.

The variables in this equation are what Geometers have called *polar co-ordinates*. The center A of the circle $O\,G\,O$ is called the *pole*, the line $A\,M$, which always passes through this point, is the *radius vector* and performs the part of the ordinate of the curve, while the arc $O\,N$ is equivalent to the abscissa.

The spiral we have been considering, and which bears the name of the *spiral of Archimedes*, is only a particular case of the curves represented by the equation $u=a\,t^n$, n having all possible values given to it. If $n=-1$, we have $u\,t=a$, an equation which belongs to the *hyperbolic spiral*.

If, in place of the distance $A\,M$, we were to take for u the part $M\,N$ of the radius vector, comprised between M and the circumference of the circle $O\,G\,O$, the equation $u^2=a\,t$ would be that of the *parabolic spiral*, or the curve formed by wrapping the axis of a parabola round the circle $O\,G$; the ordinates would then be perpendicular to the circumference of the circle, and would coincide in direction with the radii.

As long as n is positive, the spirals given by the equation $u=a\,t^n$ have their origin at A; but when n is negative, u, at first infinite when t is nothing, diminishes as this angle increases, and at each revolution the describing point approaches A without ever being able to arrive at it.

105. When curves are referred to polar co-ordinates, the first differential of the radius vector $A\,M$, fig. 31, is FIG. 31.
the part which is cut off from the succeeding radius vector, $Q\,M'$ by the arc of the circle $M\,Q$, described about the point A, as a center with the radius $M\,A$. This small arc is consi-

dered as a straight line (74), and the triangle $M\,Q\,M'$, as a rectilineal one, which gives $M\,M' = \sqrt{\overline{Q\,M}^2 + \overline{Q\,M'}^2}$. When we measure the angle $M\,A\,M'$, by an arc of a circle $N\,N'$, described with a radius $A\,N$, equal to unity, we have $Q\,M = u\,d\,t$, and $Q\,M'$, being equal to $d\,u$, we find $M\,M' = \sqrt{d\,u^2 + u^2\,d\,t^2}$.

106. If we draw $A\,T$ parallel to the chord of the small arc $Q\,M$, and produce the side $M\,M'$ of the polygon inscribed in the curve, until it meets that line, we shall have, from the similitude of the triangles $M'\,Q\,M$, and $M'A\,T$,

$$\frac{Q\,M'}{Q\,M} = \frac{A\,M'}{A\,T}.$$

When we take the limits, the chord may be taken for the arc, the angle $Q\,M\,A$ may be considered as a right angle, the line $M\,T$ as a tangent to the curve, and $A\,T$ as perpendicular to $A\,M$, which, in that case, coincides with $A\,M'$, and we have

$$\frac{d\,u}{u\,d\,t} = \frac{u}{A\,T};$$

whence we derive

$$A\,T = \frac{u^2\,d\,t}{d\,u}.$$

107. The second differential d^2u, being considered as the difference between two succeeding first differentials (62), will be represented by $M''\,Q' - M'\,Q$; and we must observe, that when we suppose the arc $N\,N'$ constant, or when we make the angle t vary always by the same quantity, the arcs $Q\,M$, $Q\,M'$ are not equal, because their radii are different.

From these considerations we may deduce expressions for the tangents, normals, &c. but it is more convenient to transfer to curves referred to polar co-ordinates, the expressions of the sub-tangents, tangents, &c. found relative to the rectangular co-ordinates; because this course of proceeding will give occasion to transform the co-ordinates of

the first system into those of the second, and to show how we can pass from one to the other.

This is so much the more useful, because algebraic curves are frequently referred to polar co-ordinates; and this is particularly the case with those of the second degree, their focus being taken for the pole.

108. For the sake of simplicity, let A, fig. 30, be the origin of the rectangular co-ordinates, Fig. 30.

$$AP=x, \qquad PM=y;$$

and in order to fix the position of the axis AB, let m denote the arc QO, contained between this axis and the point O, which is the origin of the arc t. Drawing the line PM perpendicular to AB, and observing that the angle MAP is measured by the arc NQ equal to $t-m$, we shall have

$$\overline{AM}^2 = \overline{AP}^2 + \overline{PM}^2,$$
$$AP = AM \cos NQ,$$
$$PM = AM \sin NQ;$$

whence

$$u=\sqrt{x^2+y^2},$$
$$x=u\cos(t-m),$$
$$y=u\sin(t-m).$$

By means of the two latter values we may change any algebraic equation between x and y into another, which contains only the sine and cosine of arc t, and the radius vector u. These values also give us

$$\cos(t-m)=\frac{x}{u}, \qquad \sin(t-m)=\frac{y}{u};$$

whence we may deduce values of $\cos t$ and $\sin t$, expressed in terms of x, y, u, $\sin m$ and $\cos m$, which being substituted in any given equation between u, $\sin t$, and $\cos t$, will lead to a result containing only x and y, since u may be changed into $\sqrt{x^2+y^2}$.

If, for the sake of brevity, we suppose the line AB to coincide with AO, we have simply

$$\cos t = \frac{x}{u}, \ \sin t = \frac{y}{u}.$$

When the equation between u and t, which it is proposed to transform also, contains the arc t, it is not possible to obtain an algebraic relation between x and y, since none such exists between the arc t, and its sine and cosine; but in this case we may arrive, as we shall presently see, at a differential equation, which only contains x, y, $d\,x$, and $d\,y$.

We may deduce, from the values of x, y, and u, given above

$$d\,u = d.\sqrt{x^2+y^2},$$

$$d\,x = d\,u\cos(t-m) - u\,d\,t\sin(t-m),$$

$$d\,y = d\,u\sin(t-m) + u\,d\,t\cos(t-m);$$

if we eliminate $d\,u$ from the two latter equations, we shall have

$$d\,t = \frac{d\,y\cos(t-m) - d\,x\sin(t-m)}{u},$$

and substituting for $\cos(t-m)$, $\sin(t-m)$, and u their values, we have

$$d\,t = \frac{x\,d\,y - y\,d\,x}{x^2+y^2}.$$

We can, therefore, exterminate from the equation between u and t, and from its differential, the quantities u, $\cos t$, $\sin t$, $d\,u$ and $d\,t$; and the two results obtained will only contain t, which may be made to disappear by elimination.

Take, for example, the equation $u = a\,t^n$, which gives

$$u^{\frac{1}{n}} = a^{\frac{1}{n}}\,t, \qquad \frac{1}{n}\,u^{\frac{1}{n}-1}\,d\,u = a^{\frac{1}{n}}\,d\,t;$$

the expressions of u, $d\,u$, and $d\,t$, being independent of the angle m, we have, by substitution and reduction to a common denominator,

$$\frac{1}{n}(x^2+y^2)^{\frac{1}{2n}}(x\,dx+y\,dy)=a^{\frac{1}{n}}(x\,dy-y\,dx).$$

From this expression we may determine the sub-tangents, tangents &c. of spirals, by making use of the formulæ of No. 65; it will, however, be more simple, and at the same time more general, to transform these formulæ into others, containing only the variables u and t, which may be done as follows.

109. The expression for the sub-tangent becomes, by putting for y and $\frac{dx}{dy}$, their values,

$$PT=u\sin(t-m)\,\frac{du\cos(t-m)-u\,dt\sin(t-m)}{du\sin(t-m)+u\,dt\cos(t-m)}.$$

This result may be much simplified by observing that the situation of the line of the abscissæ, on which the distance PT is measured, is arbitrary, and that we may consequently always take for m such an arc that QN shall be equal to $\frac{\pi}{2}$, in which case the ordinate PM coincides with the radius vector, AM; also, $\cos(t-m)=0$ $\sin(t-m)=1$, and PT becomes $AT=-\frac{u^2dt}{du}$.

The tangent may be constructed by drawing through the point A a perpendicular to the radius vector AM, and measuring off in this line the value of AT, given by the above formulæ.

If we apply this formula to the equation $u=a\,t^n$, we shall find

$$AT=-\frac{u^2}{na\,t^{n-1}}=-\frac{a}{n}t^{n+1}.$$

In the case of the spiral of Conon, we have $n=1$, and $a=\frac{1}{2\pi}$; and consequently,

$$AT=-\frac{t^2}{2\pi}$$

From this expression we see, that when $t=2\pi$, or after one revolution of the radius vector, the sub-tangent is equal in length to the circumference of the circle, at the end of two revolutions the sub-tangent will be equal to four times that quantity; and so on, as Archimedes remarked.

When $n=-1$, which is the case of the hyperbolic spiral, we have $AT'=a$, that is to say, the sub-tangent of this curve is constant.

It is not necessary to consider particularly the expressions of the normal and sub-normal, because they may readily be obtained when the sub-tangent is known.

It may, however, be observed, that $\frac{AT'}{AM}=\frac{u\,dt}{du}$ expresses the tangent of the angle, which the radius vector AM makes with the right line $T'M$, which touches the curve at the point M, and that we have

$$T=M\sqrt{\overline{AM}^2+\overline{AT'}^2}=u\sqrt{1+\frac{u^2\,dt^2}{du^2}}.$$

110. If in the differential of the arc AM, which is

$$dz=\sqrt{dx^2+dy^2} \quad (75),$$

we substitute for dx and dy their values, expressed in terms of the polar co-ordinates, we shall have

$$dz=\sqrt{du^2+u^2\,dt^2},$$

as might be immediately deduced, by considering the curvilineal triangle $MM'Q$, fig. 31, as rectilineal and right-
FIG. 31. angled, from which form it differs less, the nearer the points M and M' approach.

111. The differential of the area ADM, taken relative to polar co-ordinates, is not a trapezium, as in the case of parallel ordinates; but a sector, as AMM'. Taking the limits, the ratio of this sector to the differential NN' will be the limit of the ratio of the sector AMQ, $AM'R$, between which it is comprehended, and which tend towards

equality, to the same differential $N N'$. From this we may conclude, that the area $A D M$, being represented by s, we shall have

$$\frac{d s}{d t} = \frac{A M \times M Q}{2 M N} = \frac{u^2}{2}, \text{ whence } d s = \frac{u^2 d t}{2}.$$

The expression for the sector $d s$, in terms of the rectangular co-ordinates, is frequently required, and it is therefore proper to notice it. It may readily be deduced from the preceding, by putting for $d t$ and u^2 their values, found in Art. 108, when we find

$$d s = \frac{x d y - y d x}{2}.$$

112. Let us now consider the radius of curvature; and here we must observe, that the formula

$$-\frac{(d x^2 + d y^2)^{\frac{3}{2}}}{d x \, d^2 y} \qquad (94),$$

supposes the increment $d x$ constant for all the differentiations performed on y, and that, since the polar co-ordinates t and u are functions of x and y, they are implicitly functions of x, and consequently they, as well as their differentials, vary, when the latter quantity undergoes any change. We must, therefore, differentiate the two equations

$$d x = d u \cos (t - m) - u d t \sin (t - m),$$
$$d y = d u \sin (t - m) + u d t \cos (t - m),$$

and making $d y$, $d u$, and $d t$, vary at the same time, we have

$$0 = d^2 u \cos (t - m) - 2 d u d t \sin (t - m) - u d^2 t \sin (t - m) - u d t^2 \cos (t - m),$$
$$d^2 y = d^2 u \sin (t - m) + 2 d u d t \cos (t - m) + u d^2 t \cos (t - m) - u d t^2 \sin (t - m).$$

The situation of the line $A B$, fig. 30, being arbitrary, we may, for the sake of simplifying these expressions, sup- FIG. 30.

pose it perpendicular to $A\,M$ (109), and consequently $t-m = \frac{\pi}{2}$, hence

$$\sin(t-m)=1, \qquad \cos(t-m)=0,$$

$$d\,x=-u\,d\,t, \qquad d\,y=d\,u,$$

$$0=-2\,d\,u\,d\,t-u\,d^2\,t, \qquad d^2\,y=d^2 u-u\,d\,t^2,$$

$$\frac{(d\,x^2+d\,y^2)^{\frac{3}{2}}}{d\,x\,d^2\,y}=-\frac{(d\,u^2+u^2\,d\,t^2)^{\frac{3}{2}}}{u\,d\,t\,(d^2 u-u\,d\,t^2)}=M\,F.$$

When this formula is employed, it is necessary to make both the differentials $d\,u$ and $d\,t$, of the polar ordinates u and t, vary, subjecting $d^2\,t$ to the condition

$$0=-2\,d\,u\,d\,t-u\,d^2\,t,$$

which establishes a relation between $d^2\,t$ and $d\,u\,d\,t$, that is to say, we must substitute in the expression of $d^2\,u$, instead of $d^2\,t$, its value, deduced from the above expression.

113. Instead of calculating the expressions for the co-ordinates α and β of the evolute (94), it is usual, when polar co-ordinates are employed to determine the position of the center of the osculating circle, by means of that of the normal, and by the distance of $M\,E$, comprehended between the point M and the base of the perpendicular $E\,F$, drawn from the center F of the osculating circle on the line $A\,M$, by which means the construction of the radius of curvature is sometimes rendered more elegant.

The line $A\,M$, being taken as the axis of ordinates y, the part $A\,E$ will represent the ordinate β of the evolute, and consequently

$$M\,E=A\,M-A\,E=y-\beta=-\frac{d\,x^2+d\,y^2}{d^2\,y}:$$

therefore $$M\,E=-\frac{(d\,u^2+u^2\,d\,t^2)}{d^2 u-u\,d\,t^2}.$$

114. As an application of the preceding formulæ, let us take the logarithmic spiral, whose equation is $t=\mathrm{l}\,u$.

Differentiating, we have (27),

$$d\,t = M\frac{d\,u}{u}, \text{ whence } \frac{u\,d\,t}{d\,u} = M,$$

which shows that at all the points in this curve, the tangent makes the same angle with the radius vector.

Differentiating again the equation $u\,d\,t - M\,d\,u = 0$; supposing $d\,t$ and $d\,u$ variable, we shall have

$$u\,d^2\,t + d\,u\,d\,t - M\,d^2\,u = 0;$$

and substituting for $u\,d^2\,t$, its value, $-2\,d\,u\,d\,t$ (112), it becomes

$$-d\,u\,d\,t - M\,d^2\,u = 0;$$

and if we substitute in the expressions of $M\,F$ and $M\,E$ this value of $d^2\,u$, and then that of $d\,t$, expressed in terms of $d\,u$, we shall have

$$M\,F = \frac{u\sqrt{1+M^2}}{M}, \qquad M\,E = u = A\,M.$$

From this it follows, that the straight line $A\,F$, fig. 32, drawn perpendicular to the radius vector $A\,M$, will meet the normal $M\,F$, at the center of the osculating circle, or at the corresponding point of the evolute. FIG. 32.

This evolute will be a spiral similar to the given arc; for the angle $A\,F\,M$, being equal to $T'M\,A$, will be the same for all points in the curve $F\,Z$, as well as for those of the curve $A\,X$.

On the Manner of changing the Independent Variable, or the Means of converting the Differential, which has been considered as constant, into another, which shall not be so.

115. In treating of the radius of curvature for polar coordinates, we have considered the variables u and t as implicit functions of x; and we have consequently made the two diffe-

rentials dt and du vary at the same time; since, however, we may consider the equation of the given curve relative to its polar co-ordinates u and t, independently of its rectangular ones x and y, we may also regard u as a function of t, and may take dt for the constant increment of this latter variable, which is then independent on any. Considered in this point of view, we ought to make $d^2 t=0$; but it will be necessary first of all to alter the formulæ of Nos. 112. and 113, in which we have always considered x as the independent variable, and that dx was constant.

It may be observed, that considering t and u as functions of the variable x, we have

$$dt=p\,dx, \qquad du=q\,dx,$$

and consequently

$$\frac{du}{dt}=\frac{q}{p}.$$

Differentiating each side of this equation on the same hypothesis, where dt and du are considered as implicit functions of x and of dx, we have

$$d\left(\frac{du}{dt}\right)=\frac{p\,dq-q\,dp}{p^2}=\frac{\frac{dt}{dx}\,\frac{d^2u}{dx}-\frac{du}{dx}\,\frac{d^2t}{dx}}{\left(\frac{dt}{dx}\right)^2},$$

which becomes

$$d\left(\frac{du}{dt}\right)=\frac{dt\,d^2u-du\,d^2t}{dt^2},$$

the same as would result from the immediate differentiation of the fraction $\frac{du}{dt}$ (12). On this hypothesis the differential coefficient of the function $\frac{du}{dt}$, or the limit of the ratio of $d.\frac{du}{dt}$ to dt, is no longer expressed by $\frac{d^2u}{dt^2}$, as it is when it is considered as the independent variable; but by

$$\frac{1}{dt}d\left(\frac{du}{dt}\right)=\frac{dt\,d^2u-du\,d^2t}{dt^3}.$$

If then we make

$$\frac{du}{dt}=m, \qquad \frac{dm}{dt}=n,$$

and consider m and n as implicit functions of t, we shall have

$$n=\frac{dt\,d^2u-du\,d^2t}{dt^3};$$

and by the equation $0=2\,du\,dt^2+u\,d^2t$ of No. 112, we have

$$d^2t=-\frac{2\,du\,dt}{u}, \text{ and } n=\frac{u\,d^2u+2\,du^2}{u\,dt^2}.$$

Now it follows, from the nature of the Differential Calculus, that all the expressions furnished by this calculus, ought to be independent of the value of the increments, and ought consequently to be transformable into others, which contain only the determinate functions m, n, &c. We have, in fact, from the preceding operations,

$$du=m\,dt, \qquad d^2u=\frac{n\,u\,dt^2-2\,du^2}{u}=\frac{(n\,u-2\,m^2)\,dt^2}{u};$$

and substituting these in the expressions of MF and ME, we shall obtain

$$MF=\frac{(m^2+u^2)^{\frac{3}{2}}}{2\,m^2-n\,u+u^2}.$$

$$ME=\frac{u\,(m^2+u^2)}{2\,m^2-n\,u+u^2},$$

which formulæ are freed from the increments, and only contain the functions m and n, which are the limits of their ratios; but when t is taken for the independent variable, the functions m and n are represented by $\frac{du}{dt}$ and $\frac{d^2u}{dt^2}$; and we have consequently

$$MF=\frac{(d u^2+u^2 d t^2)^{\frac{3}{2}}}{2 d u^2 d t-u d t d^2 u+u^2 d t^3}$$

$$M E=\frac{u (d u^2+u^2 d t^2)}{2 d u^2-u d^2 u+u^2 d t^2},$$

which formulæ suppose u to be a function of x. In applying them we must, therefore, when we differentiate the given equation between u and t, make $d t$ constant.

116. It is frequently useful to perform an operation, which is the inverse of the preceding, that is to say, to transform a differential expression, taken on the hypothesis of y being a function of x into another, in which x and y are both considered as functions of some third variable, as z, which is supposed independent.

The coefficient $p=\frac{d y}{d x}$ then becomes

$$p=\frac{\frac{d y}{d z}}{\frac{d x}{d z}};$$

$d y$ and $d x$ must both be considered as functions of z, and differentiated accordingly, which gives

$$d p=d\left(\frac{d y}{d x}\right)=\frac{d x d^2 y-d y d^2 x}{d x^2}:$$

then making $d p=q d x$, we shall find

$$q=\frac{1}{d x} d \left(\frac{d y}{d x}\right) = \frac{d x d^2 y-d y d^2 x}{d x^3}.$$

Pursuing the same plan, we shall have

$$d q=d \left[\frac{1}{d x} d \left(\frac{d y}{d x}\right)\right]=d \left(\frac{d x d^2 y-d y d^2 x}{d x^3}\right)$$

$$=\frac{d x^2 d^3 y-3 d x d^2 x d^2 y+3 d y d^2 x^2-d x d y d^3 x}{d x^4},$$

and putting $d q=r d x$, we shall obtain

$$r = \frac{dx^2 d^3 y - 3\, dx\, d^2 y + 3\, dy\, d^2 x^2 - dx\, dy\, d^3 x}{dx^5}.$$

It is thus that the quantities p, q, r, &c. which are implicit functions of x, may be expressed by means of dx, dy, $d^2 x$, &c. considered as functions of z, substituting these values in any formula which contains only differential coefficients, it may be transformed in the general manner, which was proposed.

The expressions for the radius of curvature, for instance

$$\gamma = -\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\, d^2 y},$$

being put under the form

$$\gamma = -\frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}} = -\frac{(1+p^2)^{\frac{3}{2}}}{q},$$

will become

$$\gamma = -\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\, d^2 y - dy\, d^2 x}.$$

117. The expressions of q, r, &c. are indeterminate, as long as no relation is assigned between the variables x, y, and z; but the effect of any relation will be, to establish a dependence between $d^2 x$ and $d^2 y$, since z may also be considered as a function of x and y, dz is also a function of these variables and their differentials, and the supposition of dz, being constant, involves that of $d^2 z = 0$.

It is not even necessary for obtaining this latter, to know the primitive relation between x, y, and the variable z, which is considered as independent; it is sufficient to have the expression of dz.

If, for example, we take for that variable the arc of the given curve, we then have (75.)

$$dz=\sqrt{dx^2+dy^2};$$

and differentiating dx and dy, considered as functions of z, we have

$$dx\,d^2x+dy\,d^2y=0;$$

eliminating, by means of this equation and its differentials, the differentials d^2x, d^3x, &c. from the expressions of q, r, &c. we shall have the forms which the differential coefficients assume, when x and y vary in consequence of a change of the arc z, or when we consider that arc as the independent variable, or when, in other terms, its differential is constant.

Geometrical considerations correspond very clearly with this circumstance; for it is apparent, that in order to particularise the polygon $M\,M'\,M''$, &c. fig. 2, which we propose to inscribe in any given curve, $C\,M$, that we must assign some law for the succession of the angles of the polygon. We have first taken the differences of the abscissæ $P\,P'$, $P'\,P''$, &c. equal to each other; but this law might be changed for any other: suppose, for example, that the sides $M\,M'$, $M'\,M''$, should be equal.

FIG. 2.

We are also at liberty to suppose

$$dz=dx, \qquad \text{or } dz=dy;$$

whence there results

$$d^2x=0, \qquad \text{or } d^2y=0;$$

and by means of these hypothesis, we alternately take x and y for the independent variable, that is to say, we consider y as a function of x, or x as a function of y. In the first case

$$q=\frac{d^2y}{dx^2}, \text{ and in the second } q=-\frac{dy\,d^2x}{dx^3}.$$

If we put this latter value in the expression

$$\gamma=-\frac{(1+p^2)^{\frac{3}{2}}}{q},$$

we shall immediately transform it into one in which x is considered as a function of y, and which is

$$\gamma = \frac{(d\,x^2 + d\,y^2)^{\frac{3}{2}}}{d\,y\,d^2\,x}.$$

All which precedes affects only the symbols, and is in fact nothing more than a peculiar manner of writing the differential coefficients; for whether y varies on account of the change which x undergoes, or on account of that to which another variable z is submitted, on which x depends, it is in both cases the same; for the limits which are independent of the value of the increments. Also, when we differentiate an equation between x and y, making $d\,x$ and $d\,y$ both vary, we may then transform the result into differential coefficients, by means of the formulæ of No. 116, as we should also do by differentiating after the manner of No. 115: by both these methods we should arrive at the same result. The formulæ obtained by the first method are somewhat more elegant, because the two variables are there treated symmetrically.

In the first chapter of the treatise on the Differential and Integral Calculus may be found some important details on this subject, which had not been given by any one previous to the publication of that work.

118. From what we have just seen it appears, that we can always differentiate a system of two equations, containing three variables, from which system it results, that any two of these equations are always determinate functions of the third. If $U=0$ and $V=0$, denote two equations between x, y, and z, we may take the successive differentials, by making those of the two indeterminates, which are considered as functions of the third, vary at the same time.

If we have three equations $U=0$, $V=0$, and $W=0$, between four variables, t, x, y, z, three of these variables will necessarily be determined by the fourth; consequently they will be functions of it, and their differentials ought, therefore, to vary.

Generally, when there are a number m equations between $m+1$ variables determining m of them by means of the remaining one, they ought only to be considered as containing functions of that variable; we must, therefore, in the successive differentiations of these equations, make the differentials of those indeterminates vary, which are considered as functions of that variable which is independent, or whose differential is constant.

119. When we have equations of this nature we may always deduce from them a single result between any two of them, by a process which I shall explain for two equations between three indeterminates, and which it will be easy to extend to as many as may be required.

Let $U=0$ and $V=0$ be these two equations, one of the mth order, and the other of the nth, let them contain the variables x, y, and t, and their differentials; and from these we propose to eliminate t: the first may contain, besides the variable t, the differentials dt, d^2t, ... d^mt, and the second may contain dt, d^2t, ... d^nt. As we are not possessed of the primitive equations, nor of all their differentials of inferior orders, we must necessarily procure some new equations to eliminate the quantities dt, d^2t, &c. which may be done by differentiating n times the equation $U=0$, and m times the equation $V=0$. By this means we shall obtain $n+m$ new equations; and we shall have for the whole number $n+m+2$, including the two given ones: the quantities to be eliminated are t, dt, d^2t, ... $d^{m+n}t$, in number $m+n+1$; there will remain, therefore, a final equation between x and y, and their differentials.

If dt were constant, it would appear, that by differentiating one of the given equations once, we might eliminate t and dt, since we should then have three equations; but it ought to be observed, that the differentials d^2x, d^2y, contain t implicitly, since, in that case, we have considered x and y as functions of that variable (118): we must, there-

fore, consider as constant, the differential of that one of the variables we wish to preserve.

On the Differentiation of Functions of two or more Variables.

120. When we have one equation between three variables, we must first arbitrarily assign the values of any two of them, in order to determine the third, which consequently is a function of the two first. If we have, for example, the equation

$$x^2+y^2+z^2=a^2,$$

we cannot obtain z, without first having assigned values to x and y; but it is proper to observe, that the quantities x and y, having no relation established between them, the second may remain constant, although the first should change, and reciprocally.

It follows from hence, that the value of z may vary in several ways; 1st, in consequence of a change in the value of x or y; 2dly, by the coincidence of both these circumstances. In the first case, the quantity y, or the quantity x, being considered as constant, the proposed equation is, in fact, an equation of two variables; thus, when x alone changes, we have

$$x\,dx+z\,dz=0; \text{ whence } x+z\frac{dz}{dx}=0,$$

and when y changes, it becomes

$$y\,dy+z\,dz=0; \text{ whence } y+z\frac{dz}{dy}=0.$$

We have then, successively

$$dz=-\frac{x\,dx}{z}, \qquad dz=-\frac{y\,dy}{z};$$

but it must be observed, that the first of these equations is relative to the particular variation of x, and the second is re-

lative to that of y, which is usually expressed by saying, that one is the partial differential relative to x, and the other the partial differential relative to y.

The meaning of the proposed question will prevent us from confounding them, and they are otherwise sufficiently distinguished, by observing the differential of the independent variable, which affects them.

The analogous differential coefficients are

$$\frac{dz}{dx} = -\frac{x}{z}, \text{ and } \frac{dz}{dy} = -\frac{y}{z}.$$

In general, when a function of several variables is concerned, it should be remembered, that in $\frac{dz}{dx}$, dz is the partial differential of z, relative to x, whilst in $\frac{dz}{dy}$, dz is the partial differential relative to y.

121. If f (x, y) represent any function of x and y; supposing at first, that the variable x alone changes, and becomes $x+h$, we must consider y as a constant quantity and treat the proposed function in the same manner as function of x; we shall therefore have, by No. 21, putting u for f (x, y)

$$f(x+h, y) = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

In order to find what the proposed function becomes when y alone receives an increment k, we must consider x as constant, and f (x, y), or u, as a function of y; from which we have

$$f(x, y+k) = u + \frac{du}{dy}\frac{k}{1} + \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \frac{d^3u}{dy^3}\frac{k^3}{1.2.3} + \&c.$$

In case the two quantities x and y vary at the same time, and become $x+h$ and $y+k$, as we have not assigned any particular form to the function f (x, y), it is not possible to make the two substitutions indicated at the same time; but it is easy to perceive, that we shall arrive at the

same result, by first putting $x+h$ for x, and afterwards writing $y+k$ for y, in the developement obtained by the first substitution.

We have already found

$$f(x+h, y)=u+\frac{du}{dx}\frac{h}{1}+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\&c.$$

u denoting f (x, y). In order to develope the coefficient of the different terms of this series, having regard to the change which y undergoes, we may first observe, that in each of them x may be considered as a constant quantity, and that we may consequently treat them as functions of y alone. According to this f (x, y) or u, will become

$$u+\frac{du}{dy}\frac{k}{1}+\frac{d^2u}{dy^2}\cdot\frac{k^2}{1.2}+\frac{d^3u}{dy^3}\cdot\frac{k^3}{1.2.3}+\&c.$$

If in this developement we write $\frac{du}{dx}$ instead of u, the result will be what the function $\frac{du}{dx}$ becomes, when y is changed into $y+k$, that is to say,

$$\frac{du}{dx}+\frac{d\left(\frac{du}{dx}\right)}{dy}\frac{k}{1}+\frac{d^2\left(\frac{du}{dx}\right)}{dy^2}\frac{k^2}{1.2}+\frac{d^3\left(\frac{du}{dx}\right)}{dy^3}\frac{k^3}{1.2.3}+\&c.$$

But since the expression $\frac{d\left(\frac{du}{dx}\right)}{dy}$ indicates two differentiations made successively on the function u, in the first considering x alone as variable, and in the second making y only vary; this expression assumes a more simple form, when written thus: $\frac{d^2u}{dy\,dx}$. In the same manner $\frac{d^2\left(\frac{du}{dx}\right)}{dy^2}$ is represented by $\frac{d^3u}{dy^2\,dx}$; and generally, by $\frac{d^{n+m}u}{dy^n\,dx^m}$, is meant the differential coefficient of the nth order, of

the function $\frac{d^m u}{d x^m}$, supposing that y is the only variable in it, whilst that function is itself the differential coefficient of the order m of the proposed function, supposing x only to vary.

This being premised, the substitution of $y+k$ instead of y will change

$$\frac{du}{dx} \text{ into } \frac{du}{dx} + \frac{d^2 u}{dy\,dx}\frac{k}{1} + \frac{d^3 u}{dy^2 dx}\frac{k^2}{1.2} + \frac{d^4 u}{dy^3\,dx}\frac{k^3}{1.2.3} + \&c.$$

$$\frac{d^2 u}{dx^2} \text{ into } \frac{d^2 u}{dx^2} + \frac{d^3 u}{dy\,dx^2}\frac{k}{1} + \frac{d^4}{dy^2\,dx^2}\frac{k^2}{1.2} + \frac{d^5 u}{dy^3\,dx^2}\frac{k^3}{1.2.3} + \&c.$$

$$\frac{d^3 u}{dx^3} \text{ into } \frac{d^3 u}{dx^3} + \frac{d^4 u}{dy\,dx^3}\frac{k}{1} + \frac{d^5 u}{dy^2\,dx^3}\frac{k^2}{1.2} + \frac{d^6 u}{dy^3\,dx^3}\frac{k^3}{1.2.3} + \&c.$$

&c. &c.

Substituting these values in the developement of $f(x+h, y)$ and arranging it so that all those terms in which the sum of the exponents of h and k are equal, shall be in the same vertical column, we shall have

$$\left.\begin{aligned} f(x+h, y+k) = u + \frac{du}{dy}\frac{k}{1} &+ \frac{d^2 u}{dy^2}\frac{k^2}{1.2} + \frac{d^3 u}{dy^3}\frac{k^3}{1.2.3} + \&c. \\ + \frac{du}{dx}\frac{h}{1} &+ \frac{d^2 u}{dy\,dx}\frac{k}{1}\frac{h}{1} + \frac{d^3 u}{dy^2\,dx}\frac{k^2}{1.2}\frac{h}{1} + \&c. \\ &+ \frac{d^2 u}{dx^2}\frac{h^2}{1.2} + \frac{d^3 u}{dy\,dx^2}\frac{k}{1}\frac{h^2}{1.2} + \&c. \\ &+ \frac{d^3 u}{dx^3}\frac{h^3}{1.2.3} + \&c. \\ &+ \&c. \end{aligned}\right\}$$

122. This developement has been obtained by first putting $x+h$ instead of x, and *then* $y+k$ instead of y; but we might have proceeded in an inverse order, and have begun by the substitution of $y+k$ for y; $f(x, y)$ would then have become

$$f(x, y+k),$$

or

$$u+\frac{du}{dy}\,\frac{k}{1}+\frac{d^2u}{dy^2}\,\frac{k^2}{1.2}+\frac{d^3u}{dy^3}\,\frac{k^3}{1.2.3}+\&c.$$

The substitution of $x+h$ for x, in this series, would have changed u into

$$u+\frac{du}{dx}\,\frac{h}{1}+\frac{d^2u}{dx^2}\,\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\,\frac{h^3}{1.2.3}+\&c.$$

and also

$$\frac{du}{dy}\text{ into }\frac{du}{dy}+\frac{d^2u}{dx\,dy}\,\frac{h}{1}+\frac{d^3u}{dx^2dy}\,\frac{h^2}{1.2}+\frac{d^4u}{dx^3dy}\,\frac{h^3}{1.2.3}+\&c.$$

$$\frac{d^2u}{dy^2}\text{ into }\frac{d^2u}{dy^2}+\frac{d^3u}{dx\,dy^2}\,\frac{h}{1}+\frac{d^4u}{dx^2dy^2}\,\frac{h^2}{1.2}+\frac{d^5u}{dx^3dy^2}\,\frac{h^3}{1.2.3}+\&c.$$

$$\frac{d^3u}{dy^3}\text{ into }\frac{d^3u}{dy^3}+\frac{d^4u}{dx\,dy^3}\,\frac{h}{1}+\frac{d^5u}{dx^2dy^3}\,\frac{h^2}{1.2}+\frac{d^6u}{dx^3dy^3}\,\frac{h^3}{1.2.3}+\&c.$$

and we should therefore have found

$$\left.\begin{array}{l}
(x+h, y+k)=u+\frac{du}{dx}\frac{h}{1}+\frac{d^2u}{dx^2}\,\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\,\frac{h^3}{1.2.3}+\&c.\\
\qquad\qquad+\frac{du}{dy}\frac{k}{1}+\frac{d^2u}{dx\,dy}\,\frac{h}{1}\,\frac{k}{1}+\frac{d^3u}{dx^2dy}\,\frac{h^2}{1.2}\,\frac{k}{1}+\&c.\\
\qquad\qquad\qquad+\frac{d^2u}{dy^2}\,\frac{k^2}{1.2}+\frac{d^3u}{dx\,dy^2}\,\frac{h}{1}\,\frac{k^2}{1.2}+\&c.\\
\qquad\qquad\qquad\qquad+\frac{d^3u}{dy^3}\,\frac{k^3}{1.2.3}+\&c.\\
\qquad\qquad\qquad\qquad\qquad+\&c.
\end{array}\right\}$$

It is evident, that this second developement ought to be identical with the former; since it is indifferent, whether we first change x into $x+h$, and then y into $y+k$, or whether we make these substitutions in an inverse order, as in either case, we obtain $f(x+h, y+k)$.

If we compare, in these two developements, those terms which are affected with the same powers of h and k, we shall find the following series of equations:

$$\frac{d^2 u}{dy\,dx}=\frac{d^2 u}{dx\,dy}$$

$$\frac{d^3 u}{dy\,dx^2}=\frac{d^3 u}{dx^2\,dy}$$

$$\frac{d^3 u}{dy^2\,dx}=\frac{d^3 u}{dx\,dy^2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^{m+n} u}{dy^n\,dx^m}=\frac{d^{m+n} u}{dx^m\,dy^n}.$$

From the first of these it follows, that *the differential coefficient of the second order of any function of two variables, taken first with respect to one of these variables, and then with respect to the other, is the same in whatever order the differentiations are performed.*

Take, for example, $u=x^m y^n$; if we first differentiate it, considering x only as variable, we have $\frac{du}{dx}=m\,x^{m-1}\,y^n$; then, differentiating this result, with respect to y only, we obtain $\frac{d^2 u}{dy\,dx}=m\,n\,x^{m-1}y^{n-1}$; and by performing these operations in an inverse order, we find

$$\frac{du}{dy}=n\,x^m y^{n-1},\text{ and }\frac{d^2 u}{dx\,dy}=m\,n\,x^{m-1}y^{n-1};$$

and the final result is the same in both cases.

The remaining equations of the series given above are only consequences of the first.

123. Subtracting f (x,y) or u from f $(x+h,y+k)$, we find

$$\left.\begin{aligned}\mathrm{f}\,(x+h,y+k)-\mathrm{f}\,(x,y)=\frac{du}{dx}\frac{h}{1}+\frac{d^2u}{dx^2}\frac{h^2}{1.2}&+\text{ \&c.}\\ +\frac{du}{dy}\frac{k}{1}+\frac{d^2u}{dy\,dx}\frac{k}{1}\frac{h}{1}&+\text{ \&c.}\\ +\frac{d^2u}{dy^2}\frac{k^2}{1.2}&+\text{ \&c.}\\ &+\text{ \&c.}\end{aligned}\right\}$$

If we extend the definition (5) of the differential of a function of one variable to those of two variables, we shall perceive, that that of f (x, y), or of u, consists of two terms, which form the first column of the preceding developement; and by changing h into dx, and k into dy, we have

$$d\,\mathrm{f}(x, y) = du = \frac{du}{dx}dx + \frac{du}{dy}dy.$$

From this it follows, that the complete differential of a function of two variables, consists of two parts, viz. $\frac{du}{dx}dx$, or the differential taken on the supposition that x alone is variable, and $\frac{du}{dy}dy$, or the differential taken when y only is variable.

We may, therefore, apply to functions of two variables the rules which have been given (No. 10, 11, &c.) for those which depend only on one; and for this purpose *we must differentiate the given function, first with respect to one of the variables, and then with respect to the other; and the sum of these two results will be the complete differential required.*

124. It will not be necessary to give many examples relative to the differentiation of functions of two variables, since it is reducible to that of functions which contain only one; the following will be sufficient:

We may perceive immediately, from the rule above given, that

$$d(x+y) = dx + dy$$

$$d\,.\,xy = y\,dx + x\,dy$$

$$d\,.\,\frac{x}{y} = \frac{dx}{y} - \frac{x\,dy}{y^2} = \frac{y\,dx - x\,dy}{y^2}.$$

Again, let 1st, $u = x^m y^n$, we have

$$\frac{du}{dx}dx = m\,x^{m-1}y^n dx$$

$$\frac{du}{dy}dy = n\,x^m y^{n-1}\,dy;$$

therefore,

$$du = m x^{m-1} y^n dx + n x^m y^{n-1} dy = x^{m-1} y^{n-1}(m y\,dx + n x\,dy)$$

2dly. Let $u = \frac{ay}{\sqrt{x^2+y^2}} = ay\,(x^2+y^2)^{-\frac{1}{2}}$; then we have

$$\frac{du}{dx}dx = -\frac{ayxdx}{(x^2+y^2)^{\frac{3}{2}}}.$$

$$\frac{du}{dy}dy = \frac{a\,dy}{(x^2+y^2)^{\frac{1}{2}}} - \frac{ay^2\,dy}{(x^2+y^2)^{\frac{3}{2}}};$$

therefore

$$du = -\frac{a\,y\,x\,dx}{(x^2+y^2)^{\frac{3}{2}}} + \frac{a\,dy}{(x^2+y^2)^{\frac{1}{2}}} - \frac{ay^2\,dy}{(x^2+y^2)^{\frac{3}{2}}},$$

or by reduction

$$= \frac{-a\,x\,y\,dx + a\,x^2\,dy}{(x^2+y^2)^{\frac{3}{2}}}.$$

3dly. Let $u = \text{arc tan}\,\frac{x}{y}$, which is the expression for the arc of a circle whose radius is unity, and tangent $\frac{x}{y}$. In order to differentiate this, we must make $\frac{x}{y} = z$, and must find, from No. 35, the differential of the arc, whose tangent is expressed by z; the result is $\frac{dz}{1+z^2}$: we have, therefore, $du = \frac{dz}{1+z^2}$; and putting for z and dz their values, we shall have

$$du = \frac{\frac{y\,dx - x\,dy}{y^2}}{1+\frac{x^2}{y^2}} = \frac{y\,dx - x\,dy}{y^2+x^2}.$$

125. The manner in which the differentials of functions, which depend on more than one variable, are written, gives rise to some important remarks. We have already seen (120), that in this case $\frac{d\,u}{d\,x}\,d\,x$ must not be confounded with $d\,u$, which it might be if u contained only the variable x, because the expression $\frac{d\,u}{d\,x}$ has, in the case of u being a function of two variables a particular meaning; it denotes the differential coefficient taken on the hypothesis of x only being variable; or, it is the quotient of the first term of the developement of the difference taken on that hypothesis, divided by the increment $d\,x$; and $\frac{d\,u}{d\,y}\,d\,y$ signifies the same with respect to y.

The quantities $\frac{du}{dx}$, $\frac{du}{dy}$ are commonly called *partial differences* of the first order of the function u; and generally $\frac{d^{m+n}u}{dx^m\,dy^n}$ represents one of those of the order $m+n$, which arises by differentiating m times with respect to x, and n times with respect to y.

It ought, however, to be observed, that the term *partial difference* is not a correct one, for the formulæ which are thus denoted do not express the difference of two quantities. The real partial differences of u are

$$\mathrm{f}\,(x+h,\,y)-\mathrm{f}\,(x,\,y),$$
$$\mathrm{f}\,(x,\,y+k)-\mathrm{f}\,(x,\,y).$$

The first of these is taken, with respect only to the change of x, and the second by supposing y only to vary. The expressions

$$\frac{d\,u}{d\,x}\,h,\ \frac{d\,u}{d\,y}\,k,\ \text{or}\ \frac{d\,u}{d\,x}\,d\,x,\ \ \frac{du}{dy}\,d\,y,$$

which are the first terms in the developements of these differences, ought to be called partial differentials; and $\frac{d\,u}{d\,x}$,

$\frac{du}{dy}$, are always the differential coefficients of the first order of the given function. It should, however, be observed, that a function of one variable has only one differential coefficient in each order (17), while a function of two variables has two differential coefficients for the first order, three for the second, &c.

These coefficients may be deduced in the following manner, from the two first, beginning with

$$du = \frac{du}{dx} dx + \frac{du}{dy} dy;$$

then taking the differentials of the functions $\frac{du}{dx}$ and $\frac{du}{dy}$, which must be treated as functions of two variables, we have

$$d\frac{du}{dx} = \frac{d^2 u}{dx^2} dx + \frac{d^2 u}{dy\,dx} dy,$$

$$d\frac{du}{dy} = \frac{d^2 u}{dx\,dy} dx + \frac{d^2 u}{dy^2} dy;$$

and since the second differential is nothing more than the differential of the first, we shall have

$$d^2 u = \frac{d^2 u}{dx^2} dx^2 + 2\frac{d^2 u}{dx\,dy} dx\,dy + \frac{d^2 u}{dy^2} dy^2,$$

considering dx and dy as constant quantities, and observing, that the differential coefficients whose denominators contain only the products of dx and dy differently arranged are identical (122).

If we differentiate the differential coefficients which occur in the preceding result, we shall have

$$d.\frac{d^2 u}{dx^2} = \frac{d^3 u}{dx^3} dx + \frac{d^3 u}{dy\,dx^2} dy,$$

$$d.\frac{d^2 u}{dx\,dy} = \frac{d^3 u}{dx^2\,dy} dx + \frac{d^3 u}{dy\,dx\,dy} dy,$$

$$d.\frac{d^2 u}{d y^2} = \frac{d^3 u}{d x\, d y^2} d x + \frac{d^3 u}{d y^3} d y,$$

and consequently

$$d^3 u = \frac{d^3 u}{d x^3} d x^3 + \frac{3\, d^3 u}{d x^2 d y} d x^2 d y + \frac{3\, d^3 u}{d x\, d y^2} d x d y^2 + \frac{d^3 u}{d y^3} d y^3.$$

This formation may easily be continued, and the analogy of the results with the powers of a binomial will doubtless be observed.

It may be noticed, that the series of No. 123. is the same with that in No. 22, when we substitute $d x$ for h, and $d y$ for k; so that if we denote $f(x+h, y+k)$ by u', we still have

$$u' - u = \frac{d u}{1} + \frac{d^2 u}{1.2} + \frac{d^3 u}{1.2.3} + \text{\&c.},$$

which formula is quite as general as that of No. 123; since the increments $d x$ and $d y$ are arbitrary.

126. It is easy to extend these considerations to functions of any number of variables, and to convince ourselves that if we have

$$u = f(t, x, y, z),$$

there will result

$$d u = \frac{d u}{d t} d t + \frac{d u}{d x} d x + \frac{d u}{d y} d y + \frac{d u}{d z} d z,$$

denoting by

$$\frac{d u}{d t}, \quad \frac{d u}{d x}, \quad \frac{d u}{d y}, \quad \frac{d u}{d z},$$

the differential coefficients of the function u, taken on the supposition that t or x, or y or z, only varies.

This notation, which owes its origin to Fontaine, is the most simple, and the most expressive of any which has yet been proposed to denote the same operations. Euler, fearing, lest the differential coefficient $\frac{d u}{d x}$ should be con-

founded with the ratio of the complete differential $d\,u$, to the differential $d\,x$, which ratio is equivalent to

$$\frac{\frac{d\,u}{d\,t}d\,t+\frac{d\,u}{d\,x}\,d\,x+\frac{d\,u}{d\,y}\,d\,y+\frac{d\,u}{d\,z}\,d\,z}{d\,x},$$

denoted this ratio by $\frac{d\,u}{d\,x}$, whilst he expresses the differential coefficient by $\left(\frac{d\,u}{d\,x}\right)$. The nature of the subject almost always renders this distinction superfluous. Fontaine, however, had provided against any case where this might be absolutely necessary, by proposing that the ratio should be written thus: $\frac{1}{d\,x}d\,u$; and knowing that this ratio occurs much less frequently than the differential coefficient, he gave to this latter the most simple sign, which is conformable to the theory of all nomenclatures, and is exactly contrary to what Euler did.

Observing that we never employ the ratio of the differentials of two quantities, without supposing, at least implicitly, that one of them is a function of the other, and that the expression

$$\frac{\frac{d\,u}{d\,t}\,d\,t+\frac{d\,u}{d\,x}d\,x+\frac{d\,u}{d\,y}\,d\,y+\frac{d\,u}{d\,z}\,d\,z}{d\,x},\text{ or }\frac{1}{d\,x}\,d\,u,$$

has no meaning, unless we consider the variables t, y, and z as implicitly depending on x; I have proposed to write this expression thus:

$$\frac{d\,(u)}{d\,x},$$

enclosing the function between two parentheses, to show that not only all the terms explicitly containing x may vary but also all those quantities which may implicitly depend on it. This notation has, as well as that of Fontaine, the

advantage of preserving the most simple sign for the case which most frequently occurs.

127. Let $u=0$ be an equation, containing x, y, and z; if we consider x and y as independent variables, z will be a function of both; and when x receives any increment, y being supposed constant, z will undergo a corresponding change. On this hypothesis, the equation $u=0$ ought to be considered as an equation between two variables, x and z; we shall have, therefore (38),

$$\frac{du}{dx}+\frac{du}{dz}\frac{dz}{dx}=0,$$

and from this may be deduced the differential coefficient of z, relative to the variability of x. It must here be remembered, according to the distinction which has been made (120), that in $\frac{dz}{dx}$, dz is the partial differential of z, taken with respect to the change of x alone.

It is evident, that if we had made y vary by differentiating the proposed equation, as if it contained only y and z, we should have had

$$\frac{du}{dy}+\frac{du}{dz}\frac{dz}{dy}=0.$$

If we multiply the first of these equations just found by dx, and the second by dy, and if we then add them together, there will result

$$\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}\left(\frac{dz}{dx}dx+\frac{dz}{dy}dy\right)=0;$$

but $\frac{dz}{dx}dx+\frac{dz}{dy}dy$, is nothing more than the complete differential of z (123): we have, therefore

$$\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz=0;$$

that is to say, we may equate the first differential of the

equation $u=0$, taken with respect to the three variables x, y, and z to zero. It must not be forgotten, that this differential ought to be considered as equivalent to two equations; since, by substituting in it instead of dz, its value $\frac{dz}{dx}dx+\frac{dz}{dy}dy$, those quantities which multiply dx and dy, must, on account of the independence of the increments, be separately equal to zero.

128. The equations which give the coefficients of the higher orders may be deduced by differentiating the equations

$$\frac{du}{dx}+\frac{du}{dz}\cdot\frac{dz}{dx}=0,$$

$$\frac{du}{dy}+\frac{du}{dz}\cdot\frac{dz}{dy}=0.$$

The first may be represented by $\frac{d(u)}{dx}=0$, and the second by $\frac{d(u)}{dy}=0$, agreeably to the notation adopted in No. 126; these equations again contain the three variables x, y, and z, and may be treated in the same manner as the proposed one. And considering first the change which x undergoes, not only will z vary, but also its coefficient of the first order, $\frac{dz}{dx}$ which will give rise to the coefficient of the second order $\frac{d^2z}{dx^2}$. Therefore, by differentiating $\frac{d(u)}{dx}$ with respect to x, we shall have, as in equations of two variables,

$$\frac{d^2(u)}{dx^2}, \text{ or } \frac{d^2u}{dx^2}+2\frac{d^2u}{dx\,dz}\frac{dz}{dx}+\frac{d^2u}{dz^2}\frac{dz^2}{dx^2}+\frac{du}{dz}\frac{d^2z}{dx^2}=0.$$

If we differentiate $\frac{d(u)}{dx}$, with respect to y and z, or $\frac{d(u)}{dy}$ with respect to x and z, observing that, in the first case,

$\frac{dz}{dx}$ gives $\frac{d^2 z}{dy\,dx}$, and in the second $\frac{dz}{dy}$ gives $\frac{d^2z}{dx\,dy}=\frac{d^2 z}{dy\,dx}$,

we shall have in both cases the same result, which will be $\frac{d^2(u)}{dx\,dy}$, or

$$\frac{d^2 u}{dx\,dy}+\frac{d^2 u}{dz\,dy}\,\frac{dz}{dx}+\frac{d^2 u}{dz\,dx}\,\frac{dz}{dy}+\frac{du}{dz}\,\frac{d^2 z}{dx\,dy}+\frac{d^2 u}{dz^2}\,\frac{dz}{dx}\,\frac{dz}{dy}=0.$$

Lastly, the equation $\frac{d(u)}{dy}=0$, differentiated, considering y and z only as variables, will produce

$$\frac{d^2(u)}{dy^2}, \text{ or } \frac{d^2 u}{dy^2}+2\,\frac{d^2 u}{dy\,dz}\,\frac{dz}{dy}+\frac{d^2 u}{dz^2}\,\frac{dz^2}{dy^2}+\frac{du}{dz}\,\frac{d^2 z}{dy^2}=0.$$

But since z is a function of x, and of y, u ought to be considered as a function of these variables, and we shall consequently have (125)

$$d^2(u)=\frac{d^2(u)}{dx^2}\,dx^2+\frac{2\,d^2(u)}{dx\,dy}\,dx\,dy+\frac{d^2(u)}{dy^2}\,dy^2=0;$$

and in fact, if we substitute for

$$\frac{d^2(u)}{dx^2},\quad \frac{d^2(u)}{dx\,dy},\quad \frac{d^2(u)}{dy^2},$$

the results found above, and if we change

$$\frac{dz}{dx}\,dx+\frac{dz}{dy}\,dy,$$

into the total differential dz of the first order, and also

$$\frac{d^2 z}{dx^2}\,dx^2+\frac{2\,d^2 z}{dx\,dy}\,dx\,dy+\frac{d^2 z}{dy^2}\,dy^2,$$

into the total second differential $d^2 z$, the final equation will be the same as that which we should have obtained if we had differentiated

$$\frac{du}{dx}\,dx+\frac{du}{dy}\,dy+\frac{du}{dz}\,dz=0,$$

considering dx and dy as constant, and z as a function of x and y.

129. These considerations may easily be extended to any order of differentiation, and to any number of variables; for the whole consists in determining those which are independent, which can only be done from the nature of the question which led to the proposed equations; and we must then differentiate with respect to each of these variables, treating the subordinate variables as implicit functions of the independent ones.

If, for example, we have the two equations,

$$u=0, \qquad v=0,$$

between the five variables, s, t, x, y, and z, we see that three of these variables are independent. Supposing then that y and z are functions of the other variables, s, t, and x, given by the proposed equations, we must successively differentiate u and v, with respect to s, with respect to t, and with respect to x; and we shall have

$$\frac{du}{ds}+\frac{du}{dy}\frac{dy}{ds}+\frac{du}{dz}\frac{dz}{ds}=0,$$

$$\frac{du}{dt}+\frac{du}{dy}\frac{dy}{dt}+\frac{du}{dz}\frac{dz}{dt}=0,$$

$$\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}+\frac{du}{dz}\frac{dz}{dx}=0.$$

If we multiply each of these equations respectively by ds, dt, dx, and add the results together, putting dy instead of

$$\frac{dy}{ds}ds+\frac{dy}{dt}dt+\frac{dy}{dx}dx,$$

and dz instead of

$$\frac{dz}{ds}ds+\frac{dz}{dt}dt+\frac{dz}{dx}dx,$$

we shall have

$$\frac{du}{ds}\,ds+\frac{du}{dt}\,dt+\frac{du}{dx}\,dx+\frac{du}{dy}\,dy+\frac{du}{dz}\,dz=du=0.$$

A similar result may be deduced from the equation $v=0$; and it follows, that in differentiating the equations $u=0$ and $v=0$, with respect to all the variables, s, t, u, x, y, and z, and in substituting instead of dy and dz, the expressions for those differentials, considered as functions of three variables (126), we must make the coefficient of the differential of each independent variable separately equal to zero.

By considering the differential coefficients themselves as new functions of the independent variables, we shall have no difficulty in investigating the differentials of higher orders; we shall therefore conclude what remains concerning the formation of differential equations by some remarks on the elimination of constant quantities and of functions.

130. The equation $u=0$, between x, y, and z, having two differentials of the first order, $\frac{d\,(u)}{dx}=0$, and $\frac{d\,(u)}{dy}=0$, it is evident, that we can eliminate two constant quantities between these three equations, and the result will express the relation between the variables x, y, z, and the coefficients $\frac{dz}{dx}$, or $\frac{dz}{dy}$, independently of the particular values of the quantities eliminated.

If to the above equations we join the three of the second order,

$$\frac{d^2\,(u)}{dx^2}=0,\quad \frac{d^2\,(u)}{dx\,dy}=0,\quad \frac{d^2\,(u)}{dy^2}=0,$$

we shall have six equations, from which we may eliminate five quantities, and so on.

131. This leads to a very important remark, that we may eliminate from an equation of three or more variables, functions, whose form is absolutely unknown.

Take, for example, the equation $z=\mathrm{f}(ax+by)$, in which the characteristic f denotes a function, whose form is indeterminate. From this may be deduced an equation between $\frac{dz}{dx}$ and $\frac{dz}{dy}$, which is independent of that function, and which is equally adapted to $z=ax+by$, to $z=\sqrt{ax+by}$, to $z=\sin(ax+by)$, and in general to all functions of the quantity $ax+by$. Make $ax+by=t$, then the given equation becomes $z=\mathrm{f}(t)$, and consequently we have $dz=\mathrm{f}'(t)\,dt$, $\mathrm{f}'(t)$, denoting $\frac{d\,\mathrm{f}(t)}{dt}$; but

$$dz=\frac{dz}{dx}dx+\frac{dz}{dy}dy,$$

$$dt=\frac{dt}{dx}dx+\frac{dt}{dy}dy,$$

whence

$$\frac{dz}{dx}=\mathrm{f}'(t)\frac{dt}{dx},\qquad \frac{dz}{dy}=\mathrm{f}'(t)\frac{dt}{dy}:$$

putting for $\frac{dt}{dx}$ and $\frac{dt}{dy}$, their values, a and b, and then eliminating $\mathrm{f}'(t)$, we shall have

$$b\frac{dz}{dx}-a\frac{dz}{dy}=0.$$

This equation expresses a character by which we may distinguish, whether any proposed quantity is a function of $ax+by$ or not; for from its formation it ought to become identical, whenever we substitute in it instead of $\frac{dz}{dx}$ and $\frac{dz}{dy}$, their values deduced from any function of $ax+by$. Let us suppose that we are unacquainted with the origin of the polynomial $a^2x^2+2abxy+b^2y^2$: by equating it to z, and differentiating, we shall find

$$\frac{dz}{dx}=2a^2x+2aby,\qquad \frac{dz}{dy}=2abx+2b^2y;$$

these values substituted in the equation $b\frac{dz}{dx}-a\frac{dz}{dy}=0$, render it identical: from this we conclude, that the polynomial represented by z, is a function of $ax+by$, which is otherwise apparent, since

$$a^2x^2+2abxy+b^2y^2=(ax+by)^2.$$

In general, when we have any equation $u=0$ between x, y, z, and any indeterminate function represented by $\mathrm{f}(t)$, and in which only t is given in terms of x, y, and z, we may always eliminate $\mathrm{f}(t)$ and $\mathrm{f}'(t)$, by means of the equations

$$u=0,\quad \frac{d(u)}{dx}=0,\quad \frac{d(u)}{dy}=0.$$

When we proceed to the second order, the number of equations becoming greater, it is possible, in many cases, to eliminate two unknown functions; but we shall not enter into these details, nor into those which relate to equations containing more than three variables.

132. Very little need here be said concerning the manner of reducing into series, functions of two variables, because it most frequently happens, that they are only developed in respect to one of the variables which they contain, the other being supposed to have some constant value; and in this case they may be treated as functions of one variable. It may, however, be useful to show how the formula of No. 121. may be employed for developing functions of two variables, in the same manner as that of No. 21. is applied to those which contain only one.

If we make $x=0$, and $y=0$, in the formula of No. 121, that is, in u and in each of its differential coefficients, it will give the developement of $\mathrm{f}(h, k)$ arranged according to the powers of h and k; but we may put x instead of h, and y instead of k, and the result will be

$$f(x, y) = u + \frac{1}{1}\left\{\frac{du}{dx}x + \frac{du}{dy}y\right\}$$
$$+ \frac{1}{1\,.\,2}\left\{\frac{d^2u}{dx^2}x^2 + 2\frac{d^2u}{dx\,dy}xy + \frac{d^2u}{dy^2}y^2\right\}$$
$$+ \&c.$$

observing that x and y must be put equal to zero, both in u and in all the differential coefficients deduced from it.

We might also obtain the developement of $f(x, y)$, by differentiation, as we have arrived at that of $f(x)$ in No. 19; for if we suppose

$$u = A + Bx + Cy + Dx^2 + Exy + Fy^2 + \&c.$$

the letters A, B, C, &c. will denote quantities independent on x and y; and if we differentiate this equation with respect to x and to y several times successively, so as to form the differential coefficients

$$\frac{du}{dx}, \quad \frac{du}{dy}, \quad \frac{d^2u}{dx^2}, \quad \frac{d^2u}{dx\,dy}, \ \&c.$$

we shall have, by equating x and y to zero, after the differentiations,

$$\frac{du}{dx} = B, \qquad \frac{du}{dy} = C,$$

$$\frac{d^2u}{dx^2} = 1\,.\,2\,D, \qquad \frac{d^2u}{dx\,dy} = 1\,.\,1\,E, \quad \frac{d^2u}{dy^2} = 1\,.\,2\,F, \ \&c.$$

the value of A may be found, by seeking that of the function u, when x and y are put equal to zero.

Of the Maxima *and* Minima *of Functions of two Variables.*

133. If, in a function of two variables, we consider one as constant, and if we give to the other an infinite number of different values, for each of these values the given function will have one or more values, and amongst

these some may be *maxima*, and others *minima*, which may be determined by making the differential coefficient of the function taken relative to the quantity which is supposed to change, equal to zero (48).

Thus if we suppose u to be a function of x and y, and if we suppose y to be constant, and make $\frac{d\,u}{d\,y}=0$, we shall obtain those values of x, which make u the greatest or the least, when y retains the same value.

The result which we thus obtain is still indefinite, because it may vary on account of the changes which the other variable y undergoes; and it consequently only determines the relative *maxima* and *minima*, amongst which there exists a certain number which exceed, or which are less than the others, and which correspond to determinate values of y. These latter, which are completely determined, are the absolute *maxima* and *minima* of the given function; they may readily be discovered, by eliminating x from the function u, by means of the equation $\frac{d\,u}{d\,x}=0$, which will render u a function of y only, and denoting the result by v, we must then make $\frac{d\,v}{d\,y}=0$, which equation will determine the values of y (48).

We may obtain an equation equivalent to $\frac{d\,v}{d\,y}=0$, without eliminating x; for this purpose we must observe, that the equation $\frac{d\,u}{d\,x}=0$, which arises from the condition of the *maximum* or *minimum*, relative to x, establishes a relation between the variables x and y; so that the former of these may be considered as a function of the latter. By differentiating upon this supposition, we have (126)

$$\frac{d\,(u)}{d\,y}=\frac{d\,u}{d\,x}\,\frac{d\,x}{d\,y}+\frac{d\,u}{d\,y}=0,$$

which result is reduced to $\frac{du}{dy}=0$, because from the condition relative to x, we have already found $\frac{du}{dx}=0$.

Those values of x and y, which correspond to the absolute *maxima* and *minima* of the function u, may therefore be determined by means of the equations

$$\frac{du}{dx}=0, \qquad \frac{du}{dy}=0.$$

By extending these considerations to functions of any number of variables, we shall find, that in order to discover the absolute *maxima* and *minima* of these functions, we must make their differential coefficients of the first order, taken with respect to each of the variables on which they depend, separately equal to zero.

134. The characters which distinguish the *maxima* from the *minima*, in functions of several variables, may be deduced from principles analogous to those which have been employed with respect to functions of one variable (49); but the application is more complicated: we shall, therefore, confine ourselves to that which concerns functions of two variables.

Let u be a function of x and y, and, for the sake of brevity, let

$$A, \quad B, \quad C, \quad \&c.$$

denote the series of functions

$$u, \quad \frac{du}{dx}, \quad \frac{du}{dy}, \quad \frac{d^2u}{dx^2}, \quad \frac{d^2u}{dx\,dy}, \quad \frac{d^2u}{dy^2};$$

the result of the substitution of $x+h$, $y+k$ in u, will be (121)

$$A+Bh+Ck$$
$$+\frac{1}{1.2}\left\{Dh^2+2Ehk+Fk^2\right\}$$
$$+\ \&c.$$

the value of this expression ought to be less than that of A, in the case of an absolute *maximum*, and greater in the case of an absolute *minimum*, whatever be the signs of h and k, provided they express very small quantities. But since the terms of the first order $B\,h$, $C\,k$, which may become greater than the sum of all the others, by taking h and k of a proper degree of smallness, change their sign at the same time with those quantities themselves, it may be shown from a reasoning similar to that employed in No. 48, that they must vanish in the case of a *maximum*, or a *minimum*; we have, therefore

$$B=\frac{d\,u}{d\,x}=0, \qquad C=\frac{d\,u}{d\,y}=0, \text{\&c.}$$

which is the same result as that of the preceding rule.

These conditions being fulfilled by the values of x and y, which are determined from them; it is, in the next place, necessary, that the coefficients D, E, and F, should not vanish at the same time; and moreover, that the sign of the quantity of the second order, which forms the second line in the above developement, should be independent of any relation that may be established between h and k, and their signs.

We know, from the theory of algebraic equations, that every expression of the same form as the first member, when the second vanishes, cannot pass from positive to negative, without becoming zero in the interval; and that when they have only imaginary roots, they do not change their sign, whatever value may be given to the unknown quantity: from this it follows, that the quantity

$$D\,h^2+2\,E\,h\,k+F\,k^2,$$

being made equal to zero, and solved as an equation with respect to one of the indeterminate quantities, h or k, will only give imaginary roots; and we may conclude, that it will preserve the same sign, whatever may be the value of

the indeterminate quantities. Taking the value of h, for instance, we find

$$h=\frac{k(-E\pm\sqrt{-FD+E^2})}{D},$$

which result will be imaginary, if the quantity comprehended under the radical is negative, that is to say, if we have $FD > E^2$. We should observe, that this condition, which is necessary, in order that a *maximum* or *minimum* should exist, supposes, that the quantities F and D are of the same sign. In that case, the quantity $Dh^2+2Ehk+Fk^2$ cannot change its sign; and since it is reduced to Dh^2, when $k=0$, D must be positive, in order that it may be positive, or that a *minimum* may exist; and there will be a *maximum* in the contrary case.

135. In order to prove *a posteriori*, that when the conditions we have just found are fulfilled, the sum of the terms of the second order of the series $A+Bh+Ck+$ &c. will preserve the same sign, whatever may be the values of h and k: it is sufficient to observe, that the first member of the equations of the second degree, whose roots are

$$h=-\alpha\pm\sqrt{-\beta^2},$$

having the form

$$(h+\alpha)^2+\beta^2,$$

is the sum of two squares, and that consequen tly itcannot change its sign.

Making

$$\alpha=\frac{Ek}{D}, \qquad \beta^2=\frac{(FD-E^2)k^2}{D^2},$$

the expression of h leads to

$$\left(h+\frac{Ek}{D}\right)^2+\frac{(FD-E^2)k^2}{D^4}=h^2+\frac{2E}{D}hk+\frac{F}{D}k^2$$

$$=\frac{1}{D}(Dh^2+2Ehk+Fk^2),$$

and it appears, that the quantity

$$Dh^2+2Ehk+Fk^2$$

cannot change its sign, as long as the quantity $FD-E^2$ remains positive; because the square of $h+\frac{Ek}{D}$ is necessarily positive, whatever may be the signs of the quantities h and k.

Euler, in his Differential Calculus, only pointed out the necessity of having D and F both positive or both negative at the same time. Lagrange was the first who showed that this condition alone is not sufficient, and to him we are indebted for the theory which has just been explained.

If the coefficients of the second order vanish at the same time with those of the first, there can be no *maximum* or *minimum*, unless the coefficients of the third order disappear also; and the terms of the fourth order form a quantity, whose sign is independent of h and k. The consideration of the imaginary factors which this condition will render necessary, will lead to results analogous to the preceding. Finally, it may be observed, that whatever may be the result on substituting the values of x and y, relative to the *maximum* and *minimum*, in u and in its differential coefficients, it must always happen that the results derived from the supposition of $x\pm h$, $y\pm k$, shall be all less, or all greater than u; and that the different methods which are proper for ascertaining whether this takes place, will also be proper for ascertaining the existence of a *maximum* or of a *minimum*.

136. As an example, let us take the following question, similar to that of No. 50.

To divide a quantity a into three parts, x, y, $a-x-y$, such that the product $x^m y^n (a-x-y)^p$ may be a *maximum*.

We then have

$$u=x^m y^n (a-x-y)^p$$

$$\frac{du}{dx}=x^{m-1}y^n(a-x-y)^{p-1}\{ma-mx-my-px\}=0$$

$$\frac{du}{dy}=x^m y^{n-1}(a-x-y)^{p-1}\{na-nx-ny-py\}=0;$$

the factors $ma-mx-my-px$, and $na-nx-ny-py$, being made equal to zero, give

$$x=\frac{ma}{m+n+p},\ y=\frac{na}{m+n+p},\ a-x-y=\frac{pa}{m+n+p}.$$

In order to discover whether these values belong to a *maximum* or a *minimum*, we must substitute them in the general expressions of

$$\frac{d^2u}{dx^2},\quad \frac{d^2u}{dx\,dy},\quad \frac{d^2u}{dy^2};$$

and, for the sake of brevity, putting $m+n+p=q$, we shall find

$$D=-(m+p)\left(\frac{ma}{q}\right)^{m-1}\left(\frac{na}{q}\right)^{n}\left(\frac{pa}{q}\right)^{p-1}$$

$$E=-\frac{mna}{q}\left(\frac{ma}{q}\right)^{m-1}\left(\frac{na}{q}\right)^{n-1}\left(\frac{pa}{q}\right)^{p-1}$$

$$F=-(n+p)\left(\frac{ma}{q}\right)^{m}\left(\frac{na}{q}\right)^{n-1}\left(\frac{pa}{q}\right)^{p-1}.$$

The quantities D and F are negative, and we may readily convince ourselves that they fulfil the condition $DF-E^2>0$, when the exponents m, n, p, are positive: thus we have obtained the *maximum* which was required.

General Remarks upon the Application of the Differential Calculus to the Theory of Curves of Double Curvature, and to curve Surfaces.

137. Considerable details have been given concerning this theory, in the *Treatise on the Differential and Integral Calculus:* it contains an extract of some length, from the

researches of Monge, connected with, and enlarged by, those of Euler, and of other Geometers. At present we shall confine ourselves to the most simple problems relating to this subject.

It is known (Trig. App.) that two equations between three unknown quantities may be represented by a curve of double curvature, whilst one equation between three variables belongs to a curve surface. In applying the Differential Calculus to curves of double curvature, we may consider them as the limits of polygons, three consecutive sides of which are not in the same plane. The prolongation of one of these sides gives the tangent in the same manner as it does in curves described on a plane.

We may easily obtain the equations of the tangent MT', fig. 33, by observing that its projections are themselves tangents to those of the curve XM; and since it is sufficient to be acquainted with two projections of this straight line, we will fix on those which fall on the plane of the x and z, and on the plane of the y and z. Denoting the co-ordinates of any point in the tangent by x', y', z', those of the point M being x, y, z, the equation of the line $T''M''$, which is a tangent to the projection $X''M''$, will be FIG. 33.

$$z'-z=\frac{dz}{dx}(x'-x) \qquad (67),$$

and that of $T'''M'''$, which is a tangent to $X'''M'''$, will be

$$z'-z=\frac{dz}{dy}(y'-y).$$

Suppose that we have deduced the values of y and z in terms of x, from the equations of the curves $X''M''$, and $X'''M'''$; and that, after the substitution of these values in the equations of the tangent we eliminate x, we shall have the relation which exists between the co-ordinates x', y', and z' of the tangent TM, whatever be the position of the point M, and consequently we shall have

the equation of the curve surface, formed by all the tangents of the curve XM. From this we may easily discover whether the curve is wholly situated in the same plane or not; for in the former case the curve surface just alluded to will be a plane, and in the latter case it will be a curve surface.

138. Two successive tangents, TM and tm, determine the situation of the plane which passes through two successive sides, and which is called the Osculating Plane. Its equation may be found, by supposing it to pass through three successive points of the given curve. Let, therefore, $Ax'+By'+Cz'+D=0$, be its equation (Trig. App.); it is first necessary, that the equation $Ax+By+Cz+D=0$, should be satisfied, since this plane must contain the point whose co-ordinates are x, y, and z; and in order that the two succeeding points may be situated on it, it is also necessary that the first and second differential of its equation should hold good at the same time with those of the equations of the given curve.

We might take one of the differentials dx, dy, or dz, for constant (118); but the result will be more symmetrical if we treat them all as variable; and we shall have

$$A\,dx+B\,dy+C\,dz=0,\quad A\,d^2x+B\,d^2y+C\,d^2z=0,$$

whence we deduce

$$\frac{A}{C}=\frac{dy\,d^2z-dz\,d^2y}{dx\,d^2y-dy\,d^2x},\quad \frac{B}{C}=\frac{dz\,d^2x-dx\,d^2z}{dx\,d^2y-dy\,d^2x}:$$

and subtracting the equation

$$Ax+By+Cz+D=0,$$

from

$$Ax'+By'+Cz'+D=0,$$

and then putting for $\frac{A}{C}$ and $\frac{B}{C}$, their values found above, and making the denominators disappear, we shall find the following result, whose form is remarkable:

$$(z'-x)(dy\,d^2z-dz\,d^2y)+(y'-y)(dz\,d^2x-dx\,d^2z)$$
$$+(z'-z)(dx\,d^2y-dy\,d^2x)=0.$$

By substituting in this for any two of the three co-ordinates, their values deduced from the equations of the given curve, we shall have the equation of the osculating plane, expressed in terms of the remaining co-ordinate.

139. The expression for the differential of the arc of a curve of double curvature is

$$\sqrt{dx^2+dy^2+dz^2};$$

which will readily appear by finding the distance of the points M and m, whose respective co-ordinates are

$$x,\ y,\ z,\ x+dx,\ y+dy,\ \text{and}\ z+dz.$$

140. To draw a normal to a curve of double curvature is an indeterminate problem, for there exist an infinite number of right lines which pass through the point of contact, and which are at the same time perpendicular to the tangent. The whole assemblage of these right lines forms a plane perpendicular to the tangent, and which is called the Normal Plane: its equation will be (Trig. App.)

$$(x'-x)\frac{dx}{dz}+(y'-y)\frac{dy}{dz}+(z'-z)=0,$$

whence

$$(x'-x)\,dx+(y'-y)\,dy+(z'-z)\,dz=0.$$

141. Let x, y, and z be the co-ordinates of a point M, fig. 34, situated on any curve surface, we may consider the ordinate $M'M=z$, as a function of the two abscissæ, $AP=x$, and $PM'=y$. When x alone varies and becomes $x+h$, we shall have, for the development of the ordinate $m'm$, which is taken in the section QMm, made by a plane parallel to that of x and z, and passing through the proposed point, the series FIG. 34.

$$z+\frac{dz}{dx}\frac{h}{1}+\frac{d^2z}{dx^2}\frac{h^2}{1.2}+\frac{d^3z}{dx^3}\frac{h^3}{1.2.3}+\&c.$$

If y change, and become $y+k$, while x remains constant, we shall have the ordinate $n'n$, taken in the section PMn, made by a plane parallel to that of y and z, and passing through the given point. The developement of this ordinate will be

$$z+\frac{dz}{dy}\frac{k}{1}+\frac{d^2z}{dy^2}\frac{k^2}{1.2}+\frac{d^3z}{dy^3}\frac{k^3}{1.2.3}+\&c.$$

By making x and y vary at the same time, we pass from any point M, to any other point N, and this may be effected in two different ways, either by substituting $y+k$, instead of y, in the former of the above-written developements, or by putting $x+h$ instead of x, in the latter.

By one of these operations, we pass from the ordinate $m'm$, to the ordinate $N'N$, in the section pmN, and by the other we pass from $n'n$ to $N'N$, in the section qnN. It is evident that these two sections ought to meet in the point N, without which the proposed surface would not be continuous: the results given in No. 121. and 122. must, therefore, be identical. The equation $\frac{d^2z}{dx\,dy}=\frac{d^2z}{dy\,dx}$, to which this circumstance relates, is therefore nothing more than an expression of the law of continuity.

When, in the series

$$\begin{aligned}z&+\frac{dz}{dx}h+\frac{dz}{dy}k\\&+\frac{1}{2}\left(\frac{d^2z}{dx^2}h^2+2\frac{d^2z}{dx\,dy}hk+\frac{d^2z}{dy^2}k^2\right)\\&+\&c.\end{aligned}$$

which represents the developement of the value of z correspondent to $x+h$ and $y+k$, we cease to consider the quantities h and k as independent of each other, and establish a relation between them, we then fix the direction of the plane, drawn perpendicularly to that of x and y, by means of the two points M and N, for

$$\frac{k}{h}=\frac{N'\,m'}{M'\,m'}=\tan N'\,M'\,m'.$$

From these considerations, and from those which have been mentioned in No. 127, it follows, that if $u=0$ represent the equation of the curve surface, the differential equations

$$\frac{d\,u}{d\,x}+\frac{du}{d\,z}\,\frac{d\,z}{d\,x}=0, \text{ and } \frac{d\,u}{d\,y}+\frac{d\,u}{d\,z}\,\frac{d\,z}{d\,y}=0,$$

belong respectively to the two sections $Q\,M\,m$ and $P\,M\,n$; the co-ordinate y only enters into the first as an arbitrary constant, which determines the position of the cutting plane; and it is the same with respect to the co-ordinate x in the second equation. The $d\,z$ of the first equation must not be confounded with that of the second; for they are both only partial differentials, as has been remarked in No. 120: the complete differential, which is the sum of the terms of the first order, is

$$d\,z=\frac{d\,z}{d\,x}d\,x+\frac{d\,z}{d\,y}d\,y=p\,d\,x+q\,d\,y,$$

p representing $\frac{d\,z}{d\,x}$, and q representing $\frac{d\,z}{d\,y}$.

When we have simply $d\,z=p\,d\,x$, $d\,z$ is the differential of the ordinate of the section parallel to the plane of x and z; and similarly $d\,z=q\,d\,y$ is that of the ordinate of the section, parallel to the plane of y and z.

If we take $d\,y=\alpha\,d\,x$, the complete differential $d\,z=d\,x\,(p+\alpha\,q)$, will belong to the ordinate of the section made by the plane $M'\,M\,N\,N'$, perpendicular to that of x and y, supposing $N'\,m'=\alpha\times M'\,m'$. We shall find analogous circumstances, by taking each of the variables x and y successively, for ordinates, and considering the two others as abscissæ.

142. We may discover the equation of the tangent plane by subjecting it to pass through the two straight

lines MT and Mt, which respectively touch the sections QMm and PMn, at the point M (Comp. des Elem. de Geom.) Since the functions $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are the differential coefficients of the ordinate z, considered successively relative to each of these sections; and since the right lines MT and Mt are moreover parallel to the planes of x and z, and of y and z, it is easy to perceive that the equations of MT will be

$$z'-z=\frac{dz}{dx}(x'-x), \quad y'-y=0,$$

and that those of Mt will be

$$z'-z=\frac{dz}{dy}\,(y'-y), \quad x'-x=0.$$

If now we represent the equation of the tangent plane by

$$z'-z=A\,(x'-x)+B\,(y'-y),$$

it is evident that this equation ought to agree with the preceding ones; and in order to this, if we make successively $y'-y=0$ and $x'-x=0$, we ought to find

$$z'-z=\frac{dz}{dx}\,(x'-x), \text{ and } z'-z=\frac{dz}{dy}\,(y'-y);$$

but from the same suppositions we have

$$z'-z=A\,(x'-x), \text{ and } z'-z=B\,(y'-y):$$

whence we conclude that

$$A=\frac{dz}{dx}=p, \quad \text{and} \quad B=\frac{dz}{dy}=q,$$

and consequently

$$z'-z=p\,(x'-x)+q\,(y'-y).$$

It might be supposed that the tangent plane thus determined, only touches the surface at the two sections we have considered, but, by differentiating its equations with

respect to x' and y' only, we find $dz' = p\,dx' + q\,dy'$, which proves, that when we take $dx = dx'$, and $dy = dy'$, we shall have $dz = dz'$; and consequently, that the points on the given surface, which immediately surround the point M, coincide with all those of the tangent plane, at least whilst we only consider quantities of the first order. From this it follows, that any plane whatever drawn through the point M, cuts the given surface in a curve, which has two points common with the tangent plane, or (which amounts to the same thing), which has for its tangent the intersection of this plane with the cutting plane (67.)

143. The normal to a surface being perpendicular to the tangent plane, has for its equations (Trig. App.)

$$x' - x + p\,(z' - z) = 0,$$
$$y' - y + q\,(z' - z) = 0.$$

We might also arrive at these equations, by observing that the normal is the shortest line which can be drawn from any given point to the surface; for if x', y', z', denote the co-ordinates of this point, the distance from it to that point of the given surface whose co-ordinates are x, y, and z, will be expressed by

$$\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2},$$

and on account of the dependence of z, this will be a function of x and y only, when the point at which the normal meets the surface is considered as unknown. Differentiating it on this hypothesis (133), in order to find the *minimum*, we shall have

$$(x'-x)\,dx + (z'-z)\,dz = 0$$
$$(y'-y)\,dy + (z'-z)\,dz = 0,$$

which results are similar to the preceding, when $p\,dx$ and $q\,dy$ are successively put for dz.

144. It is proper to remark, that when we seek the *maximum* or *minimum* of a function of z, dependent on x and y, we in effect suppose that the tangent plane of the surface which the equation between x, y, and z represents, is parallel to the plane of the x and y, since we make at once $\frac{dz}{dx}=0$, and $\frac{dz}{dy}=0$. This circumstance is analogous to what we have seen in Nos. 133. and 144.

PART II.

ON THE INTEGRAL CALCULUS.

On the Integration of Rational Functions involving only one Variable.

145. THE Integral Calculus is the inverse of the Differential; its object being to ascend from the differential coefficients to the functions from which they are derived. The exposition of the principles of this Calculus presents divisions somewhat analogous to those presented by the Differential Calculus. It may happen that the differential coefficients of the function sought for, may be given immediately by the independent variables, or that we have only an equation between these differential coefficients, and one or more of the variables themselves. The first case being the most simple, is that which we shall treat of first.

When the differential coefficient of the first order of a function of x is given in terms of x, we have $\frac{dy}{dx}=X$, or $dy=X\,dx$; the function sought for is consequently that whose differential is $X\,dx$; and we represent it as follows: $y=\int X\,dx$, the characteristic $\int$ being the inverse of the characteristic d*. To find this function we must invert

* The letter $\int$ was employed by the first writers on the Integral Calculus, as the initial of the word *summa*, inasmuch as, accord-

ing

the rules of differentiation; but in order to proceed systematically, we shall treat successively of the different forms which the function X may admit, and which may be classed as follows: rational functions,

$$Ax^m + Bx^n + Cx^p + \&c. = U.$$

$$\frac{Ax^m + Bx^n + Cx^p + \&c.}{A'x^{m'} + B'x^{n'} + C'x^{p'} + \&c.} = \frac{U}{V}.$$

irrational functions,

$$U.V^{\frac{m}{n}},$$

transcendental functions,

$$f(U, \mathrm{l}\,U), \quad f(U, \sin V), \&c.$$

146. The differential of $Ax^m + B$, being $mAx^{m-1}dx$, we may from thence infer, that the integral of $ax^n dx$ is $\frac{ax^{n+1}}{n+1} + B$; for if we compare $ax^n dx$ with $mAx^{m-1}dx$, we have $m-1=n$, and $mA=a$ or $A=\frac{a}{m}=\frac{a}{n+1}$.

It follows, from this example, that when

$$dy = ax^n dx, \quad y = \frac{ax^{n+1}}{n+1} + B,$$

or, *to integrate a differential of one term, such as* $ax^n dx$, *we must increase the exponent of the variable by unity, and then divide by the new exponent index, and by* dx.

---

ing to the principles of Leibnitz, the differentials represented the indefinite small increments of the variables, and consequently each variable itself is the sum of the infinite number of these increments, which it has received from its origin to the moment in which we consider it; and it is on this account that they gave to the function which we call *primitive*, the name of *integral*, considering it as the result of the aggregation of all the differentials: these names being properly understood, we may make use indifferently of either one or the other.

The constant B is of arbitrary value (8). We may give it a form similar to that of the first term; for if we represent by b that value of x, which makes the function y equal to nothing, we shall have $\frac{a b^{n+1}}{n+1} + B = 0$, or $B = -\frac{a b^{n+1}}{n+1}$, and consequently

$$y = \frac{a\,(x^{n+1} - b^{n+1})}{n+1},$$

a result which only differs from the former in the form which is given to the constant.

147. Before we proceed further, it may be proper to examine a particular case, in which the value of y, found as above, becomes $\frac{0}{0}$; it is that in which $n = -1$, for then we have

$$y = \frac{a\,(x^0 - b^0)}{0} = \frac{a\,(1-1)}{0} = \frac{0}{0}.$$

To find the true value of this function, we must revert to the rule in No. 52; and as we have seen, by this rule, page 65, that $\frac{a^x - b^x}{x}$ became $\mathrm{l}\,a - \mathrm{l}\,b$, we shall have, in the example before us, when we substitute corresponding letters, $y = a\,(\mathrm{l}\,x - \mathrm{l}\,b)$; but when $n = -1$, we have $dy = a x^{-1} dx$; and therefore $dy = \frac{a\,dx}{x}$ gives

$$y = a\,(\mathrm{l}\,x - \mathrm{l}\,b), \quad \text{or } y = a\,\mathrm{l}\,x + B.$$

We might have drawn the same conclusion from No. 27, since it there appears, that $d\,\mathrm{l}\,x = \frac{dx}{x}$. The exception to the rule in No. 146, presented in this case, arises from the impossibility of expressing the transcendental quantity $\mathrm{l}\,x$, in a finite number of algebraical terms.

The whole difficulty of the integration of functions of one variable, consists in the discovery of such transformations as are proper to reduce the proposed functions to one or

more terms, involving simply powers of the variable and constant quantities, to each of which the rule in the preceding No. may be applied.

148. It is evident, at first sight, that

$$dy = a x^m dx + b x^n dx + c x^p dx \ldots\ldots\ldots$$

gives

$$y = \frac{a x^{m+1}}{m+1} + \frac{b x^{n+1}}{n+1} + \frac{c x^{p+1}}{p+1} + \&c. \ldots \ldots + B.$$

We only add one arbitrary constant, for it is easily seen that if a constant was added to each integral, they would together be equivalent to one constant quantity, which is equal to their sum.

In general, since we have seen, No. 10, that

$$d(u+v-w) = du + dv - dw,$$

we may thence include, that

$$\int(du+dv-dw) = \int du + \int dv - \int dw,$$

and that

$$\int(P dx + Q dx - R dx) = \int P dx + \int Q dx - \int R dx.$$

We will, in this place, notice a consequence of this rule, which will be of great use in what follows. In integrating separately each term of the differential $d. uv = u dv + v du$ (11), there results $uv = \int u dv + \int v du$, which gives the relation between the primitive functions of the differentials $u dv$, $v du$, insomuch that one being known, the other is known also. We have, for example, $\int u dv = uv - \int v du$.

The differential

$$d. \frac{u}{v} = \frac{du}{v} - u \frac{dv}{v^2} \text{ (12)},$$

will give, in a similar manner,

$$\frac{u}{v} = \int \frac{du}{v} - \int u \frac{dv}{v^2};$$

from whence we get

$$\int u \frac{dv}{v^2} = -\frac{u}{v} + \int \frac{du}{v}.$$

It is proper to observe, that this result is a consequence of the preceding; for by substituting in the value of $\int u\,dv$, found above, $\frac{dv}{v^2}$ in the place of dv, which is the same thing as changing v into $-\frac{1}{v}$.

$$\left(\text{for } \frac{dv}{v^2} = v^{-2}dv = -d.\,v^{-1} = -d.\frac{1}{v}\right),$$

we shall have

$$\int u \frac{dv}{v^2} = -\frac{u}{v} + \int \frac{du}{v}.$$

Also, since $d.\,a\,u = a\,du$, we may conclude, that $\int a\,X\,dx = a\int X\,dx$, and consequently that all such constant quantities may be brought from under the sign $\int$ of integration.

149. If there were given $dy = (ax+b)^m dx$, we might expand the given power, and integrate every term in the result; but it is proper to observe that we may obtain the integral without effecting this developement. It is sufficient to make $ax+b=z$, which gives $x = \frac{z-b}{a}$, and $dx = \frac{dz}{a}$; by substituting in the expression for dy, we find $dy = \frac{z^m dz}{a}$, and consequently $y = \frac{z^{m+1}}{a(m+1)} + B$. Putting for z its value, we find, that when

$$dy = (ax+b)^m dx, \quad y = \frac{(ax+b)^{m+1}}{a(m+1)} + B.$$

If we had $dy = (ax^n+b)^m x^{n-1} dx$, this transformation would still succeed; for by making $ax^n+b=z$, we have $nax^{n-1}dx = dz$, and therefore

$$x^{n-1}dx=\frac{dz}{na},\ dy=\frac{z^m dz}{na},\ \text{and } y=\frac{z^{m+1}}{na(m+1)}+B,$$

whence it appears that, when

$$dy=(ax^n+b)^m x^{n-1}dx,\quad y=\frac{(ax^n+b)^{m+1}}{na(m+1)}+B.$$

150. We will now proceed to fractional functions, and to begin with a very simple case, we will suppose that $dy=\frac{Ax^m dx}{(ax+b)^n}$; making $ax+b=z$, we find

$$x=\frac{z-b}{a},\quad dx=\frac{dz}{a},$$

and consequently

$$dy=\frac{A(z-b)^m dz}{a^{m+1}z^n};$$

expanding $(z-b)^m$, multiplying the result by dz, and then dividing by z^n, we shall have a series of terms involving simply powers of the variable and constant quantities, which may be integrated by the rule in No. 146.

Let us take, for example, the case of $m=3$, and $n=2$; then

$$dy=\frac{A(z-b)^3 dz}{a^4 z^2}=\frac{A}{a^4}[zdz-3bdz+3b^2z^{-1}dz-b^3z^{-2}dz]:$$

applying to each of these terms the general rule, we have

$$y=\frac{A}{a^4}\left[\frac{z^2}{2}-3bz+3b^2 1z+b^3z^{-1}\right]+B.$$

We will now substitute for z its value, and we shall find that when

$$dy=\frac{Ax^3dx}{(ax+b)^2}$$

$$y=\frac{A}{a^4}[\tfrac{1}{2}(ax+b)^2-3b(ax+b)+3b^2 1(ax+b)+b^3(ax+b)^{-1}]+$$

We might deduce, without difficulty, the general formula; and if we had

$$dy = \frac{A x^n dx + B x^p dx + C x^q d x \ldots\ldots\ldots}{(a x + b)^m},$$

we may write it thus,

$$dy = \frac{A x^n dx}{(ax+b)^m} + \frac{B x^p dx}{(ax+b)^m} + \frac{C x_q dx}{(ax+b)^m} + \&c.$$

where every term may be integrated in the same manner as the first.

151. All differentials, which are rational functions, may be comprehended under this general form,

$$\frac{(A x^m + B x^n + C x^p + \&c.) dx}{A' x^{m'} + B' x^{n'} + C' x^{p'} \ldots \ldots},$$

which, for the sake of brevity, we shall represent by $\frac{U dx}{V}$. Now, in the first place, we may remark that the greatest exponent of the powers of x in the numerator may be supposed to be less than that of its powers in the denominator; for if it were not so, by dividing U by V, and calling Q the quotient of the division, and R the remainder, we should have $\int \frac{U dx}{V} = \int Q dx + \int \frac{R dx}{V}$; but Q being a rational and integral function of x, the integral of $Q dx$ may be found by the immediate application of the rule in No. 146; and there will remain nothing to find but $\int \frac{R dx}{V}$, a quantity in which the function R is of lower dimensions with respect to x, than the function V. The most general form, therefore, which the function $\frac{U dx}{V}$ can assume, will be

$$\frac{(A x^{n-1} + B x^{n-2} + C x^{n-3} \ldots\ldots + T) dx}{x^n + A' x^{n-1} + B' x^{n-2} + C' x^{n-3} \ldots\ldots + T'}.$$

The general method of integrating differentials of the above form, consists in decomposing them into others,

whose denominators are more simple, which we designate by the name of *partial fractions*, and which we obtain in the following manner:

If we make the denominator of the proposed fraction equal to nothing, we shall form the equation

$$x^n + A'x^{n-1} + B'x^{n-2} + \&c. \ldots\ldots T' = 0,$$

and supposing that we have determined all the roots of this equation, we may represent them by

$$-a, \quad -a', \quad -a'', \quad -a''', \quad \&c.$$

supposing them all to be different from each other: by this means, the first side of the proposed equation will assume the form of a product of n factors

$$x+a, \quad x+a', \quad x+a'', \quad x+a''', \quad \&c.$$

This done, we shall consider the proposed fraction as the sum of the fractions

$$\frac{Ndx}{x+a}, \quad \frac{N'dx}{x+a'}, \quad \frac{N''dx}{x+a''}, \quad \&c.$$

whose denominators are the factors of the denominator of the proposed fraction, and whose numerators are undetermined constants.

We will suppose, as an illustration of the above, that the differential which it is proposed to integrate, is the following:

$$\frac{(Ax^2+Bx+C)\,dx}{x^3+A'x^2+B'x+C'},$$

and that we know that

$$x^3+A'x^2+B'x+C' = (x+a)(x+a')(x+a'').$$

Reducing to the same denominator the fractions

$$\frac{Ndx}{x+a}, \quad \frac{N'dx}{x+a'}, \quad \frac{N''dx}{x+a''},$$

and adding them together, we shall have

$$\frac{[N(x+a')(x+a'')+N'(x+a)(x+a'')+N''(x+a)(x+a')]\,dx}{(x+a)(x+a')(x+a'')}$$

The denominator is the same as that of the proposed fraction, and the numerator will necessarily be a function of a degree next inferior to that of the denominator, that is, of the second degree; for, by performing the multiplications, we have

$$[(N+N'+N'')x^2+\{N(a'+a'')+N'(a+a'')+N''(a+a')\}x$$
$$+Na'a''+N'aa''+N''aa']dx.$$

This function being compared with the numerator of the proposed fraction, gives these three equations:

$$N+N'+N''=A',$$
$$N(a'+a'')+N'(a+a'')+N''(a+a')=B,$$
$$Na'a''+N'aa''+N''aa'=C,$$

which are only of the first degree, with respect to the undetermined quantities N, N', N''; resolving these equations, we shall have

$$\frac{(Ax^2+Bx+C)dx}{x^3+A'x^2+B'x+C'}=\frac{Ndx}{x+a}+\frac{N'dx}{x+a'}+\frac{N''dx}{x+a''},$$

Making $x+a=z$, we see that the differential $\frac{Ndx}{x+a}$, will be transformed into $\frac{Ndz}{z}$ the integral of which is $N\,\mathrm{l}\,z$, or $N\,\mathrm{l}\,(x+a)$. We find, in the same way, that

$$\int\frac{N'dx}{x+a'}=N'\,\mathrm{l}\,(x+a'),\quad \int\frac{N''dx}{x+a''}=N''\,\mathrm{l}\,(x+a'');$$

and we shall therefore have

$$\int\frac{(Ax^2+Bx+C)dx}{x^3+A'x^2+B'x+C'}$$
$$=N\,\mathrm{l}\,(x+a)+N'\,\mathrm{l}\,(x+a')+N''\,\mathrm{l}\,(x+a'')+\text{const.}$$
$$=\mathrm{l}\,\{(x+a)^N(x+a')^{N'}(x+a'')^{N''}\}+\text{const.}$$

This process, which may be easily extended to the general formula cited at the beginning of this article, shows

that the integration of rational fractions, in the cases where the denominators are decomposable into factors which are real and unequal, has no difficulty independent of this decomposition, and consequently depends upon the numerical resolution of equations.

152. In what precedes we have supposed all the factors of the denominator of the proposed fraction, to be unequal; for if this be not the case, the decomposition of this fraction can no longer be effected under the above-mentioned form. We see this immediately in $\frac{Ax+B}{(x+a)^2}$, which we cannot represent by $\frac{N}{x+a}+\frac{N'}{x+a}$, since these two fractions, when combined, form the simple one, $\frac{N+N'}{x+a}$.

If the denominator $x^n+A'x^{n-1}+\&c. \ldots\ldots +T$, of the proposed fraction, contain a factor $(x+a)^p$, it will be necessary to assume for this factor, a partial fraction of the form

$$\frac{(Px^{p-1}+Qx^{p-2}+Rx^{p-3}+\&c.\ldots\ldots Y)\,dx}{(x+a)^p};$$

we should determine the coefficients of its numerator, by reducing this and the other partial fractions to the same denominator, and then comparing the sum of the numerators with that of the proposed fraction (151.).

We might then integrate, by the rule in No. 150; but it is easily seen, that we may substitute for the fraction

$$\frac{(Px^{p-1}+Qx^{p-2}\ldots\ldots\ldots+Y)\,dx}{(x+a)^p}$$

the expression

$$\frac{N\,dx}{(x+a)^p}+\frac{N'\,dx}{(x+a)^{p-1}}+\frac{N''\,dx}{(x+a)^{p-2}}+\frac{N'''\cdots dx}{x+a};$$

for, by reducing all the terms of this expression to the same denominator, the numerator which we obtain will be of the

same form with that of the first fraction. This done, let $x+a=z$, and we have

$$\int\frac{N\,dx}{(x+a)^p}=\int\frac{N\,dz}{z^p}=\frac{Nz^{-p+1}}{1-p}=\frac{N}{(1-p)\,(x+a)^{p-1}};$$

we shall find, in the same way,

$$\int\frac{N'\,dx}{(x+a)^{p-1}}=\frac{N'}{(2-p)\,(x+a)^{p-2}},$$

and so on for the others: all these integrals will be algebraical, except the last $\int\frac{N'''\cdots dx}{x+a}$, which will involve a logarithm.

153. If the values of a, a', a'', &c. were imaginary, they would introduce expressions of this nature into the numerators of the partial fractions. We might, indeed, make these impossible parts disappear; but this would make the calculation very complicated, and we may avoid this difficulty by decomposing the denominator of the proposed fraction into real factors, either of the first or second degree, which is in all cases possible (*Compl. d'Alg.* 27). The factors of the second degree, which contain the impossible roots, may be represented by

$$x^2+2\,\alpha\,x+\alpha^2+\beta^2;$$

and if there are several of these factors which are equal to each other, the denominator of the proposed fraction will have factors of the form

$$(x^2+2\,\alpha\,x+\alpha^2+\beta^2)^q.$$

To the simple factor $x^2+2\,\alpha\,x+\alpha^2+\beta^2$, there will correspond the partial fraction

$$\frac{(Kx+L)\,dx}{x^2+2\,\alpha\,x+\alpha^2+\beta^2},$$

and to factors of the second kind, the fraction

$$\frac{(Q'\,x^{2q-1}+R'\,x^{2q-2}\ldots+Y')\,dx}{(x^2+2\,\alpha\,x+\alpha^2+\beta^2)^q};$$

but to facilitate the integration, and to preserve the analogy with the formulas in the preceding No. we may substitute for this last the following expression:

$$\frac{(Kx+L)\,dx}{(x^2+2\alpha x+\alpha^2+\beta^2)^q}+\frac{(K'x+L')\,dx}{(x^2+2\alpha x+\alpha^2+\beta^2)^{q-1}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{(K''\cdots x+L''\cdots)\,dx}{x^2+2\alpha x+\alpha^2+\beta^2}.$$

The coefficients of the numerators may be determined in the manner we have indicated in Nos. 151, 152.

To integrate the fraction

$$\frac{(Kx+L)\,dx}{x^2+2\alpha x+\alpha^2+\beta^2},$$

we observe that

$$x^2+2\alpha x+\alpha^2+\beta^2=(x+\alpha)^2+\beta^2;$$

and if we make

$$x+\alpha=z,$$

there will arise

$$\frac{(Kx+L)\,dx}{(x+\alpha)^2+\beta^2}=\frac{(Kz+L-K\alpha)\,dz}{z^2+\beta^2}=\frac{(Kz+L_1)\,dz}{z^2+\beta^2},$$

by putting

$$L-K\alpha=L_1.$$

But

$$\frac{(Kz+L_1)\,dz}{z^2+\beta^2}=\frac{Kz\,dz}{z^2+\beta^2}+\frac{L_1\,dz}{z^2+\beta^2}.$$

The first part of the second member of this equation is an algebraical integral; for by making $z^2+\beta^2=u$, we have $z\,dz=\frac{du}{2}$, which gives

$$\int\frac{Kz\,dz}{z^2+\beta^2}=\frac{K}{2}\int\frac{du}{u}=K.\frac{1}{2}\,\mathrm{l}\,u=K\,\mathrm{l}\sqrt{z^2+\beta^2}.$$

With respect to the second part, if we make $z=\beta u$, we shall have

$$\frac{L_1 dz}{z^2+\beta^2} = \frac{L_1}{\beta} \frac{du}{u^2+1};$$

but we have seen in No. 33, that $\frac{du}{1+u^2}$ is the differential of the arc whose tan. $=u$: therefore

$$\int \frac{L_1}{\beta} \frac{du}{u^2+1} = \frac{L_1}{\beta} \text{arc}(\tan. = u) + \text{const.}$$

$$= \frac{L_1}{\beta} \text{arc} \left(\tan. = \frac{z}{\beta}\right) + \text{const.}$$

Adding together these two results, we shall get

$$\int \frac{(Kz+L_1)dz}{z^2+\beta^2} = K \, 1 \sqrt{z^2+\beta^2} + \frac{L_1}{\beta} \text{arc} \left(\tan. = \frac{z}{\beta}\right)$$
$$+ \text{const.}$$

It is proper to remark, that the sine of the arc whose tangent is $\frac{z}{\beta}$, is $\frac{z}{\sqrt{z^2+\beta^2}}$, and its cosine $\frac{\beta}{\sqrt{z^2+\beta^2}}$; for this consideration affords the means of presenting the proposed integral under different forms, by designating the arc by its sine or its cosine.

When we replace the value of z, we have

$$\int \frac{(Kx+L)dx}{x^2+2\alpha x+\alpha^2+\beta^2} = \text{const.}$$

$$+K \, 1 \sqrt{x^2+2\alpha x+\alpha^2+\beta^2} + \frac{L-K\alpha}{\beta} \text{arc} \left(\tan. = \frac{x+\alpha}{\beta}\right).$$

To integrate the differential

$$\frac{(Kx+L)dx}{(x^2+2\alpha x+\alpha^2+\beta^2)^q},$$

we shall at once make $x+\alpha=z$, and $L-K\alpha=L_1$; by this means we shall only have to find $\int \frac{(Kz+L_1)dz}{(z^2+\beta^2)^q}$, which may also be written thus:

$$K\int\frac{z\,dz}{(z^2+\beta^2)^q}+L_1\int\frac{dz}{(z^2+\beta^2)^q}.$$

The first part is integrable immediately; for it is obvious that by making $z^2+\beta^2=u$, we have $z\,dz=\frac{du}{2}$, and consequently

$$K\int\frac{z\,dz}{(z^2+\beta^2)^q}=\frac{K}{2}\int\frac{du}{u^q}=\frac{K\,u^{-q+1}}{2\,(1-q)};$$

and the integration of the second part we may make dependent on that of the formula $\frac{dz}{(z^2+\beta^2)^{q-1}}$, in which the index of the denominator is less by unity than in the first.

154. In short, if we assume the equation

$$\int\frac{dz}{(z^2+\beta^2)^q}=\frac{G\,z}{(z^2+\beta^2)^{q-1}}+H\int\frac{dz}{(z^2+\beta^2)^{q-1}};$$

where G and H are indeterminate quantities, and if we take the differentials of each member, and reduce all the terms of the result to the same denominator, we shall be able to suppress this denominator, as well as the common factor dz; there will result

$$1=G\,(z^2+\beta^2)-2\,(q-1)\,G\,z^2+H\,(z^2+\beta^2);$$

and then comparing similar terms, we shall form two equations

$$1=G\beta^2+H\beta^2,\quad G-2\,(q-1)\,G+H=0,$$

which will give

$$G=\frac{1}{(2q-2)\,\beta^2},\quad H=\frac{(2q-3)}{(2q-2)\,\beta^2};$$

we consequently shall have

$$\int\frac{dz}{(z^2+\beta^2)^q}=\left\{\frac{1}{(2q-2)\beta^2}\cdot\frac{z}{(z^2+\beta^2)^{q-1}}\right.$$

$$\left.+\frac{(2q-3)}{(2q-2)\,\beta^2}\int\frac{dz}{(z^2+\beta^2)^{q-1}}\right\}\ldots\ldots\ldots\ldots(a.)$$

This formula furnishes the means of depressing to unity the index of the denominator of the proposed fraction; for if we put $q-1$ in the place of q, we shall find

$$\int \frac{dz}{(z^2+\beta^2)^{q-1}} = \frac{1}{(2q-4)\beta^2} \frac{z}{(z^2+\beta^2)^{q-2}}$$
$$+\frac{(2q-5)}{(2q-4)\beta^2} \int \frac{dz}{(z^2+\beta^2)^{q-2}};$$

substituting this value in the same equation (a), there will result

$$\int \frac{dz}{(z^2+\beta^2)^q} = \left\{ \frac{1}{(2q-2)\beta^2} \cdot \frac{z}{(z^2+\beta^2)^{q-1}} \right.$$
$$+\frac{1.(2q-3)}{(2q-2)(2q-4)\beta^4} \cdot \frac{z}{(z^2+\beta^2)^{q-2}}$$
$$\left. +\frac{(2q-3)(2q-5)}{(2q-2)(2q-4)\beta^4} \int \frac{dz}{(z^2+\beta^2)^{q-2}} \right\} \ldots (b.)$$

We may obtain, in the same manner, the value of $\int \frac{dz}{(z^2+\beta^2)^{q-2}}$, by changing q into $q-2$; if we then substitute this value in the equation (b), we shall have

$$\int \frac{dz}{(z^2+\beta^2)^q} = \left\{ \frac{1}{(2q-2)\beta^2} \cdot \frac{z}{(z^2+\beta^2)^{q-1}} \right.$$
$$+\frac{1.(2q-3)}{(2q-2)\ (2q-4)\beta^4} \cdot \frac{z}{(z^2+\beta^2)^{q-2}}$$
$$+\frac{1.(2q-3)(2q-5)}{(2q-2)(2q-4)(2q-6)\beta^6} \cdot \frac{z}{(z^2+\beta^2)^{q-3}}$$
$$\left. +\frac{(2q-3)(2q-5)(2q-7)}{(2q-2)(2q-4)(2q-6)\beta^6} \int \frac{dz}{(z^2+\beta^2)^{q-3}} \right\} \ldots (c.)$$

If we were again to deduce from equation (a) the value of $\int \frac{dz}{(z^2+\beta^2)^{q-3}}$, we should obtain a new value of

$\int \frac{dz}{(z^2+\beta^2)^q}$, which would depend upon $\int \frac{dz}{(z^2+\beta^2)^{q-1}}$. By continuing to operate in this manner, we should form a series, which must terminate when we arrive at the term $\int \frac{dz}{z^2+\beta^2}$; for the term following, which would involve $\int \frac{dz}{(z^2+\beta^2)^0}$, would have an infinite coefficient. This may be seen by making $q=2$ and $q=3$, in the equations (*b*) and (*c*); and it is easy to discover the reason of this circumstance; for if we could arrive at the term of which we have just spoken, we should then have a complete algebraical integral of the proposed fraction, since

$$\int \frac{dz}{(z^2+\beta^2)^0} = \int dz.$$

We here see the origin of a method of integration, as fertile as it is elegant: it is that by which we pass from one integral to another. We shall explain it hereafter, in a manner more general.

In comparing the results of the preceding Nos. we shall undoubtedly have remarked, that differentials which present themselves under the form of rational fractions, may be always integrated, either algebraically, or by means of logarithms or circular arcs; and that no other preparation is necessary, than to decompose them into fractions whose denominators are either binomial or trinomial quantities. We have hitherto only indicated for this purpose, the method of indeterminate coefficients, it being that which presents itself the first; but there are many other which require less complicated calculations.

155. Let us resume the fraction $\frac{U}{V}$. Let $x+a$ be one of the unequal factors of the denominator V, so that we have $V=(x+a)Q$, where Q does not involve $x+a$ as a

factor: if we make $\frac{U}{V}=\frac{A}{x+a}+\frac{P}{Q}$, P being an indeterminate function of x, but into which this quantity does not enter as a divisor, we shall have $U=AQ+P(x+a)$, and consequently $P=\frac{U-AQ}{x+a}$. Since P ought to be an integral function of x, the quantity $U-AQ$, which is also rational and integral, must be divisible by $x+a$; or, which is the same thing, must vanish when we substitute for x the value $-a$, which makes $x+a$ equal to nothing; designating then by u and by q, the values of U and Q after this substitution, which does not affect the value of the indeterminate quantity A, that being independent of x, there will result $u-Aq=0$, and consequently $A=\frac{u}{q}$.

The factor Q is found by dividing V by $x+a$; but its value q, relative to the hypothesis of $x+a=0$, is obtained immediately by differentiating the equation $V=(x+a)Q$, from whence there arises

$$\frac{dV}{dx}=Q+(x+a)\frac{dQ}{dx};$$

now if in this equation we make $x+a=0$, and represent by v the value of $\frac{dV}{dx}$ in this case, we shall have $v=q$, and therefore $A=\frac{u}{v}$.

The quantity $A=\frac{u}{q}$ will always have a finite value; for the numerator and denominator cannot become equal to nothing, since the proposed fraction is reduced to its most simple terms, and therefore the function U cannot contain the factor $x+a$, which makes part of the denominator, and which only appearing in it once, does not enter

into Q. By properly applying this reasoning to all the different cases which may present themselves, we shall readily discover that the decomposition of any rational fraction, under the forms indicated above, is always possible.

156. Let us now see how we may find the numerators of the partial fractions, which correspond to equal factors of the first degree. In this case we have $V=Q(x+a)^n$, and we must suppose

$$\frac{U}{V}=\frac{A_0}{(x+a)^n}+\frac{A_1}{(x+a)^{n-1}}+\frac{A_2}{(x+a)^{n-2}}\dots+\frac{A_{n-1}}{x+a}+\frac{P}{Q};$$

by reducing the fractions to the same denominator, we shall get

$$U=Q\,[A_0+A_1(x+a)+A_2(x+a)^2\dots+A_{n-1}(x+a)^{n-1}]+P(x+a)^n$$

$$P=\frac{U-Q\,[A_0+A_1(x+a)+A_2(x+a)^2\dots+A_{n-1}(x+a)^{n-1}]}{(x+a)^n};$$

and since P must be an integral function, it is necessary that the numerator of the expression for it should be divisible n times successively by $x+a$; this numerator will vanish therefore, when we put $-a$ in the place of x. We see immediately, that it reduces itself in this case to $U-QA_0$; but that $U-QA_0$ may be divisible by $x+a$, it is necessary that, preserving the same notation as in the preceding Nos. we should have $u-qA_0=0$, or $A_0=\frac{u}{q}$.

This value will change the quantity $U-QA_0$ into $U-\frac{u}{q}Q$, which is divisible by $x+a$; and we shall have, by effacing at the same time this factor in the denominator, and making for greater brevity, $U-\frac{u}{q}Q=U_1(x+a)$,

$$P=\frac{U_1-Q\,[A_1+A_2(x+a)\dots\dots+A_{n-1}(x+a)^{n-2}]}{(x+a)^{n-1}}.$$

Then in order to obtain A_1, we must make $x+a=0$, and representing by u_1, the value of U_1, arising from the substitution of $-a$ in the place of x, we shall have $u_1-qA_1=0$, or $A_1=\frac{u_1}{q}$.

Substituting for A_1 its value in U_1-QA_1 there will arise the quantity $U_1-\frac{u_1}{q}Q$, which vanishing when $x+a=0$, will be divisible by $x+a$, and consequently P will reduce itself to

$$P=\frac{U_2-Q\,[A_2+A_3\,(x+a)\ldots.+A_{n-1}\,(x+a)^{n-3}]}{(x+a)^{n-2}},$$

U_2 representing the quotient of the division of $U_1-\frac{u_1}{q}Q$, by $x+a$. By continuing the same process, and using the same notation, we shall again find $u_2-q\,A_2=0$, and therefore $A_2=\frac{u_2}{q}$, and so on for the others.

The Differential Calculus very much facilitates the preceding operations. In fact, if we differentiate $n-1$ times in succession, the equation

$$U=Q\,[A_0+A_1\,(x+a)+A_2\,(x+a)^2\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+A_{n-1}\,(x+a)^{n-1}]+P\,(x+a)^n,$$

and then make $x+a=0$, both in this equation, and in those which we thus deduce from it, there will arise

$$U=A_0Q$$
$$dU=A_0\,d\,Q+A_1\,Q\,d\,x$$
$$d^2U=A_0\,d^2\,Q+2\,A_1\,d\,Q\,d\,x+2\,A_2\,Q\,d\,x^2$$
$$d^3U=A_0\,d^3\,Q+3\,A_1\,d^2\,Q\,d\,x+6\,A_2\,d\,Q\,d\,x^2+6\,A_3\,Q\,d\,x^3,$$
&c.

equations which determine each of the unknown quantities, A_0, A_1, A_2, &c. by means of those which precede it; it being well understood, that we substitute after each differentiation, $-a$ in the place of x.

The most simple method of obtaining the value of Q, in this case, is to divide V by $(x+a)^n$; nevertheless we may arrive at the same result by differentiation, as in the preceding No.; for since we have $V=Q\,(x+a)^n$, by differentiating n times successively both the members of this equation, and then making $x+a=0$, we shall find $d^n V=1\,.\,2\ldots n\,Q\,dx^n$ (52), and consequently $Q=\dfrac{d^n V}{1\,.\,2\ldots.n\,dx^n}$.

We shall arrive at the expression for the differentials of Q, upon the hypothesis that $x+a=0$, by taking successively the differentials of the orders $n+1$, $n+2$, &c. of the equation $V=Q\,(x+a)^n$; for it is readily seen, according to the remark in No. 52, that in this case $d^{n+1}V=d^{n+1}\,.\,Q\,(x+a)^n$, for example, becomes $d^{n+1}V=1\,.\,2\,.\,3\ldots.(n+1)\,d\,Q\,dx^n$. It follows from this, that we may express the indeterminate quantities A_1, A_2, A_3, &c. by the assistance of the differentials of the numerator U, and of those of the denominator V, of the proposed fraction.

157. The process in No. 155. a little modified, serves also to find the numerator of a partial fraction of the form

$$\frac{A\,x+B}{x^2+2\,\alpha\,x+\alpha^2+\beta^2}.$$

Assuming

$$\frac{U}{V}=\frac{A\,x+B}{x^2+2\,\alpha\,x+\alpha^2+\beta^2}+\frac{P}{Q};$$

and reducing the second member of the equation to the same denominator as the first, we find

$$U=Q\,(A\,x+B)+P\,(x^2+2\,\alpha\,x+\alpha^2+\beta^2),$$

from which we get

$$P=\frac{U-Q\,(A\,x+B)}{x^2+2\alpha x+\alpha^2+\beta^2}.$$

As P ought always to be an integral function with respect

to x, the quantity $U-Q(Ax+B)$, must be divisible by $x^2+2\alpha x+\alpha^2+\beta^2$; it must, therefore, contain among its factors, those of this last quantity, and must vanish under the same circumstances. But the factors of $x^2+2\alpha x+\alpha^2+\beta^2$, are $x+\alpha+\beta\sqrt{-1}$, $x+\alpha-\beta\sqrt{-1}$; and if these be made each $=0$, we shall have $x=-(\alpha+\beta\sqrt{-1})$, $x=-(\alpha-\beta\sqrt{-1})$: these values being substituted in $U-Q(Ax+B)$, ought to cause this quantity to vanish. If then we denote by $u\pm u'\sqrt{-1}$, and by $q\pm q'\sqrt{-1}$, what U and Q respectively become after this substitution, we shall have

$$u\pm u'\sqrt{-1}-(q\pm q'\sqrt{-1})[-A(\alpha\pm\beta\sqrt{-1})+B]=0.$$

This equation is two-fold, in consequence of the sign $\pm$, by which several of its terms are affected; and it is equivalent to those that would be formed by putting the real and imaginary part separately equal to nothing: from this consideration, we shall have

$$u+q\alpha A-q'\beta A-qB=0,$$
$$u'+q\beta A+q'\alpha A-q'B=0,$$

equations which will give us the values of A and B.

We may find q and q' very nearly in the same way as we found q, in No. 155. In fact, if we differentiate each side of the equation

$$Q(x^2+2\alpha x+\alpha^2+\beta^2)=V,$$

and afterwards make

$$x^2+2\alpha x+\alpha^2+\beta^2=0,$$

there will result

$$Q(2x\,dx+2\alpha\,dx)=dV, \text{ or } Q=\frac{dV}{2x\,dx+2\alpha\,dx};$$

substituting in the place of x, its two values $-(\alpha\pm\beta\sqrt{-1})$, representing by $v\pm v'\sqrt{-1}$, what the expression $\frac{dV}{dx}$ be-

comes by this substitution, and writing $q \pm q'\sqrt{-1}$, in the place of Q, we shall get

$$q \pm q'\sqrt{-1} = \frac{v \pm v'\sqrt{-1}}{\mp 2\beta\sqrt{-1}};$$

multiplying both the terms of the fraction which forms the second member of the equation by $\sqrt{-1}$, and then equating the real and imaginary terms on each side of the equation, we shall have

$$q = -\frac{v'}{2\beta}, \qquad q' = \frac{v}{2\beta}.$$

158. If the factor $x^2 + 2\alpha x + \alpha^2 + \beta^2$, which, for brevity's sake, we will represent by R, is found several times in the denominator V, so that

$$V = Q(x^2 + 2\alpha x + \alpha^2 + \beta^2)^n = Q R^n,$$

we must assume, in this case (153.),

$$\frac{U}{V} = \frac{A_0 x + B_0}{R^n} + \frac{A_1 x + B_1}{R^{n-1}} + \frac{A_2 x + B_2}{R^{n-2}} \dots + \frac{P}{Q};$$

reducing this expression to a common denominator, and thence deducing the value of P, we obtain

$$P = \frac{U - Q\,[A_0 x + B_0 + (A_1 x + B_1)R + (A_2 x + B_2)R^2 + \dots]}{R^n}$$

By reasoning in this as in the preceding cases, it may be concluded, that the numerator of this expression ought to vanish by the substitution of $-(\alpha \pm \beta\sqrt{-1})$, which also renders $x = 0$; and using the same notation as before, we shall have, from this substitution,

$$u \pm u'\sqrt{-1} - (q \pm q'\sqrt{-1})\,[-A_0(\alpha \pm \beta\sqrt{-1}) + B_0] = 0,$$

which will furnish for the determination of A and B, the same equations as in the preceding No. Having found the values of these quantities, we substitute them in the numerator of P; and the terms $U - Q(A_0 x + B_0)$,

being divisible by R, or $x^2+2\alpha x+\alpha^2+\beta^2$, the whole expression will be so likewise. Denoting by U_1, the quotient of the division of $U-Q(A_0x+B_0)$ by $x^2+2\alpha x+\alpha^2+\beta^2$, we have

$$P=\frac{U_1-Q[A_1x+B_1+(A_2x+B_2)R+\ldots\ldots]}{R^{n-1}}.$$

If, in this new numerator, we put for x, its values derived from the equation $R=0$, and make the result $=0$, we shall determine A_1 and B_1, in the same way as we have before determined A and B; and we must continue to operate in the same way, to determine the values of A_2, B_2, A_3, B_3, &c.

This case is quite analogous to that which has been considered in No. 156; and the Differential Calculus is equally applicable to one as to the other, by means of the equation

$$U=Q[A_0x+B_0+(A_1x+B_1)R+(A_2x+B_2)R^2+\ldots]$$
$$+PR^n,$$

and its differentials, in which, as far as the $n-1$th order inclusively, the term PR^n disappears, by making $R=0$. We shall obtain in this manner the equations

$$U=(A_0x+B_0)Q$$
$$dU=(A_0x+B_0)dQ+A_0Q\,dx+(A_1x+B_1)Q\,dR,\ \text{\&c.}$$

each of which becomes two-fold, when we substitute for x the two values of which it is susceptible in virtue of the equation $R=0$, or $x^2+2\alpha x+\alpha^2+\beta^2=0$. By making the real and imaginary part separately equal to nothing, we shall obtain a sufficient number of equations to determine A_0, B_0, A_1, B_1, &c.

It may yet be remarked, that from the equation

$$V=Q(x^2+2\alpha x+\alpha^2+\beta^2)^n,$$

we find

$$Q=\frac{d^n V}{d^n.(x^2+2\alpha x+\alpha^2+\beta^2)^n},$$

when we suppose

$$x^2 + 2\alpha x + \alpha^2 + \beta^2 = 0.$$

We shall find $d\,Q$, $d^2\,Q$, &c. upon the same hypothesis, by means of $n+1$, $n+2$, &c. differentiations of the equation

$$V = Q\,(x^2 + 2\,\alpha\,x + \alpha^2 + \beta^2)^n,$$

and by suppressing all those terms which this hypothesis makes equal to nothing.

159. We shall now proceed to give some applications of the foregoing theory. Let there be the fraction $\frac{d\,x}{x^8 + x^7 - x^4 - x^3}$: the factors of its denominator are easily discovered; for it may be put under this form

$$x^3\,(x^5 + x^4 - x - 1) = x^3\,(x+1)\,(x^4 - 1).$$

The factor $x^4 - 1$ may be resolved into $x^2 - 1$ and $x^2 + 1$, or $x - 1$, $x + 1$, and $x^2 + 1$: we have therefore

$$x^8 + x^7 - x^4 - x^3 = x^3\,(x-1)\,(x+1)^2\,(x^2+1);$$

and consequently the proposed fraction may be decomposed as follows (151, 152, 153):

$$\frac{A\,d\,x}{x-1} + \frac{B\,d\,x}{(x+1)^2} + \frac{C\,d\,x}{x+1}$$

$$+ \frac{D\,d\,x}{x^3} + \frac{E\,d\,x}{x^2} + \frac{F\,d\,x}{x} + \frac{(G\,x + H)\,d\,x}{1+x^2}.$$

By reducing these fractions to the same denominator, and comparing the result with $\frac{d\,x}{x^8 + x^7 - x^4 - x^3}$, we might determine the unknown numerators; but we shall rather make use of the methods already explained.

For this purpose we shall consider separately the four factors,

$$x - 1, \quad (x+1)^2, \quad x^3, \quad \text{and } x^2 + 1,$$

which compose the denominator of the proposed fraction; to

the first corresponds a fraction of the form $\frac{A}{x-1}$, whose denominator being made equal to zero, gives $x=1$; the quantities $U=1$, and $\frac{dV}{dx}=\frac{d,(x^8+x^7-x^4-x^3)}{dx}$, become 1 and 8; we have therefore (155) $A=\frac{1}{8}$, and the first partial fraction is $\frac{1}{8}\cdot\frac{1}{x-1}$.

To the factor $(x+1)^2$ there correspond two partial fractions of the form $\frac{A_0}{(x+1)^2}+\frac{A_1}{x+1}$ (156). Having, in the first place found that

$$Q=\frac{x^8+x^7-x^4-x^3}{(x+1)^2}=x^6-x^5+x^4-x^3,$$

we make $x+1=0$, from whence $x=-1$, $q=4$, and $\frac{u}{q}=\frac{1}{4}$; so that the second partial fraction is $\frac{1}{4}\frac{1}{(x+1)^2}$.

In the expression for U_1, (156), putting in the place of A_0, its value $\frac{1}{4}$, we have

$$U_1=\frac{U-A_0Q}{x+1}=\frac{4-x^6+x^5-x^4+x^3}{4(x+1)}$$

$$=\frac{-x^5+2x^4-3x^3+4x^2-4x+4}{4},$$

from which there arises $\frac{u_1}{q}=\frac{18}{16}=\frac{9}{8}$; we have therefore, for the third partial fraction, $\frac{9}{8}\frac{1}{x+1}$.

To apply in this case the Differential Calculus, we must form the equation (156)

$$1=Q[A_0+A_1(x+1)]+P(x+1)^2,$$

which it will be sufficient to differentiate once; and then making $x=-1$, we shall have

$$1=A_0 Q$$

$$0=A_0\, d\, Q+A_1\, Q\, d\, x.$$

Q being $x^6-x^5+x^4-x^3$, the first of these equations gives $A_0=\frac{1}{4}$, and the second $A_1=\frac{9}{8}$.

The factor x^3 furnishes the three partial fractions

$$\frac{A_0}{x^3}+\frac{A_1}{x^2}+\frac{A_2}{x},$$

which we determine by means of the equation

$$1=Q\,[A_0+A_1\, x+A_2\, x^2]+P\, x^3,$$

and its first and second differentials. By observing that $Q=x^5+x^4-x-1$, and making $x=0$, in Q, $d\,Q$ and d^2Q, we find

$$A_0=-1,\quad A_1=1,\quad A_2=-1:$$

we have, therefore, $-\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}$.

There yet remains the partial fraction corresponding to the factor x^2+1, and whose form is $\frac{A\,x+B}{x^2+1}$. We might determine it, by subtracting all the preceding partial fractions from the one proposed: but we proceed to find it directly, from the formulæ in No. 157. In the first place we have $Q=x^6+x^5-x^4-x^3$; next, the factor x^2+1, being made equal to zero, gives $x=\pm\sqrt{-1}$, $\alpha=0$, $\beta=1$, from which we find $q\pm q'\sqrt{-1}=-2\pm 2\sqrt{-1}$, $u=1$, and $u_1=0$: the equations which determine A and B, become

$$1+2A+2B=0,\qquad 2A-2B=0;$$

and we consequently find $A=B=-\frac{1}{4}$.

We have thus decomposed the proposed fraction

$\frac{dx}{x^8+x^7-x^4-x^3}$, into the following:

$$\frac{1}{8}\,\frac{dx}{x-1}+\frac{1}{4}\,\frac{dx}{(x+1)^2}+\frac{9}{8}\,\frac{dx}{x+1}$$

$$-\frac{dx}{x^3}+\frac{dx}{x^2}-\frac{dx}{x}-\frac{1}{4}\frac{(x+1)\,dx}{x^2+1}.$$

The integration of each of these fractions is effected without difficulty, and we shall have for the result

$$\left.\begin{array}{l}\frac{1}{8}\,1\,(x-1)-\frac{1}{4}\,\frac{1}{x+1}\\ +\frac{9}{8}\,1\,(x+1)+\frac{1}{2\,x^2}-\frac{1}{x}-1\,x\\ -\frac{1}{8}\,1\,(x^2+1)-\frac{1}{4}\,\text{arc}\,(\tan=x)+\text{const.}\end{array}\right\}$$

The aggregation of all the algebraical terms will produce the fraction $\frac{2-2\,x-5\,x^2}{4\,x^2\,(1+x)}$, and that of the logarithmic terms, will give

$$\tfrac{1}{8}\,1\,(x-1)+\tfrac{1}{8}\,1\,(x+1)+1\,(x+1)-\tfrac{1}{8}\,1\,(x^2+1)-1\,x$$

$$=\tfrac{1}{8}\,1\left(\frac{x^2-1}{x^2+1}\right)+1\left(\frac{x+1}{x}\right);$$

we have, therefore, upon the whole,

$$\int\frac{dx}{x^8+x^7-x^4-x^3}=\frac{2-2\,x-5\,x^2}{4\,x^2\,(1+x)}+\frac{1}{8}\,1\left(\frac{x^2-1}{x^2+1}\right)$$

$$+1\left(\frac{x+1}{x}\right)-\tfrac{1}{4}\,\text{arc}\,(\tan=x)+\text{const.}$$

On the Integration of Irrational Functions.

160. Irrational functions ought to be considered as integrated, in all cases where by any transformation, they are made rational, or at least reduced to any number of irrational quantities, consisting of one term only; for these we can apply directly the preceding rules.

Let us take, for example,

$\dfrac{(1+\sqrt{x}-\sqrt[3]{x^2})\,dx}{1+\sqrt[3]{x}}$; it is evident, that by making $x=z^6$, all the extractions indicated by the radical signs, may be effected, and we shall thus have $\dfrac{6\,z^5 d\,z\,(1+z^3-z^4)}{1+z^2}$; dividing by $1+z^2$, there results

$$-6\left[z^7\,d\,z-z^6\,d\,z-z^5\,d\,z+z^4\,d\,z-z^2\,d\,z+d\,z-\frac{d\,z}{1+z^2}\right]$$

whose integral is

$$-6\left[\frac{z^8}{8}-\frac{z^7}{7}-\frac{z^6}{6}+\frac{z^5}{5}-\frac{z^3}{3}+z-\text{arc}\,(\tan=z)\right]+\text{const.}$$

and replacing z by its value $\sqrt[6]{x}$, we shall have

$$-\tfrac{6}{8}x\sqrt[6]{x^2}+\tfrac{6}{7}\,x\sqrt[6]{x}+x-\tfrac{6}{5}\sqrt[6]{x^5}+2\sqrt{x}-6\sqrt[6]{x}$$
$$+6\ \text{arc}\,(\tan=\sqrt[6]{x})+\text{const.}$$

161. We shall first consider those irrational functions which include the radical $\sqrt{A+B\,x+C\,x^2}$ only, and which can only appear under one or other of the forms

$X\,d\,x\sqrt{A+B\,x+C\,x^2}$ and $\dfrac{X\,d\,x}{\sqrt{A+B\,x+C\,x^2}}$, X being a rational function of x. It may be observed, that one of these forms is included in the other; for we may write the first as follows:

$$X\,d\,x\,\frac{\sqrt{A+B\,x+C\,x^2}\times\sqrt{A+B\,x+C\,x^2}}{\sqrt{A+B\,x+C\,x^2}}$$

$$=\frac{X\,(A+B\,x+C\,x^2)\,d\,x}{\sqrt{A+B\,x+C\,x^2}},$$

and the numerator of the resulting fraction then becomes a rational function.

Before we proceed to explain the method of making the expression $\sqrt{A+B\,x+C\,x^2}$ rational, with respect to x, we will put the quantity $A+B\,x+C\,x^2$, under this form,

$$C\left(\frac{A}{C}+\frac{B}{C}\,x+x^2\right);$$

and then, for the sake of brevity, making

$$C=\gamma^2, \qquad \frac{A}{C}=\alpha, \qquad \frac{B}{C}=\beta,$$

we shall have $\sqrt{A+B\,x+C\,x^2}=\gamma\sqrt{\alpha+\beta\,x+x^2}$.

This done, if we assume $\sqrt{\alpha+\beta\,x+x^2}=x+z$, and square both sides of the equation, there will result $\alpha+\beta\,x=2\,x\,z+z^2$, which will give $x=\frac{\alpha-z^2}{2\,z-\beta}$, from whence

$$\sqrt{A+B\,x+C\,x^2}=\gamma\,(x+z)=\gamma\left(\frac{\alpha-\beta\,z+z^2}{2\,z-\beta}\right).$$

$$d\,x=-\frac{2\,(\alpha-\beta\,z+z^2)\,d\,z}{(2\,z-\beta)^2}.$$

By means of these values, we shall transform the differential $\frac{X\,d\,x}{\sqrt{A+B\,x+C\,x^2}}$ into another of the form $Z\,d\,z$, Z being a rational function of z, and also real, if C be a positive quantity; for if C was negative, γ would become imaginary, and the transformed expression would become so likewise.

In this case, we have to consider $\sqrt{A+Bx-Cx^2}$; and making

$$C=\gamma^2, \quad \frac{A}{C}=\alpha, \quad \frac{B}{C}=\beta,$$

it becomes $\sqrt{\alpha+\beta x-x^2}$. The quantity $x^2-\beta x-\alpha$, may always be decomposed into real factors of the first degree; if we represent them by $x-a$ and $x-a'$, it is evident, that

$$\alpha+\beta x-x^2=-(x^2-\beta x-\alpha)=(x-a)(a'-x).$$

Then making $\sqrt{(x-a)(a'-x)}=(x-a)z$, and squaring both sides of the equation, it becomes divisible by $x-a$, and we have $a'-x=(x-a)z^2$, from which we find

$$x=\frac{az^2+a'}{z^2+1}, \quad (x-a)z=\frac{(a'-a)z}{z^2+1}, \quad dx=\frac{2(a-a')z\,dz}{(z^2+1)},$$

values which will also render rational the proposed differential.

162. We will take for our first example the differential $\frac{dx}{\sqrt{A+Bx+Cx^2}}$; the first of the preceding transformations gives $\frac{-2\,dz}{\gamma(2z-\beta)}$, whose integral is $-\frac{1}{\gamma}\,1(2z-\beta)$ + const. Substituting now for z its value $-x+\sqrt{\alpha+\beta x+x^2}$, and for α, β, and γ, the quantities which they severally represent, it will become

$$-\frac{1}{\sqrt{C}}\,1\left[\frac{2}{\sqrt{C}}\left(-\frac{B}{2\sqrt{C}}-x\sqrt{C}+\sqrt{A+Bx+Cx^2}\right)\right] + \text{const.}$$

a result to which we may give the form

$$-\frac{1}{\sqrt{C}}1\left[-\frac{B}{2\sqrt{C}}-x\sqrt{C}+\sqrt{A+Bx+Cx^2}\right]-\frac{1}{\sqrt{C}}1\frac{2}{\sqrt{C}} + \text{const.}$$

Combining the constant terms, and observing that the radical $\sqrt{C}$ is susceptible of the sign $\pm$, we shall have

$$\int\frac{dx}{\sqrt{A+Bx+Cx^2}}=\frac{1}{\sqrt{C}}l\left[\frac{B}{2\sqrt{C}}+x\sqrt{C}+\sqrt{A+Bx+Cx^2}\right]$$
$$+\text{const.}$$

168. Let us take, for a second example, $\frac{dx}{\sqrt{A+Bx-Cx^2}}$; making use of the last transformation in No. 161, we shall have $\frac{-2\,dz}{\gamma(z^2+1)}$, whose integral is

$$-\frac{2}{\gamma}\text{ arc }(\tan = z)+\text{const.}$$

Substituting for z, its value, $\frac{\sqrt{a'-x}}{\sqrt{x-a}}$, deduced from the equation $a'-x=(x-a)z^2$, and putting $\sqrt{C}$ for γ, we shall have

$$\int\frac{dx}{\sqrt{A+Bx-Cx^2}}=-\frac{2}{\sqrt{C}}.\text{arc }\left(\tan=\frac{\sqrt{a'-x}}{\sqrt{x-a}}\right)+\text{const.}$$

a and a' being the roots of the equation

$$x^2-\frac{B}{C}x-\frac{A}{C}=0.$$

If we suppose $A=C=1$, and $B=0$, the proposed differential becomes, in this particular case, $\frac{dx}{\sqrt{1-x^4}}$, and the preceding formula gives, for its integral, -2. arc $\left(\tan=\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)$+const.; for a and a', being the roots of the equation $x^2-1=0$, we must take $a=-1$, and $a'=1$, to avoid an imaginary expression.

We now proceed to shew, that this result is identical with an arc whose sine $=x$, the differential of which we

known to be $\frac{dx}{\sqrt{1-x^2}}$ &c. For this purpose we must call to mind the expression (Trig. 26.),

$$\tan 2A = \frac{2 \tan A}{1-\tan A^2},$$

from which it follows, that the arc which is double of that indicated in the preceding formula, has for its tangent $\frac{\sqrt{1-x^2}}{x}$, and that consequently it is the complement of the arc whose tangent is $\frac{x}{\sqrt{1-x^2}}$, and whose sine is x (Trig. 26).

Designating this last arc by s, we shall have

$$\int \frac{dx}{\sqrt{1-x^2}} = s - \frac{\pi}{2} + \text{const.}$$

and incorporating $-\frac{\pi}{2}$ with the arbitrary constant, there will result $\int \frac{dx}{\sqrt{1-x^2}} = s + \text{const.}$

We will also observe, that we may directly reduce the differential $\frac{dx}{\sqrt{A+Bx-Cx^2}} = \frac{dx}{\gamma\sqrt{\alpha+\beta x-x^2}}$, to that of the arc of a circle; for by making at first $x - \frac{\beta}{2} = z$, we shall get $\frac{dz}{\gamma\sqrt{\alpha+\frac{1}{4}\beta^2-z^2}}$; then putting $\alpha+\frac{1}{4}\beta^2 = g^2$, and $z = gu$, we shall obtain $\frac{du}{\gamma\sqrt{1-u^2}}$, whose integral is $\frac{1}{\gamma}$. arc ($\sin = u$)+const.

164. The integration of the expression $\frac{dx}{\sqrt{1-x^4}}$ ma

also be effected by means of logarithms, a method which leads to very remarkable imaginary expressions for the sine and cosine.

By comparing this expression with $\frac{dx}{\sqrt{A+Bx+Cx^2}}$, we find $A=1$, $B=0$, $C=-1$; and the general integral becomes (162.)

$$\frac{1}{\sqrt{-1}}\,\mathrm{l}\,(x\sqrt{-1}+\sqrt{1-x^2})+\text{const.}$$

if we represent by z the arc, of which $\frac{dx}{\sqrt{1-x^2}}$ is the differential, we shall then have

$$z=\frac{1}{\sqrt{-1}}\,\mathrm{l}\,(x\sqrt{-1}+\sqrt{1-x^2})+\text{const.}$$

But if we wish that this arc should be nothing at the same time with x, we must suppress the arbitrary constant; for, by making $x=0$, the second member reduces itself to this constant, inasmuch as $\mathrm{l}\,1=0$.

This done, and observing that if x be the sine of the arc z, then $\sqrt{1-x^2}$ is its cosine, the equation above will become

$$z\sqrt{-1}=\mathrm{l}\,(\cos z+\sqrt{-1}\sin z);$$

and if we suppose z negative, since

$$\sin(-z)=-\sin z,\quad \cos(-z)=\cos z,$$

we shall have, in this case also,

$$-z\sqrt{-1}=\mathrm{l}\,(\cos z-\sqrt{-1}\sin z),$$

a result which may be joined with the preceding, in the double equation

$$\pm z\sqrt{-1}=\mathrm{l}\,(\cos z\pm\sqrt{-1}\sin z).\ *$$

* The expression $dz=\frac{dx}{\sqrt{1-x^2}}$, is changed immediately into a loga-

Passing now in each member of the equation, from the logarithms to the numbers to which they correspond, we shall have

$$e^{\pm z\sqrt{-1}}=\cos z\pm\sqrt{-1}\sin z,$$

an equation, which furnishes the two following

$$e^{z\sqrt{-1}}=\cos z+\sqrt{-1}\sin z,$$

$$e^{-z\sqrt{-1}}=\cos z-\sqrt{-1}\sin z.$$

If we add these together, we shall find

$$\cos z=\frac{e^{z\sqrt{-1}}+e^{-z\sqrt{-1}}}{2};$$

and subtracting the second from the first, there will result

$$\sin z=\frac{e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}}{2\sqrt{-1}}.$$

These expressions are nothing more in reality than pure algebraical symbols, which represent, under an abridged form, the series in No. 36, of which we may be satisfied, by substituting for the exponential quantities $e^{z\sqrt{-1}}$, and $e^{-z\sqrt{-1}}$, their developements, formed after the series in No. 25; but these symbols, though we cannot assign their value, under any real finite form, are nevertheless of the greatest use in analysis, and exhibit all the properties of the trigonometrical lines which they represent.

---

a logarithmic differential, when we multiply both its numerator and its denominator by the factor $x\sqrt{-1}+\sqrt{1-x^2}$. There arises, from this operation,

$$dz=\frac{\frac{x\,dx\sqrt{-1}}{\sqrt{1-x^2}}+dx}{x\sqrt{-1}+\sqrt{1-x^2}}=\frac{1}{\sqrt{-1}}\cdot\frac{dx\sqrt{-1}-\frac{x\,dx}{\sqrt{1-x^2}}}{x\sqrt{-1}+\sqrt{1-x^2}};$$

where we may easily perceive, that the numerator of the second fraction is the differential of its denominator; from whence we conclude, as before, that

$$z\sqrt{-1}=\mathrm{l}\,(x\sqrt{-1}+\sqrt{1-x^2}).$$

By substituting $n\,z$ instead of z, in the equation

$$e^{\pm z\sqrt{-1}} = \cos z \pm \sqrt{-1}\,\sin z,$$

it becomes

$$e^{\pm nz\sqrt{-1}} = \cos n\,z \pm \sqrt{-1}\,\sin n\,z.$$

But we have also

$$e^{\pm nz\sqrt{-1}} = (e^{\pm z\sqrt{-1}})^n = (\cos z \pm \sqrt{-1}\,\sin z)^n;$$

and consequently

$$(\cos z \pm \sqrt{-1}\,\sin z)^n = \cos n\,z \pm \sqrt{-1}\,\sin n\,z.$$

This last equation leads to results of great importance, which we shall develope hereafter, whenever they are required. We shall here confine ourselves to the use which we may make of them, in discovering the factors of binomials, of the form $x^n \mp a^n$, since this inquiry is necessary, in the integration of rational fractions, of the form $\frac{x^m\,d\,x}{x^n \mp a^n}$.

165. The function $x^n \mp a^n$ is transformed into $a^n\,(y^n \mp 1)$, by making $x = a\,y$; and to discover its factors, it is only required to solve the equation

$$y^n \mp 1 = 0,$$

which is the same thing as

$$y^n = \pm 1.$$

The expression $y = \cos z + \sqrt{-1}\,\sin z$, satisfies this equation by a very simple determination of the arc z; for we have

$$y^n = (\cos z + \sqrt{-1}\,\sin z)^n = \cos n\,z + \sqrt{-1}\,\sin n\,z;$$

and since, putting π to denote the semi-circumference, and m any whole number, we have

$$\sin m\,\pi = 0, \qquad \cos m\,\pi = \pm 1,$$

according as m is an even or odd number, we have only to suppose $n\,z = m\,\pi$, in order to obtain $y^n = \pm 1$.

That we may distinguish more particularly the case in which m is even, from that in which it is odd, we shall

write for the first $2m$, and for the second $2m+1$; we therefore make

$$nz=2m\pi, \text{ and } nz=(2m+1)\pi.$$

From the first hypothesis, we find

$$y^n=+1, \quad y=\cos\frac{2m\pi}{n}+\sqrt{-1}\sin\frac{2m\pi}{n},$$

and from the second

$$y^n=-1, \quad y=\cos\frac{(2m+1)\pi}{n}+\sqrt{-1}\sin\frac{(2m+1)\pi}{n}.$$

By means of the indeterminate number m, ea chof these expressions for y furnishes all the values of which this quantity is susceptible; for we may take successively

$$m=0, \quad m=1, \quad m=2, \quad m=3, \text{ \&c.}$$

The first formula gives

$$y=\cos 0 . \pi=1.$$

$$y=\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n}$$

$$y=\cos\frac{4\pi}{n}+\sqrt{-1}\sin\frac{4\pi}{n},$$

&c.

and it is evident, that we shall always have different results as far as $m=n-1$; for, by supposing $n=m$, we have $y=\cos 2\pi=1$, which is the same as the first of the values already obtained; and if we suppose $m=n+1$, then

$$\cos\frac{(2n+2)\pi}{n}=\cos\left(2\pi+\frac{2\pi}{n}\right)=\cos\frac{2\pi}{n}\text{(Trig. 22.)}$$

$$\sin\frac{(2n+2)\pi}{n}=\sin\left(2\pi+\frac{2\pi}{n}\right)=\sin\frac{2\pi}{n}:$$

which leads to a value of y, the same with the second, and so on with respect to the others.

The second general formula for y, which is relative to the equation $y^n+1=0$, in the same manner will only give

different values from $m=0$, to $m=n-1$, inclusively; for if we take $m=n$, then

$$\cos\frac{(2n+1)\pi}{n}=\cos\left(2\pi+\frac{\pi}{n}\right)=\cos\frac{\pi}{n}$$

$$\sin\frac{(2n+1)\pi}{n}=\sin\left(2\pi+\frac{\pi}{n}\right)=\sin\frac{\pi}{n}.$$

166. By this mode of proceeding we shall not only obtain the roots of the equation $y^n\mp 1=0$; but, with a little attention, we shall discover that these roots may be arranged in pairs, by bringing together those which only differ in the sign of the radical $\sqrt{-1}$; for since

$$\cos(2\pi-p)=\cos p, \text{ and } \sin(2\pi-p)=-\sin p,$$

it follows, that

$$y=\cos\frac{(n+q)\pi}{n}+\sqrt{-1}\sin\frac{(n+q)\pi}{n}$$

$$=\cos\frac{(n-q)\pi}{n}-\sqrt{-1}\sin\frac{(n-q)\pi}{n}.$$

Now it is easily seen, that the numbers $n+q$ and $n-q$ are both even or both odd at the same time; we may therefore in the expressions for y, enumerated above, confine ourselves to those multiples of π which do not exceed $n\pi$, provided that we take the radical $\sqrt{-1}$ alternately $+$ and $-$, and they will consequently become

$$y=\cos\frac{2m\pi}{n}\pm\sqrt{-1}\sin\frac{(2m\pi)}{n},$$

$$y=\cos\frac{(2m+1)\pi}{n}\pm\sqrt{-1}\sin\frac{(2m+1)\pi}{n}.$$

When n is even, the values of m in the first formula ought to be all the whole numbers from 0 to $\frac{n}{2}$ inclusively, and only as far as $\frac{n-2}{2}$ in the second; and when n is

odd, the values of m must in both cases be extended as far as $\frac{n-1}{2}$.

The two values comprehended in the formula

$$y=\cos\frac{2m\pi}{n}\pm\sqrt{-1}\sin\frac{2m\pi}{n}$$

give for factors of the first degree of the quantity y^n-1, he two imaginary expressions

$$\left(y-\cos\frac{2m\pi}{n}\right)-\sqrt{-1}\sin\frac{2m\pi}{n},$$

$$\left(y-\cos\frac{2m\pi}{n}\right)+\sqrt{-1}\sin\frac{2m\pi}{n};$$

and the product of these is the expression

$$y^2-2y\cos\frac{2m\pi}{n}+1,$$

which comprehends all the real factors of the second degree.

We find, in the same manner, that the factors of the second degree of the quantity y^n+1 are

$$y^2-2y\cos\frac{(2m+1)\pi}{n}+1.$$

167. Let us take, as an example of the formula,

$$y=\cos\frac{2m\pi}{n}\pm\sqrt{-1}\sin\frac{2m\pi}{n},$$

the list of the factors of the first degree contained in the function y^6-1

$$y-1$$

$$y-\left(\cos\frac{2\pi}{6}\pm\sqrt{-1}\sin\frac{2\pi}{6}\right)$$

$$y-\left(\cos\frac{4\pi}{6}\pm\sqrt{-1}\sin\frac{4\pi}{6}\right)$$

$$y+1$$

The formula

$$y^2-2y\cos\frac{2m\pi}{n}+1$$

gives, as factors of the second degree,

$$y^2-2y+1$$

$$y^2-2y\cos\frac{2\pi}{6}+1$$

$$y^2-2y\cos\frac{4\pi}{6}+1$$

$$y^2+2y+1.$$

The first and the last of these factors are the squares of $y-1$ and $y+1$, of which only the first degree enters into the proposed function; it will be necessary then, when we employ the factors of the second degree, to replace the first and the last by

$$(y-1)(y+1), \text{ or } y^2-1.$$

We have, for the factors of the first degree of the function y^5-1

$$y-1$$

$$y-\left(\cos\frac{2\pi}{5}\pm\sqrt{-1}\sin\frac{2\pi}{5}\right)$$

$$y-\left(\cos\frac{4\pi}{5}\pm\sqrt{-1}\sin\frac{4\pi}{5}\right).$$

Those of the second degree are

$$y^2-2y+1$$

$$y^2-2y\cos\frac{2\pi}{5}+1$$

$$y^2-2y\cos\frac{4\pi}{5}+1;$$

but it must be observed, that the first factor of the second degree is the square of the factor $y-1$, which enters only once into the proposed function.

By the formula

$$y = \cos\frac{(2m+1)\pi}{n} \pm \sqrt{-1}\sin\frac{(2m+1)\pi}{n}:$$

the factors of the first degree of y^5+1, are

$$y - \left(\cos\frac{\pi}{5} \pm \sqrt{-1}\sin\frac{\pi}{5}\right)$$

$$y - \left(\cos\frac{3\pi}{5} \pm \sqrt{-1}\sin\frac{3\pi}{5}\right)$$

$$y+1;$$

and the formula $y^2 - 2y\cos\frac{(2m+1)\pi}{n} + 1$ gives us

$$y^2 - 2y\cos\frac{\pi}{5} + 1$$

$$y^2 - 2y\cos\frac{3\pi}{5} + 1,$$

$$y^2 + 2y + 1.$$

The function y^6+1 has for factors of the first degree

$$y - \left(\cos\frac{\pi}{6} \pm \sqrt{-1}\sin\frac{\pi}{6}\right)$$

$$y - \left(\cos\frac{3\pi}{6} \pm \sqrt{-1}\sin\frac{3\pi}{6}\right), \text{ or } y \mp \sqrt{-1}$$

$$y - \left(\cos\frac{5\pi}{6} \pm \sqrt{-1}\sin\frac{5\pi}{6}\right),$$

and for factors of the second,

$$y^2 - 2y\cos\frac{\pi}{6} + 1$$

$$y^2 - 2y\cos\frac{3\pi}{6} + 1, \text{ or } y^2+1$$

$$y^2 - 2y\cos\frac{5\pi}{6} + 1.$$

168. Such functions as are of the form $x^{2n} - 2p\,x^n + q$

may be treated in the same manner as those which include only two terms. By resolving them in the manner of an equation of the second degree, we shall find the factors to be

$$x^n-(p\pm\sqrt{p^2-q}),$$

which will be real, if p^2 exceed q; and by making, in that case,

$$\pm a^n=p\pm\sqrt{p^2-q},$$

we shall have functions of the form

$$x^n\mp a^n$$

to resolve into factors.

When we have $p^2<q$, we put $p=\alpha^n$, $q=\beta^{2n}$, $x=\beta y$, and the function becomes

$$\beta^{2n}y^{2n}-2\alpha^n\beta^n y^n+\beta^{2n}=\beta^{2n}(y^{2n}-\frac{2\alpha^n}{\beta^n}y^n+1);$$

but the condition $p^2<q$, or $\alpha^{2n}<\beta^{2n}$, giving $\alpha^n<\beta^n$ the quantity $\frac{\alpha^n}{\beta^n}$ is a proper fraction, and may be represented by the cosine of a given arc δ; and the proposed function will be reduced to

$$\beta^{2n}(y^{2n}-2y^n\cos\delta+1),$$

and we have only to resolve the equation

$$y^{2n}-2y^n\cos\delta+1=0.$$

We immediately find

$$y^n=\cos\delta\pm\sqrt{-1}\sin\delta;$$

then assuming

$$y=\cos z\pm\sqrt{-1}\sin z,$$

we find (164)

$$y^n=\cos nz\pm\sqrt{-1}\sin nz,$$

and composing this value of y^n with the former, we obtain

$$\cos nz=\cos\delta,\quad \sin nz=\sin\delta.$$

These relations will be satisfied, if we suppose $n z = 2 m \pi + \delta$, m being any whole number; for

$$\cos(2 m \pi + \delta) = \cos \delta, \quad \sin(m \pi + \delta) = \sin \delta;$$

we shall have, therefore,

$$z = \frac{2 m \pi + \delta}{n}, \; y = \cos \frac{2 m \pi + \delta}{n} \pm \sqrt{-1} \sin \frac{2 m \pi + \delta}{n},$$

and the factors of the first degree of the function

$$y^{2n} - 2 y^n \cos \delta + 1$$

will consequently be comprehended in this formula

$$y - \left\{ \cos \frac{2 m \pi + \delta}{n} \pm \sqrt{-1} \sin \frac{2 m \pi + \delta}{n} \right\}.$$

If we had $x^{2n} + 2 p x^n + q = 0$, we must still assume $\frac{\alpha^n}{\beta^n} = \cos \delta$; but we must take

$$y^{2n} - 2 y^n \cos(\pi - \delta) + 1;$$

since $\cos(\pi - \delta) = -\cos \delta$. This done, there will arise

$$\cos n z = \cos(\pi - \delta), \quad \sin n z = \sin(\pi - \delta);$$

and consequently

$$n z = 2 m \pi + \pi - \delta = (2 m + 1) \pi - \delta. *$$

---

* The formulæ in Nos. 166, 167, 168, comprehend implicitly the theorems of *Cotes* and *Demoivre*, and indeed more than supply the place of those theorems, which hereafter are an object of mere curiosity. We have, on this account, thought it unnecessary to insert them here. The reader will find them fully discussed in the large *Treatise on the Differential and Integral Calculus*.

On the Integration of Binomial Differentials.

169. These differentials are represented by the formula

$$x^{m-1}\,dx\,(a+bx^n)^{\frac{p}{q}},$$

whose generality will not be affected by supposing m and n to be whole numbers.

If we had, for example, $x^{\frac{1}{3}}\,dx\,(a+bx^{\frac{1}{2}})^{\frac{p}{q}}$, we may assume $x=z^6$, and the differential would become $6\,z^7\,dz\,(a+bz^3)^{\frac{p}{q}}$. We may also consider n to be, in all cases, positive, inasmuch as in the case in which we have $x^{m-1}\,dx\,(a+bx^{-n})^{\frac{p}{q}}$, we may suppose $x=\frac{1}{z}$, and the result of this substitution will be $-z^{-m-1}\,dz\,(a+b\,z^n)^{\frac{p}{q}}$.

To find in what case $x^{m-1}\,dx\,(a+bx^n)^{\frac{p}{q}}$, may become rational, we assume $a+bx^n=z^q$, so that $(a+bx^n)^{\frac{p}{q}}=z^p$; we then find

$$x^n=\frac{z_n-a}{b},\ x^m=\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}},\ x^{m-1}dx=\frac{q}{nb}z^{q-1}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1}dz,$$

and the proposed differential is transformed into

$$\frac{q}{nb}\,z^{p+q-1}\,dz\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1},$$

an expression which is evidently rational whenever $\frac{m}{n}$ is a whole number.

The differential $x^8\,dx\,(a+bx^3)^{\frac{p}{q}}$ satisfies this condition, since $m=9$, $n=3$, $\frac{m}{n}=3$; and it is transformed into

$$\frac{q}{3b} z^{p+q-1}\, dz\left(\frac{z^q-a}{b}\right)^2.$$

The expression $x^{m-1}\,dx\,(a+bx^n)^{\frac{p}{q}}$ is susceptible of another form, by making the index of d between the brackets negative, or by dividing the quantity $a+b\,x^n$ by x^n; then we have

$$x^{m-1}\,dx\,(a+b\,x^n)^{\frac{p}{q}} = x^{m-1}\,dx\,[(a\,x^{-n}+b)\,x^n]^{\frac{p}{q}}$$
$$= x^{m-1}\,dx\,(a\,x^{-n}+b)^{\frac{p}{q}}\,x^{\frac{np}{q}}$$
$$= x^{m+\frac{np}{q}-1}\,dx\,(a\,x^{-n}+b)^{\frac{p}{q}};$$

and by the process preceding, the last of these expressions may be made rational, whenever $\dfrac{m+\frac{np}{q}}{n}$ is a whole number, or what is the same thing, whenever $\frac{m}{n}+\frac{p}{q}$ is an integer.

The differential

$$x^4\,dx\,(a+b\,x^3)^{\frac{1}{3}}$$

is of this description, since

$$\frac{m}{n}=\frac{5}{3},\ \frac{p}{q}=\frac{1}{3},\ \frac{m}{n}+\frac{p}{q}=\frac{6}{3}=2.$$

In applying to the differential

$$x^{m+\frac{np}{q}-1}\,dx\,(a\,x^{-n}+b)^{\frac{p}{q}},$$

the substitution indicated for the first form of this differential, we shall make

$$a\,x^{-n}+b=z^q,$$

from which we deduce

$$a+b\,x^n=x^n\,z^q;$$

and if we transform immediately the expression $x^{m-1}\,dx$

$(a+bx^n)^{\frac{p}{q}}$, by means of the equation preceding, we shall evidently obtain the same result, as if one had at first given it the form

$$x^{m+\frac{np}{q}-1}dx\,(ax^{-n}+b)^{\frac{p}{q}}.$$

170. Since it is not possible, in every case, to integrate the expression $x^{m-1}dx\,(a+bx^n)^{\frac{p}{q}}$, the method of procedure which first suggests itself, is to endeavour to reduce it to the most simple cases which it can include, as we have done in No. 154. with respect to $\int\frac{dz}{(z^2+\beta^2)^q}$, which is reduced to $\int\frac{dz}{z^2+\beta^2}$. We shall effect this, by the assistance of the remark in No. 148., in which we observed, that $\int u\,dv=uv-\int v\,du$; for if we decompose the quantity $x^{m-1}dx\,(a+bx^n)^{\frac{p}{q}}$ into two factors, of which one being integrable, may be represented by dv, and the other by u, we shall make the integration of the proposed expression, depend on that of $v\,du$, which, in certain cases, will be more simple than the given differential, as we shall proceed to shew. This method, which is at once extensive and curious, is called *Integration by Parts.*

For the sake of abridging the results, we shall write p in the place of $\frac{p}{q}$; supposing p to represent any fractional number, the formula will then become

$$x^{m-1}dx\,(a+bx^n)^p.$$

Among the different ways of resolving this differential into factors, we shall choose that which diminishes the index of x without the parenthesis; which is effected by writing the proposed differential thus,

$$x^{m-n}\,.\,x^{n-1}dx\,(a+bx^n)^p.$$

By this means the factor $x^{n-1}\,dx\,(a+bx^n)^p$ is integrable, whatever be the value of p (149.): representing it, therefore, by dv, we have

$$v=\frac{(a+bx^n)^{p+1}}{(p+1)\,nb},\quad \text{and } u=x^{m-n};$$

whence there results

$$\int x^{m-1}\,dx\,(a+bx^n)^p=$$
$$\frac{x^{m-n}(a+bx^n)^{p+1}}{(p+1)\,nb}-\frac{m-n}{(p+1)\,nb}\int x^{m-n-1}dx\,(a+bx^n)^{p+1};$$

but

$$\int x^{m-n-1}\,dx\,(a+bx^n)^{p+1}$$
$$=\int x^{m-n-1}\,dx\,(a+bx^n)^p\,(a+bx^n)$$
$$=a\int x^{m-n-1}dx\,(a+bx^n)^p+b\int x^{m-1}\,dx\,(a+bx^n)^p.$$

Substituting now this last value in the preceding equation, and collecting into one the terms involving the integral $\int x^{m-n-1}dx\,(a+bx^n)^p$, we find

$$\left(1+\frac{m-n}{(p+1)\,n}\right)\int x^{m-1}\,dx\,(a+bx^n)^p$$
$$=\frac{x^{m-n}\,(a+bx^n)^{p+1}-a\,(m-n)\int x^{m-n-1}\,dx\,(a+bx^n)^p}{(p+1)\,nb};$$

from which we get (A)............ ...$\int x^{m-1}\,dx\,(a+bx^n)^p$

$$=\frac{x^{m-n}\,(a+bx^n)^{p+1}-a\,(m-n)\int x^{m-n-1}\,dx\,(a+bx^n)^p}{b\,(pn+m)}.$$

It is readily seen, that since we may reduce, by this formula, the determination of $\int x^{m-1}\,dx\,(a+bx^n)^p$, to that of $\int x^{m-n-1}\,dx\,(a+bx^n)^p$, we may also reduce this last to that of $\int x^{m-2n-1}\,dx\,(a+bx^n)^p$, by writing $m-n$ in the place of m, in the equation (A); then, by changing $m-n$ into $m-2n$, in this last equation, we shall be able to determine $\int x^{m-2n-1}\,dx\,(a+bx^n)^p$, by means of $\int x^{m-3n-1}\,dx\,(a+bx^n)^p$, and so on.

In general, if r denote the number of reductions, we shall at last come to $\int x^{m-rn-1}\,dx\,(a+bx^n)^p$, and the last formula will be

$$\int x^{m-(r-1)n-1}\,dx\,(a+bx^n)^p$$
$$=\frac{x^{m-rn}(a+bx^n)^{p+1}-a(m-rn)\int x^{m-rn-1}dx\,(a+bx^n)^p}{b[pn+m-(r-1)n]}.$$

It is evident, from this formula, that if m be a multiple of n, then $\int x^{m-1}dx\,(a+bx^n)^p$ will be a finite algebraical quantity; for in that case the coefficient $m-rn=0$, and therefore the term containing $\int x^{m-rn-1}\,dx\,(a+bx^n)^p$ will vanish. This result agrees with what we have already found in No. 169.

171. There is another method of reduction, by which the exponent of the quantity within the parenthesis may be diminished by unity; for this purpose it is sufficient to observe, that

$$\int x^{m-1}\,dx\,(a+bx^n)^p=\int x^{m-1}\,dx\,(a+bx^n)^{p-1}\,(a+bx^n)$$
$$=a\int x^{m-1}\,dx\,(a+bx^n)^{p-1}+b\int x^{m+n-1}\,dx\,(a+bx^n)^{p-1},$$

and that the formula (A), by changing m into $m+n$, and p into $p-1$, gives

$$\int x^{m+n-1}\,dx\,(a+bx^n)^{p-1}$$
$$=\frac{x^m(a+bx^n)_p-a\,m\int x^{m-1}\,dx\,(a+bx^n)_p{}^{-1}}{b\,(pn+m)}.$$

Substituting this value in the preceding equation, we have (B).................................. $\int x^{m-1}\,dx\,(a+bx^n)^p=$

$$\frac{x^m(a+bx^n)^p+pna\int x^{m-1}\,dx\,(a+bx^n)^{p-1}}{(pn+m)}.$$

By means of the formula (B) we may take away successively from p, as many unities as it contains; and by the application of this formula, and also of formula (A), we may make the integral $\int x^{m-1}\,dx\,(a+bx^n)^p$ depend on $\int x^{m-rn-1}\,dx\,(a+bx^n)^{p-s}$, $r\,n$ being the greatest multiple of n contained in $m-1$, and s the greatest whole number in p.

The integral $\int x^7\,dx\,(a+bx^3)^{\frac{5}{2}}$, for example, may be reduced by the formula (A), successively, to

$$\int x^4\,dx\,(a+bx^3)^{\frac{5}{2}},\quad \int x\,dx\,(a+bx^3)^{\frac{5}{2}};$$

and by the formula (B) $\int x\,dx\,(a+b\,x^3)^{\frac{5}{2}}$ is reduced to $\int x\,dx\,(a+b\,x^3)^{\frac{3}{2}}$, and that again to $\int x\,dx\,(a+b\,x^3)^{\frac{1}{2}}$.

172. It is evident, that if m and p were negative, the formulæ (A) and (B) would not answer the purpose for which they have been investigated: in that case they would increase the exponent of x without the parenthesis, as well as that of the parenthesis itself. If, however, we reverse them, we shall find that they then apply to the case under consideration.

From the formula (A) we deduce

$$\int x^{m-n-1}\,dx\,(a+b\,x^n)^p$$
$$=\frac{x^{m-n}\,(a+b\,x^n)^{p+1}-b\,(m+n\,p)\int x^{m-1}dx\,(a+b\,x^n)^p}{a\,(m-n)};$$

substitute $m+n$, in the place of m, and it becomes (C)

$$\int x^{m-1}\,dx\,(a+b\,x^n)^p$$
$$=\frac{x^m\,(a+b\,x^n)^{p+1}-b\,(m+n+n\,p)\int x^{m+n-1}dx\,(a+bx^n)^p}{a\,m},$$

a formula which diminishes the exponent of x without the parenthesis, since $m+n-1$ becomes $-m+n-1$, when we put $-m$ in the place of m.

To reverse the formula (B), we take

$$\int x^{m-1}\,dx\,(a+b\,x^n)^{p-1}$$
$$=-\frac{x^m\,(a+b\,x^n)^p-(m+n\,p)\int x^{m-1}\,dx\,(a+b\,x^n)^p}{p\,n\,a};$$

then, writing $p+1$ in the place of p, we find (D)

$$\int x^{m-1}\,dx\,(a+b\,x^n)^p$$
$$=-\frac{x^m(a+bx^n)^{p+1}-(m+n+n\,p)\int x^{m-1}dx(a+bx^n)^{p+1}}{(p+1)\,n\,a}.$$

This formula answers the object in view, since $p+1$ becomes $-p+1$, when p is negative.

The formulæ (A), (B), (C), (D), cannot be applied when their denominators vanish. This is the case with the formula (A), for example, when $m=-n\,p$; but in all such cases the proposed function is integrable, either algebraically, or by logarithms.

173. Let the formula be $\int \frac{x^{m-1}dx}{\sqrt{1-x^2}}$, where m is an integral and positive number; we find by the formula (A), making $a=1$, $b=-1$, $n=2$, $p=-\frac{1}{2}$,

$$\int \frac{x^{m-1}dx}{\sqrt{1-x^2}} = -\frac{x^{m-2}\sqrt{1-x^2}}{m-1} + \frac{m-2}{m-1}\int \frac{x^{m-3}dx}{\sqrt{1-x^2}};$$

writing m in the place of $m-1$, it becomes

$$\int \frac{x^m dx}{\sqrt{1-x^2}} = -\frac{x^{m-1}\sqrt{1-x^2}}{m} + \frac{m-1}{m}\int \frac{x^{m-2}dx}{\sqrt{1-x^2}}.$$

If we give to m successively different values, beginning with the odd numbers, we shall have

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \text{const.}$$

$$\int \frac{x^3 dx}{\sqrt{1-x^2}} = -\frac{1}{3}x^2\sqrt{1-x^2} + \frac{2}{3}\int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$\int \frac{x^5 dx}{\sqrt{1-x^2}} = -\frac{1}{5}x^4\sqrt{1-x^2} + \frac{4}{5}\int \frac{x^3 dx}{\sqrt{1-x^2}}$$

$$\int \frac{x^7 dx}{\sqrt{1-x^2}} = -\frac{1}{7}x^6\sqrt{1-x^2} + \frac{6}{7}\int \frac{x^5 dx}{\sqrt{1-x^2}}$$

&c.

From which we find

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \text{const.}$$

$$\int \frac{x^3 dx}{\sqrt{1-x^2}} = -\left(\frac{1}{3}x^2 + \frac{1.2}{1.3}\right)\sqrt{1-x^2} + \text{const.}$$

$$\int \frac{x^5 dx}{\sqrt{1-x^2}} = -\left(\frac{1}{5}x^4 + \frac{1.4}{3.5}x^2 + \frac{1.2.4}{1.3.5}\right)\sqrt{1-x^2} + \text{const.}$$

$$\int \frac{x^7 dx}{\sqrt{1-x^2}} = -\left(\frac{1}{7}x^6 + \frac{1.6}{5.7}x^4 + \frac{1.4.6}{3.5.7}x^2 + \frac{1.2.4.6}{1.3.5.7}\right)\sqrt{1-x^2} + \text{cc}$$

&c

The law of these values is evident.

Passing now to the even values of m, and supposing $m=2$, $m=4$, $m=6$, &c. we find

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}$$

$$\int \frac{x^4\,dx}{\sqrt{1-x^2}} = -\frac{1}{4}x^3\sqrt{1-x^2}+\frac{3}{4}\int\frac{x^2\,dx}{\sqrt{1-x^2}}$$

$$\int \frac{x^6\,dx}{\sqrt{1-x^2}} = -\frac{1}{6}x^5\sqrt{1-x^2}+\frac{5}{6}\int\frac{x^4\,dx}{\sqrt{1-x^2}}$$

&c.

In all these cases the proposed integrals will depend upon

$$\int \frac{dx}{\sqrt{1-x^2}} = \text{arc }(\sin = x) + \text{const. (35.)}$$

and if we represent this arc by A, we shall have

$$\int \frac{dx}{\sqrt{1-x^2}} = A + \text{const.}$$

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}A + \text{const.}$$

$$\int \frac{x^4\,dx}{\sqrt{1-x^2}} = -\left(\frac{1}{4}x^3+\frac{1.3}{2.4}x\right)\sqrt{1-x^2}+\frac{1.3}{2.4}A + \text{const.}$$

$$\int \frac{x^6\,dx}{\sqrt{1-x^2}} = -\left(\frac{1}{6}x^5+\frac{1.5}{4.6}x^3+\frac{1.3.5}{2.4.6}x\right)\sqrt{1-x^2}+\frac{1.3.5}{2.4.6}A +$$

&c.

174. We now proceed to find the formulæ for those cases in which m is a negative number. We have then, by formula (C) (172.)

$$\int \frac{x^{-m-1}\,dx}{\sqrt{1-x^2}} = -\frac{x^{-m}\sqrt{1-x^2}}{m}+\frac{m-1}{m}\int\frac{x^{-m+1}\,dx}{\sqrt{1-x^2}};$$

and writing $-m$, in the place of $-m-1$, it becomes

$$\int \frac{dx}{x^m\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{(m-1)x^{m-1}}+\frac{m-2}{m-1}\int\frac{dx}{x^{m-2}\sqrt{1-x^2}}.$$

We cannot here suppose $m=1$, since that value would render the denominator $=0$; we must therefore previously investigate the integral of $\int \frac{dx}{x\sqrt{1-x^2}}$. We shall find it easily from what has been said in No. 161; but we may also arrive at it in the following manner: make $1-x^2=z^2$; from which we have

$$x=\sqrt{1-z^2} \qquad dx=\frac{-z\,dz}{\sqrt{1-z^2}},$$

and consequently

$$\frac{dx}{x\sqrt{1-x^2}}=\frac{-dz}{1-z^2},$$

an equation, the integral of whose second member is

$$-\tfrac{1}{2}\,\mathrm{l}\,(1+z)+\tfrac{1}{2}\,\mathrm{l}\,(1-z)=-\tfrac{1}{2}\,\mathrm{l}\left(\frac{1+z}{1-z}\right);$$

and substituting for z its value, we have

$$-\tfrac{1}{2}\,\mathrm{l}\left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right);$$

multiplying by $1+\sqrt{1-x^2}$, the two members of the fraction comprised under the sign l, we shall get

$$-\tfrac{1}{2}\,\mathrm{l}\left[\frac{(1+\sqrt{1-x^2})^2}{x^2}\right]=-\tfrac{1}{2}\,\mathrm{l}\left[\left(\frac{1+\sqrt{1-x^2}}{x}\right)^2\right]$$

$$=-\mathrm{l}\left(\frac{1+\sqrt{1-x^2}}{x}\right);$$

we shall have then finally

$$\int\frac{dx}{x\sqrt{1-x^2}}=-\mathrm{l}\left(\frac{1+\sqrt{1-x^2}}{x}\right)+\text{const.}$$

And making $m=3$, $m=5$, &c. we shall find

$$\int\frac{dx}{x^3\sqrt{1-x^2}}=-\frac{\sqrt{1-x^2}}{2x^2}+\frac{1}{2}\int\frac{dx}{x\sqrt{1-x^2}}$$

$$\int\frac{dx}{x^5\sqrt{1-x^2}}=-\frac{\sqrt{1-x^2}}{4x^4}+\frac{3}{4}\int\frac{dx}{x^3\sqrt{1-x^2}}$$

$$\int \frac{dx}{x^7\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{6x^6} + \frac{5}{6}\int \frac{dx}{x^5\sqrt{1-x^2}}$$

&c.

Again, making $m=2$, $m=4$, $m=6$, &c. we shall find

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x} + \text{const.}$$

$$\int \frac{dx}{x^4\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{3x^3} + \frac{2}{3}\int \frac{dx}{x^2\sqrt{1-x^2}}$$

$$\int \frac{dx}{x^6\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{5x^5} + \frac{4}{5}\int \frac{dx}{x^4\sqrt{1-x^2}}$$

&c.

From these two series of equations we shall be able to deduce, as in the preceding No. one class of formulas integrated by logarithms, and another class which will be entirely algebraical.

On Integration by Series.

175. The integral of $X\,dx$ is easily found, when we have expanded the function X in a series, since in that case nothing remains, but to apply the rule in No. 146, to integrate every term in succession. Thus, let $X = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + \&c.$; if we multiply the two members of this equation by dx, and then separately integrate every term of the second, we shall get

$$\int X\,dx = \frac{Ax^{m+1}}{m+1} + \frac{Bx^{m+n+1}}{m+n+1} + \frac{Cx^{m+2n+1}}{m+2n+1} + \&c. + \text{const.}$$

If, in the expansion of X, we meet with any term of the form $\frac{A}{x}$, the integral corresponding to that term will be $A\,l\,x$ (147).

176. The most simple function of x that can be expanded in a series, is $\frac{1}{a+x}$, which becomes

$$\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\text{\&c.};$$

from which we have

$$\int\frac{dx}{a+x}=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\text{\&c.}+\text{const.}$$

but we also know that $\int\frac{dx}{a+x}=\mathrm{l}\,(a+x)$: therefore

$$\mathrm{l}\,(a+x)=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\text{\&c.}+\text{const.}$$

To find the nature and value of the constant quantity, we have only to make $x=0$; for then the equation becomes $\mathrm{l}\,a=\text{const.}$
and consequently

$$\mathrm{l}\,(a+x)-\mathrm{l}\,a=\mathrm{l}\left(1+\frac{x}{a}\right)=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\text{\&c.}$$

a result conformable to that in No. 29.

Let the differential be $\frac{a\,dx}{a^2+x^2}$, which may be put under the form $\dfrac{\frac{dx}{a}}{1+\frac{x^2}{a^2}}$, and which consequently belongs to an arc whose tangent $=\frac{x}{a}$: by reducing $\frac{a}{a^2+x^2}$ into a series, we find

$$\frac{a}{a^2+x^2}=\frac{1}{a}-\frac{x^2}{a^3}+\frac{x^4}{a^5}-\frac{x^6}{a^7}+\text{\&c.};$$

and by integrating each term separately, we have

$$\int\frac{a\,dx}{a^2+x^2}=\text{arc}\left(\tan=\frac{x}{a}\right)+\text{const}=$$

$$\frac{x}{a} - \frac{x^3}{3\,a^3} + \frac{x^5}{5\,a^5} - \frac{x^7}{7\,a^7} + \text{\&c.} + \text{const.}$$

If we wish to deduce, from this equation, the value of the least arc whose tangent is $\frac{x}{a}$, we must suppress the arbitrary constant, since the arc sought for is $=0$, when $x=0$, and we shall have

$$\text{arc}\left(\tan. = \frac{x}{a}\right) = \frac{x}{a} - \frac{x^3}{3a^3} + \frac{x^5}{5\,a^5} - \frac{x^7}{7a^7} + \text{\&c.}$$

a result which agrees with that in No. 37; but here the law is manifest.

By operating in the same manner upon $\frac{x^m\,d\,x}{a^n + x^n}$, we find

$$\int \frac{x^m d\,x}{a^n + x^n} = \frac{x^{m+1}}{(m+1)a^n} - \frac{x^{m+n+1}}{(m+n+1)\,a^{2n}}$$

$$+ \frac{x^{m+2n+1}}{(m+2\,n+1)\,a^{3n}} - \text{\&c.} + \text{const.}$$

177. The object of integration by series being to obtain approximate values of the integrals which we cannot obtain accurately, it is of consequence to have several series, so that we may be able to choose that which becomes convergent upon the substitution of a proposed value of x. Those series which proceed by positive and increasing powers of x, or *ascending series* in general, do not converge, unless when x is very small; whilst those which proceed by the negative powers of x, or *descending series*, become convergent only when x is very large.

To obtain a series of this kind in the example given above, we must change the order of the terms of the binomial $a^n + x^n$, or we must put x in the place of a in the developement of $\frac{1}{a^n+x^n}$, and we shall have

$$\frac{1}{x^n+a^n}=\frac{1}{x^n}-\frac{a^n}{x^{2n}}+\frac{a^{2n}}{x^{3n}}-\frac{a^{3n}}{x^{4n}}+\&c.$$

and it will become, after multiplying by $x^m\,d\,x$ and integrating,

$$\int\frac{x^m\,d\,x}{x^n+a^n}=-\frac{1}{(n-m-1)x^{n-m-1}}$$
$$+\frac{a^n}{(2n-m-1)\,x^{2n-m-1}}-\frac{a^{2n}}{(3\,n-m-1)x^{3n-m-1}}$$
$$+\ \&c.\ +\text{const.}$$

This series would fail, if any one of its denominators, which are comprised in the formula $i\,n-m-1$, should become equal to nothing, which would be the case if $m+1$ were a multiple of n: in this case the expanded differential would contain a term of the form $a^{(i-1)n}\frac{d\,x}{x}$, whose integral is $a^{(i-1)n}\,l\,x$.

If in the series above mentioned we make $m=0$, $n=2$, and $a=1$, it becomes

$$\int\frac{d\,x}{1+x^2}=-\frac{1}{x}+\frac{1}{3\,x^3}-\frac{1}{5\,x^5}+\&c.+\text{const.}$$

but although the expression $\frac{d\,x}{1+x^2}$ is the differential of the arc, whose tangent is x, we should not therefore conclude, that the preceding series is the developement of this arc, since it becomes infinite when $x=0$. The consideration of the arbitrary constant will remove this difficulty, if we consider, that, in order to know the true value of a series, we must always begin with the case, in which it is convergent. Now the series

$$-\frac{1}{x}+\frac{1}{3\,x^3}-\frac{1}{5\,x^5}+\&c.$$

converges so much the faster as x is greater, and it vanishes when x is infinite. The equation

$$\text{arc}\ (\tan.=x)=-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\text{\&c.}+\text{const.}$$

will become in this extreme case arc $=\frac{\pi}{2}=$ const., and substituting this value of the constant, we shall have

$$\text{arc}\ (\tan.=x)=\frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\text{\&c.}$$

We might also integrate the rational fraction $\frac{U\,dx}{V}$ (151), by expanding the quantity $\frac{U}{V}$ into a series; but this method would lead us to results very complicated, and seldom convergent; besides, this manner of integrating is in this case almost useless; since we know how to reduce this differential quantity to arcs of circles, or to logarithms, the values of which are easily found from the common tables.

178. The formula $x^{m-1}\,dx\,(a+bx^n)^{\frac{p}{q}}$ is easily integrated by the expansion of the quantity $(a+bx^n)^{\frac{p}{q}}$ into a series; from which we obtain

$$\int x^{m-1}\,dx\,(a+bx^n)^{\frac{p}{q}}=a^{\frac{p}{q}}\left\{\frac{x^m}{m}+\frac{pb}{qa}\frac{x^{m+n}}{m+n}\right.$$
$$\left.+\frac{p(p-q)b^2}{1.2\,q^2a^2}\frac{x^{m+2n}}{m+2n}+\frac{p(p-q)(p-2q)b^3}{1.2.3\,q^3a^3}\frac{x^{m+3n}}{m+3n}+\text{\&c}\right\}$$
$$+\text{con}$$

If we wished to have a descending series with respect to x, we must give the proposed differential the form $x^{m+\frac{np}{q}-1}\,dx\,(b+ax^{-n})^{\frac{p}{q}}$; and we should find by expanding $(b+ax^{-n})^{\frac{p}{q}}$, multiplying by $x^{m+\frac{np}{q}-1}\,dx$, and integrating the result, that

$$\int x^{m-1}dx\,(a+bx^n)^{\frac{p}{q}} = b^{\frac{p}{q}}\left\{\frac{q\,x^{m+\frac{pn}{q}}}{mq+np}\right.$$

$$\left.+\frac{pa}{qb}\,\frac{q\,x^{m+\frac{(p-q)n}{q}}}{mq+(p-q)n}+\frac{p(p-q)a^2}{1.2.q^2b^2}\,\frac{q\,x^{m+\frac{(p-2q)n}{q}}}{mq+(p-2q)n}+\&c.\right\}$$
$$+\text{ const.}$$

Whenever a and b are both positive, or q an odd number, we may make use at pleasure, of either this or the preceding series; but when q is even, the first formula will become imaginary from the factor $a^{\frac{p}{q}}$, if a^p be negative, which will also happen to the second, if b^p be negative.

179. Let us take for example $\frac{dx}{\sqrt{1-x^2}}$, a quantity which is the differential of the arc whose sine $= x$; we shall have

$$\frac{1}{\sqrt{1-x^2}} = 1+\frac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\frac{1.3.5.7}{2.4.6.8}x^8+\&c.$$

and consequently

$$\int\frac{dx}{\sqrt{1-x^2}} = x+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\&c.+\text{const.}$$

By suppressing the constant the series will vanish, when $x=0$; it will consequently express the value of the least arc, whose sine $= x$, as in No. 37.

We shall here add a few results, which are easily obtained from what has preceded, and are useful in themselves.

1st. $\frac{dx}{\sqrt{x-x^2}} = \frac{dx}{\sqrt{x}\,.\sqrt{1-x}}$; making $\sqrt{x}=u$, we have $\frac{2\,du}{\sqrt{1-u^2}}$; but from the preceding series, we have

$$\int\frac{2\,du}{\sqrt{1-u^2}} = 2\left(u+\frac{1}{2}\frac{u^3}{3}+\frac{1.3}{2.4}\frac{u^5}{5}+\frac{1.3.5}{2.4.6}\frac{u^7}{7}+\&c.\right)+\text{const}$$

therefore,

$$\int \frac{dx}{\sqrt{x-x^2}} = 2\left(1+\frac{1}{2}\frac{x}{3}+\frac{1.3}{2.4}\frac{x^2}{5}+\frac{1.3.5}{2.4.6}\frac{x^3}{7}+\&c.\right)\sqrt{x}+C$$

$$\text{2d. } dx\sqrt{2ax-x^2} = (2a)^{\frac{1}{2}}x^{\frac{1}{2}}dx\left(1-\frac{x}{2a}\right)^{\frac{1}{2}};$$

now

$$\left(1-\frac{x}{2a}\right)^{\frac{1}{2}} = 1-\frac{1}{2}\frac{x}{2a}-\frac{1.1}{2.4}\frac{x^2}{4a^2}-\frac{1.1.3}{2.4.6}\frac{x^3}{8a^3}-\&c.$$

therefore,

$$\int dx\sqrt{2ax-x^2} = \left(\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{2}\frac{2x^{\frac{5}{2}}}{5.2a}-\frac{1.1}{2.4}\frac{2x^{\frac{7}{2}}}{7.4a^2}\right.$$

$$\left.-\frac{1.1.3}{2.4.6}\frac{2x^{\frac{9}{2}}}{9.8a^3}-\&c.\right)\sqrt{2a}+\text{const.}$$

gives

$$\int dx\sqrt{2ax-x^2} = \left(\frac{1}{3}-\frac{1}{2}\frac{x}{5.2a}-\frac{1.1}{2.4}\frac{x^2}{7.4a^2}\right.$$

$$\left.-\frac{1.1.3}{2.4.6}\frac{x^3}{9.8a^3}-\&c.\right)2x\sqrt{2ax}+\text{const.},$$

3d. $\int \frac{dx}{\sqrt{1+x^2}}$ gives, by expanding, $\frac{1}{(1+x^2)^{\frac{1}{2}}}$ into a series, and integrating the terms of the expansion, when multiplied by dx,

$$\int \frac{dx}{\sqrt{1+x^2}} = x-\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}-\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\&c.$$
$$+\text{ const}$$

$$\text{4th. }\int \frac{dx}{\sqrt{x^2-1}} = 1x-\frac{1}{1.2x^2}-\frac{1.3}{2.4.4x^4}-\frac{1.3.5}{2.4.6\,6x^6}$$
$$-\&c.+\text{const.}$$

This series, which includes the transcendental quantity $1x$, is by so much the more convergent, the greater the value of x; we may obtain another series for this integral, which is entirely algebraical, and which will be the more convergent the nearer x is to unity. For this purpose we must make $x = 1+u$, a transformation which gives

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{du}{\sqrt{2u+u^2}} = \frac{1}{\sqrt{2}} \int u^{-\frac{1}{2}} du \left(1+\frac{u}{2}\right)^{-\frac{1}{2}};$$

expanding $\left(1+\frac{u}{2}\right)^{-\frac{1}{2}}$, multiplying each term by $u^{-\frac{1}{2}} du$, and integrating, we find

$$\int \frac{du}{\sqrt{2u+u^2}} =$$

$$\frac{1}{\sqrt{2}}\left(2u^{\frac{1}{2}} - \frac{1}{2}\,\frac{2u^{\frac{3}{2}}}{3.2} + \frac{1.3}{2.4}\,\frac{2u^{\frac{5}{2}}}{5.4} - \frac{1.3.5}{2.4.6}\,\frac{2u^{\frac{7}{2}}}{7.8} + \&c.\right) + \text{const.}$$

$$= \left(1 - \frac{1.u}{2.3.2} + \frac{1.3.u^2}{2.4.5.4} - \frac{1.3.5.u^3}{2.4.6.7.8} + \&c.\right)\sqrt{2u} + \text{const.}$$

and since $u = x - 1$, the terms of this series are so much the less, the less the value of $x - 1$.

180. The object of the reduction of differentials into series is to transform them into a series of terms each of which is separately integrable, and it is not always necessary for this purpose, that all the terms should be quantities of the form $A x^{\frac{m}{n}} dx$.

If we have, for example,

$$\frac{dx\sqrt{1-e^2x^2}}{\sqrt{1-x^2}},$$

when e is a very small quantity, we may expand $\sqrt{1-e^2x^2}$ into a series, converging with great rapidity, inasmuch as in the proposed differential x^2 is always < 1, on account of the radical $\sqrt{1-x^2}$; we find

$$\sqrt{1-e^2x^2} = 1 - \frac{1}{2}e^2x^2 - \frac{1.1}{2.4}e^4x^4 - \frac{1.1.3}{2.4.6}e^6x^6 - \&c.$$

and we shall have, in consequence, to integrate the expression

$$\int \frac{dx}{\sqrt{1-x}}\left(1 - \frac{1}{2}e^2x^2 - \frac{1.1}{2.4}e^4x^4 - \frac{1.1.3}{2.4.6}e^6x^6 - \&c.\right)$$

each term of which is comprised in the general formula $\int \frac{x^m dx}{\sqrt{1-x^2}}$, considered in Nos. 173, 174. Substituting in the place of

$$\int \frac{dx}{\sqrt{1-x^2}}, \quad \int \frac{x^2 dx}{\sqrt{1-x^2}}, \quad \int \frac{x^4 dx}{\sqrt{1-x^2}}, \text{ \&c.}$$

the expressions given in No. 173, we shall derive from thence

$$\int \frac{dx\sqrt{1-e^2x^2}}{\sqrt{1-x^2}} = A$$

$$+ \frac{1}{2}e^2\left\{\frac{1}{2}\sqrt{1-x^2} - A\right\}$$

$$+ \frac{1.1}{2.4}e^4\left\{\left(\frac{1}{4}x^3 + \frac{1.3}{2.4}x\right)\sqrt{1-x^2} - \frac{1.3}{2.4}A\right\}$$

$$+ \frac{1.1.3}{2.4.6}e^6\left\{\left(\frac{1}{6}x^5 + \frac{1.5}{4.6}x^3 + \frac{1.3.5}{2.4.6}x\right)\sqrt{1-x^2} - \frac{1.3.5}{2.4.6} =\right.$$

$$+ \text{ \&c.} \ldots\ldots\ldots\ldots\ldots\ldots + \text{const.}$$

We might also treat, in a similar manner, the differential

$$\frac{dx}{\sqrt{(1-x^2)(a+x)}} = \frac{dx}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{a+x}}:$$

by reducing into a series the quantity

$$\frac{1}{\sqrt{a+x}} = (a+x)^{-\frac{1}{2}}.$$

It is proper to remark, that the formula

$$\frac{du}{\sqrt{(2mu-u^2)(n-u)}},$$

which is met with in some applications of this Calculus to mechanical problems, reduces itself to

$$\frac{-1}{\sqrt{m}}\int\frac{dx}{\sqrt{(1-x^2)\ (a+x)}},$$

by making

$$m-u=mx, \text{ and } \frac{n-m}{m}=a.$$

On the Integration of Logarithmic and Exponential Quantities.

181. Let us first take the formula $\int P\,dx\ (1\,x)^n$, where P is an algebraical function of x; by applying to this the principle of reduction indicated by the formula $\int u\,dv = uv - \int v\,du$, and making for greater shortness $\int P\,dx = N$, we get

$$\int P\,dx\,(1\,x)^n = N\,(1\,x)^n - n\int\frac{dx}{x}\,(1\,x)^{n-1}\,N.$$

If we represent $\int\frac{dx}{x}N$ by M, and change in the preceding formula N into M, and n into $n-1$, we shall have

$$\int\frac{dx}{x}N\,(1\,x)^{n-1} = M\,(1\,x)^{n-1} - (n-1)\int\frac{dx}{x}(1\,x)^{n-2}M.$$

By a series of similar reductions, we shall obtain

$$\left.\begin{aligned}\int P\,dx\,(1\,x)^n = N\,(1\,x)^n - n\,M\,(1\,x)^{n-1} + n\,(n-1)\,L\,(1\,x)^{n-2}\\ -n\,(n-1)\,(n-2)\,K\,(1\,x)^{n-3} + \&c.\end{aligned}\right\},$$

and consequently the proposed integral will be algebraical, in the case in which n is a whole number, and where the quantities represented by N, M, L, K, are all algebraical functions; the examples which follow will illustrate this more completely.

182. The differential $x^m\,dx\,(1\,x)$ gives

$$\int P\,dx = \int x^m\,dx = \frac{x^{m+1}}{m+1} = N,$$

and consequently

$$x^m dx(1x)^n = \frac{x^{m+1}(1x)^n}{m+1} - \frac{n}{m+1}\int x^m dx(1x)^{n-1}.$$

If in this equation we change successively n into $n-1$, $n-2$, &c. we shall find

$$\int x^m dx(1x)^{n-1} = \frac{x^{m+1}(1x)^{n-1}}{m+1} - \frac{n-1}{m+1}\int x^m dx(1x)^{n-2},$$

$$\int x^m dx(1x)^{n-2} = \frac{x^{m+1}(1x)^{n-2}}{m+1} - \frac{n-2}{m+1}\int x^m dx(1x)^{n-3},$$

&c.

By continuing these reductions, and then ascending from the last to the first, we shall obtain the following general formula:

$$\int x^m dx(1x)^n = \frac{x^{m+1}}{m+1}$$

$$\left\{\begin{array}{l}(1x)^n - \dfrac{n}{m+1}(1x)^{n-1} + \dfrac{n(n-1)}{(m+1)^2}(1x)^{n-2} \\ \quad - \dfrac{n(n-1)(n-2)}{(m+1)^3}(1x)^{n-3} + \&\text{c.}\end{array}\right\} + \text{const.}$$

It is obvious that this series will terminate in all cases in which n is a whole positive number.

By taking $n=1$, and $n=2$, we get

$$\int x^m dx(1x) = \frac{x^{m+1}}{m+1}\left\{(1x) - \frac{1}{m+1}\right\} + \text{const.}$$

$$\int x^m dx(1x)^2 = \frac{x^{m+1}}{m+1}\left\{(1x)^2 - \frac{2}{m+1}(1x) + \frac{1}{(m+1)^2}\right\}.$$

When $m=-1$, we have

$$\int \frac{dx}{x}(1x)^n = \frac{1}{n+1}(1x)^{n+1} + \text{const.}$$

In general, the differential $\frac{dx}{x}U$, in which U denotes

an algebraical function of $l\,x$, will become algebraical by making $l\,x = u$.

When n is negative, or fractional, the series proceeds in *infinitum* by making $n = -\frac{1}{2}$, for instance, there arises

$$\int \frac{x^m\,d\,x}{\sqrt{l\,x}} =$$

$$\frac{x^{m+1}}{m+1}\left\{\frac{1}{(l\,x)^{\frac{1}{2}}} + \frac{1\,.\,2}{2\,(m+1)\;(l\,x)^{\frac{3}{2}}} + \frac{1\,.\,3}{4\,(m+1)^2\,(l\,x)^{\frac{5}{2}}}\right.$$

$$\left. + \frac{1\,.\,3\,.\,5}{8\,(m+1)^3\,(l\,x)^{\frac{7}{2}}} + \&c.\right\} + \text{const.}$$

183. Instead of making use of the formula in the preceding No. in which, when n is negative, the exponent of $l\,x$ increases without limit, we may, by a method which we shall now proceed to state, reduce the integration of the formula $\int P\,d\,x\,(l\,x)^{-n}$ to another of the form $\int V\,d\,x\,(l\,x)^{-1}$, if n is a whole number, or at least to one of the form $\int V\,d\,x\,(l\,x)^{-n+m}$, where m is the greatest whole number contained in n.

The differential $P\,d\,x\,(l\,x)^{-n}$ may be put under the form $P\,x\,.\frac{d\,x}{x}\,(l x)^{-n}$; but the integral of the factor $\frac{d\,x}{x}\,(l x)^{-n}$ being $\frac{-1}{(n-1)\,(l\,x)^{n-1}}$, we shall get

$$\int \frac{P\,d\,x}{(l\,x)^n} = -\frac{P\,x}{(n-1)\;(l\,x)^{n-1}}$$

$$+\frac{1}{n-1}\int \frac{1}{(l\,x)^{n-1}}\,d\,(P\,x).$$

If we make successively

$d\,(P\,x) = Q\,d\,x,\quad d\,(Q\,x) = R\,d\,x,\quad d\,(R\,x) = S\,d\,x$, &c.

and then change n into $n-1$, $n-2$, &c. we shall obtain, by continuing the reduction indicated above,

$$\int \frac{P\,d\,x}{(1\,x)^n} = -\frac{P\,x}{(n-1)\,(1\,x)^{n-1}} - \frac{Q\,x}{(n-1)\,(n-2)\,(1\,x)^{n-2}}$$

$$-\frac{R\,x}{(n-1)\,(n-2)\,(n-3)\,(1\,x)^{n-3}} - \&c.$$

proceeding with this series till we meet with a term

$$+\frac{1}{(n-1)\,(n-2)\ldots\ldots 1}\int \frac{V\,d\,x}{1\,x},$$ if n be a whole number,

or a term

$$+\frac{1}{(n-1)\,(n-2)\ldots\ldots\ldots(n-m)}\int \frac{V\,d\,x}{(1\,x)^{n-m}},$$

where m is the greatest whole number contained in n.

184. If we take $P=x^m$, we shall have

$$\int \frac{x^m\,d\,x}{(1\,x)^n} = -\frac{x^{m+1}}{(n-1)\,(1\,x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m\,d\,x}{(1\,x)^{n-1}};$$

and continuing this reduction, by changing n into $n-1$, $n-2$, &c. we shall get

$$\left.\begin{aligned}\int \frac{x^m\,d\,x}{(1\,x)^n} = &-\frac{x^{m+1}}{(n-1)\,(1\,x)^{n-1}} - \frac{(m+1)x^{m+1}}{(n-1)\,(n-2)\,(1x)^{n-2}} \\ &-\frac{(m+1)^2\,x^{m+1}}{(n-1)\,(n-2)\,(n-3)\,(1\,x)^{n-3}} \ldots\ldots\ldots\ldots \\ &+\frac{(m+1)^{n-1}}{(n-1)\,(n-2)\ldots\ldots 1}\int \frac{x^m\,d\,x}{1\,x}\ldots\ldots\end{aligned}\right\}$$

by supposing n to be a whole number.

The preceding formula leads us, when $m=1$, to

$$\int \frac{d\,x}{x\,(1\,x)^n} = -\frac{1}{(n-1)\,(1\,x)^{n-1}} + \text{const.}$$

It gives no result when $n=1$; but if we had at the same time $m=-1$ and $n=1$, the differential $\frac{d\,x}{x\,1\,x}$, which arises from that hypothesis, would have for its integral

$l(l x)$ + const., since, by making $l x = u$, it would be transformed into $\frac{du}{u}$.

It should seem that the integral $\int \frac{x^m dx}{lx}$, upon which $\int \frac{x^m dx}{(lx)^n}$ depends, when n is a whole number, ought to constitute a peculiar transcendant. We may reduce it however to a more simple form by making $x^{m+1} = z$; for then we have $x^m dx = \frac{dz}{m+1}$, $lx = \frac{lz}{m+1}$, and consequently $\int \frac{x^m dx}{lx} = \int \frac{dz}{lz}$.

We shall hereafter expand this latter quantity, into a series, which likewise has a relation to exponential functions; for by putting $l z = u$, we shall have $z = e^u$, $dz = e^u$ and $\int \frac{dz}{lz} = \int \frac{e^u du}{u}$, a formula whose exact integral has not yet been found.

185. We shall now proceed to the integration of exponential functions: we shall remark in the first place that the equation $d . a^x = a^x dx\, l a$ (27.), gives

$a^x dx = \frac{1}{la} d . a^x$, and consequently $\int a^x dx = \frac{a^x}{la}$ + const.

We also infer from this that $dx = \frac{d . a^x}{a^x la}$; by this means the differential $V dx$ becoming $\frac{V d . a^x}{a^x la}$, is changed into $\frac{V du}{u la}$, when we make $a^x = a$, and is entirely algebraical with respect to u, when V is an algebraical function of a^x. We find by this transformation

$$\frac{a^x dx}{\sqrt{1+a^{2x}}} = \frac{du}{1a\sqrt{1+u^2}}.$$

186. Let us consider the differential $Pa^x dx$: we shall decompose it into two factors $a^x dx$ and P; the integral of the first is $\frac{1}{1a}a^x$; and we consequently have

$\int Pa^x dx = \frac{1}{1a}Pa^x - \frac{1}{1a}\int a^x dP$. Making

$$dP = Q\,dx,\quad dQ = R\,dx,\quad dR = S\,dx, \text{ \&c.}$$

and continuing the preceding reduction, we shall obtain this series:

$$\int Pa^x dx = \frac{1}{1a}Pa^x - \frac{1}{(1a)^2}Qa^x + \frac{1}{(1a)^3}Ra^x \ldots$$

$$\ldots \pm \frac{1}{(1a)^n}\int Va^x dx,$$

the sign + corresponding to the case in which n is odd, and the sign − to that in which n is even.

By integrating at first the part $P\,dx$ of the proposed differential, $Pa^x\,dx$, and making $\int P\,dx = N$, we get this reduction:

$$\int Pa^x dx = a^x N - (1a)\int Na^x dx;$$

and by proceeding with it, and supposing $\int N\,dx = M$, $\int M\,dx = L$, we find

$$\int Pa^x dx = a^x N - (1a)\,a^x M + (1a)^2 a^x L \ldots \pm (1a)^n \int Ga^x dx.$$

187. The application of the first formula in the preceding will lead to an exact integral, whenever P is a rational and integer function; for in that case the number of the quantities $Q = \frac{dP}{dx}$, $R = \frac{dQ}{dx}$, $S = \frac{dR}{dx}$, &c. will be limited; the last will be constant (18.), and consequently $\int Va^x dx$ will be changed into $V\int a^x dx = V\frac{a^x}{1a}$ + const.

Let us take, for example, $P = x^n$: the equation

$$\int P a^x dx = \frac{1}{1a} P a^x - \frac{1}{1a} \int a^x dP$$

becomes, in this case,

$$\int a^x x^n dx = \frac{a^x x^n}{1a} - \frac{n}{1a} \int a^x x^{n-1} dx;$$

from this we deduce

$$\int a^x x^{n-1} dx = \frac{a^x x^{n-1}}{1a} - \frac{n-1}{1a} \int a^x x^{n-2} dx,$$

and by continuing this process, we obtain, when n is a whole positive number,

$$\int a^x x^n dx =$$

$$\frac{a^x}{1a}\left\{x^n - \frac{n x^{n-1}}{1a} + \frac{n(n-1)}{(1a)^2} x^{n-2} - \frac{n(n-1)(n-2)}{(1a)^3} x^{n-3}\right.$$

$$\left. \ldots\ldots\ldots\ldots \pm \frac{n(n-1)\ldots 1}{(1a)^n}\right\} + \text{const.}$$

188. The second formula in No. 186. only applies to those cases in which the quantities $N = \int P dx$, $M = \int N dx$, $L = \int M dx$, &c. may be obtained algebraically: it may be successfully applied to the example in the preceding No., when n is a whole negative number. We have then

$$P = \frac{1}{x^n}, \quad N = \int P dx = -\frac{1}{(n-1)x^{n-1}}$$

$$\int \frac{a^x dx}{x^n} = -\frac{a^x}{(n-1)x^{n-1}} + \frac{1a}{n-1} \int \frac{a^x dx}{x^{n-1}},$$

from whence

$$\int \frac{a^x dx}{x^n} = -\frac{a^x}{(n-1)x^{n-1}} - \frac{a^x 1a}{(n-1)(n-2)x^{n-2}}$$

$$-\frac{a^x (1a)^2}{(n-1)(n-2)(n-3)x^{n-3}} \cdots - \frac{a^x (1a)^{n-1}}{(n-1)(n-2)\ldots 1 . x}$$

$$+\frac{(1\,a)^{n-1}}{(n-1)(n-2)\,..1}\int\frac{a^x\,d\,x}{x}+\text{const.}$$

We cannot proceed with the reduction beyond $\int\frac{a^x\,d\,x}{x}$; for the equation

$$\int\frac{a^x\,d\,x}{x^n}=-\frac{a^x}{(n-1)x^{n-1}}+\frac{1\,a}{n-1}\int\frac{a^x\,d\,x}{x^{n-1}},$$

gives no result, when $n=1$.

We again in this example meet with the transcendant $\int\frac{a^x\,d\,x}{x}$, of which we have spoken in No. 184; and if we could obtain the expression for it, we should at once obtain the integral $\int a^x\,x^n\,d\,x$ for all the cases in which n is a whole number.

189. When n is a fraction, the two series which we have made use of in the two preceding Nos. will not terminate. If we had for example $n=-\frac{1}{2}$, we should find by the first

$$\int\frac{a^x\,d\,x}{\sqrt{x}}=\frac{a^x}{1\,a\sqrt{x}}\left\{1+\frac{1}{2\,x\,1\,a}+\frac{1\,.\,3}{4\,x^2\,(1\,a)^2}+\frac{1\,.\,3\,.\,5}{8\,x^3\,(1\,a)^3}+\&c.\right\}+\text{const.}$$

and by the second

$$\int\frac{a^x\,d\,x}{\sqrt{x}}=2\,a^x\sqrt{x}\left\{\frac{1}{1}-\frac{2\,x\,1\,a}{1\,.\,3}+\frac{4\,x^2\,(1\,a)^2}{1\,.\,3\,.\,5}-\frac{8\,x^3\,(1\,a)^3}{1\,.\,3\,.\,5\,.\,7}+\&c.\right\}+\text{const.}$$

It is necessary to observe, that in the case in which the proposed formula is $\int a^x\,x^{n+\frac{p}{q}}\,d\,x$, n being a whole number, we may reduce it to $\int a^x\,x^{\frac{p}{q}}\,d\,x$, by means of the first

series, if n be positive, and by means of the second, if n be negative.

190. Replacing a^x in the function $\int P\,a^x\,d\,x$, by its developement (27.), we shall have

$$\int P a^x d\,x = \int P\,d\,x + \frac{1\,a}{1}\int P\,x\,d\,x + \frac{(1\,a)^2}{1\,.\,2}\int P\,x^2\,d\,x +$$
$$\frac{(1\,a)^3}{1.2.3}\int P\,x^3\,d\,x + \frac{(1\,a)^4}{1\,.\,2\,.\,3\,.\,4}\int P\,x^4\,d\,x + \&c.$$

a result which will furnish us with a new developement of $\int P a^x d\,x$, whenever we can determine the functions

$$\int P\,d\,x,\ \int P\,x\,d\,x, \ldots \int P\,x^n\,d\,x,\ \&c.$$

If $P = x^n$, we shall get

$$\int a^x x^n d\,x = \frac{x^{n+1}}{n+1} + \frac{x^{n+2}\,1\,a}{1\,(n+2)} + \frac{x^{n+3}\,(1\,a)^2}{1\,.\,2\,(n+3)}$$
$$+ \frac{x^{n+4}\,(1\,a)^3}{1\,.\,2\,.\,3\,(n+4)} + \&c. + \text{const.}$$

and in this series, we must put $1\,x$ in the place of $\dfrac{x^{n+i}}{n+i}$, when n is a whole negative number, and equal to $-i$.

The application of this method to the integral $\displaystyle\int \frac{a^x\,d\,x}{x}$, gives the developement

$$\int \frac{a^x\,d\,x}{x} = 1\,x + \frac{x\,1\,a}{1\,.\,1} + \frac{x^2\,(1\,a)^2}{1\,.\,2\,.\,2} + \frac{x^3\,(1\,a)^3}{1\,.\,2\,.\,3\,.\,3}$$
$$+ \frac{x^4\,(1\,a)^4}{1\,.\,2\,.\,3\,.\,4\,.\,4} + \&c. + \text{const.}$$

If we suppose $a^x = z$, from which we get $x = \dfrac{1\,z}{1\,a}$, and $1\,x = 1\,1\,z - 1\,1\,a$, this result will be transposed into

$$\int \frac{d\,z}{1\,z} = 1\,1\,z + \frac{1\,z}{1} + \frac{1}{2}\,\frac{(1\,z)^2}{1\,.\,2} + \frac{1}{3}\,\frac{(1\,z)^3}{1\,.\,2\,.\,3}$$
$$+ \frac{1}{4}\,\frac{(1\,z)^4}{1\,.\,2\,.\,3\,.\,4} + \&c. + \text{const.}$$

191. There is yet another method of integrating an exponential function, such as $\frac{e^x x\,dx}{(1+x)^2}$; it is to compare it with the differential of the function $e^x P$, which is $e^x(dP+P\,dx)$, and in which P represents an algebraical function of x. The example proposed being very simple, it is sufficient to make $1+x=z$: we have then

$$\frac{e^x x\,dx}{(1+x)^2}=\frac{e^{z-1}(z-1)\,dz}{z^2}=\frac{1}{e}\left\{e^z\left(\frac{1}{z}dz-\frac{dz}{z^2}\right)\right\};$$

and with a little attention we easily discover that $\frac{dz}{z^2}$ being the differential of $\frac{1}{z}$, we must take $P=\frac{1}{z}$, from which we get the integral $\frac{e^z}{ez}$ + const. Replacing z by its value, we find $\int\frac{e^x x\,dx}{(1+x)^2}=\frac{e^x}{1+x}$ + const.

On the Integration of Circular Functions.

192. Let us take the formula $\int X\,dx$ arc $(\sin=x)$; if we at first integrate the factor $X\,dx$, and observe that $d\,.$ arc $(\sin=x)=\frac{dx}{\sqrt{1-x^2}}$, making also $\int X\,dx=V$, we shall have

$$\int X\,dx\,.\,\text{arc}\,(\sin=x)=V.\,\text{arc}\,(\sin=x)-\int\frac{V\,dx}{\sqrt{1-x^2}}:$$

the integration of the proposed formula will then be reduced to that of an algebraical function, if V be algebraical.

Taking for example $\int x^n\,d\,x$ arc $(\sin = x)$, we shall find $V = \frac{x^{n+1}}{n+1}$, and $\int x^n\,d\,x$ arc $(\sin = x) =$

$$\frac{x^{n+1}}{n+1} \text{arc}\,(\sin = x) - \frac{1}{n+1}\int \frac{x^{n+1}\,d\,x}{\sqrt{1-x^2}}:$$

the formula $\int \frac{x^{n+1}\,d\,x}{\sqrt{1-x^2}}$ has been already considered in Nos. 173. and 174.

193. Since $d\,.\,\text{arc}\,(\cos = x) = -\frac{d\,x}{\sqrt{1-x^2}}$,

$$d.\ \text{arc}\,(\tan = x) = \frac{d\,x}{1+x^2},$$

we shall have, in the same manner as above,

$$\int X\,d\,x\,.\,\text{arc}\,(\cos = x) = V.\ \text{arc}\,(\cos = x) + \int \frac{V\,d\,x}{\sqrt{1-x^2}}$$

$$\int X\,d\,x\,.\,\text{arc}\,(\tan = x) = V.\ \text{arc}\,(\tan = x) - \int \frac{V\,d\,x}{1+x^2};$$

and the integration of these formulæ will depend merely upon that of an algebraical function, whenever V is algebraical.

194. If z represents an arc, whose sine, cosine, or tangent are expressed by a function of x, that is to say, if we have $d\,z = X_1\,d\,x$, X_1 being a given function of x, we shall obtain $\int X\,z^n\,d\,x$ by a process similar to that in the preceding articles. Let $\int X\,d\,x = V$, we have

$$\int z^n\,X\,d\,x = V z^n - n\int V z^{n-1}\,d\,z;$$

and substituting for $d\,z$ its value, we get

$$\int z^n\,X\,d\,x = V z^n - n\int z^{n-1}\,V X_1\,d\,x.$$

By following this method, we shall diminish continually the exponent of z, which will finally $= 0$, whenever n is a whole positive number.

The most simple case is that in which $X=1$, or where z is an arc whose sine $=x$; we then find succesively

$$X_1=\frac{1}{\sqrt{1-x^2}},\ V=n,\ \int z^n\,dx=x\,z^n-n\int z^{n-1}\frac{x\,dx}{\sqrt{1-x^2}},$$

$$=\int\frac{x\,dx}{\sqrt{1-x^2}}=-\sqrt{1-x^2},\int z^{n-1}\,dx=-z^{n-1}\sqrt{1-x^2}+(n-1)\int z^{n-2}$$

and these values give $\int z^n\,dx=$

$$z^n x+n\,z^{n-1}\sqrt{1-x^2}-n\,(n-1)\,z^{n-2}x$$
$$-n\,(n-1)\,(n-2)\,z^{n-3}\sqrt{1-x^2}+\&c.$$

a series which terminates when n is a whole positive number.

If we had $X\,dx=dz$, or $X=X_1$, the integral $\int X\,z^n dx$ would be changed into $\int z^n\,dz=\frac{z^{n+1}}{n+1}+\text{const.}$; and if we substitute for z^n any algebraical function of z, the integral considered with reference to z would be comprehended in some one or other of the formulæ which are enumerated in the preceding pages.

195. Before we proceed to the more general functions of the quantities z, $\sin z$, $\cos z$, &c. we must call to mind that by Nos. 32, 33, we have

$$d\,.\sin nz=\ n\,dz\cos nz,\quad\text{and}\ \int dz\cos nz=\ \frac{1}{n}\sin\ nz+\text{const.}$$

$$d\,.\cos nz=-n\,dz\sin nz,\qquad \int dz\sin nz=-\frac{1}{n}\cos nz+\text{const.}$$

$$d\,.\tan.nz=\ \frac{n\,dz}{(\cos nz)^2},\qquad \int\frac{dz}{(\cos nz)^2}=\ \frac{1}{n}\tan.\,nz+\text{const.}$$

$$d\,.\cot nz=-\frac{n\,dz}{(\sin nz)^2},\qquad \int\frac{dz}{(\sin nz)^2}=-\frac{1}{n}\cot nz+\text{const.}$$

$$d\,.\sec nz=\ \frac{n\,dz\sin nz}{(\cos nz)^2},\qquad \int\frac{dz\sin nz}{(\cos nz)^2}=\ \frac{1}{n}\sec nz+\text{const.}$$

$$=\frac{1}{n\cos nz}+\text{const.}$$

$$d\,.\,\text{cosec}\, n z = -\frac{n\,d z \cos n z}{(\sin n z)^2}, \quad \int \frac{d z \cos n z}{(\sin n z)^2} = -\frac{1}{n}\text{cosec}\, n z + \text{const}$$
$$= -\frac{1}{n \sin n z} + \text{const.}$$

196. From these integrations results that of the expressions

$$d z\,(A + B \sin z + C \sin 2 z + D \sin 3 z + \&c.)$$
$$d z\,(A + B \cos z + C \cos 2 z + D \cos 3 z + \&c.)$$

which give

$$A z - B \cos z - \tfrac{1}{2} C \cos 2 z - \tfrac{1}{3} D \cos 3 z - \&c. + \text{const.}$$
$$A z + B \sin z + \tfrac{1}{2} C \sin 2 z + \tfrac{1}{3} D \sin 3 z + \&c. + \text{const.}$$

197. It is of great importance to observe, that we may reduce every rational function of $\sin z$ and $\cos z$, to terms of the form

$$A \sin m z, \quad \text{or } A \cos m z.$$

This operation by which the integration of all differentials of this kind is reducible to that of the formula in the preceding No. likewise facilitates the numerical estimation of the formulæ resulting; for in many cases the use of the sines and the cosines of multiple arcs is more commodious than that of the powers of those quantities.

The formulæ (Trig. 26.)

$$\sin a \cos b = \tfrac{1}{2} \sin (a+b) + \tfrac{1}{2} \sin (a-b)$$
$$\cos a \sin b = \tfrac{1}{2} \sin (a+b) - \tfrac{1}{2} \sin (a-b)$$
$$\sin a \sin b = -\tfrac{1}{2} \cos (a+b) + \tfrac{1}{2} \cos (a-b)$$
$$\cos a \cos b = \tfrac{1}{2} \cos (a+b) + \tfrac{1}{2} \cos (a-b)$$

are the elements of the transformation to which we have just alluded; for if we take $b=a$ in the two last, they will give

$$\sin a^2 = -\tfrac{1}{2} \cos 2 a + \tfrac{1}{2}$$
$$\cos a^2 = \tfrac{1}{2} \cos 2 a + \tfrac{1}{2},$$

by observing that $\cos (a-b) = \cos 0 = 1$. Again, since

$$\sin a^3 = \sin a^2 . \sin a, \quad \cos a^3 = \cos a^2 . \cos a,$$

we shall have

$$\begin{aligned}\sin a^3 &= (-\tfrac{1}{2}\cos 2a + \tfrac{1}{2}) \sin a \\ &= -\tfrac{1}{2}\cos 2a \sin a + \tfrac{1}{2}\sin a \\ \cos a^3 &= (\tfrac{1}{2}\cos 2a + \tfrac{1}{2}) \cos a \\ &= \tfrac{1}{2}\cos 2a \cos a + \tfrac{1}{2}\cos a.\end{aligned}$$

These two results include the products

$$\cos 2a \sin a, \text{ and } \cos 2a \cos a,.$$

which we shall be able to express in terms of sines of the multiples of a, by the last of the formulæ enumerated above, by simply making $b = 2a$.

This operation is very simple, and it is evident that in this way, as we have just seen, we may proceed

from $\sin a^3$ to $\sin a^4$, to $\sin a^5$, &c.

from $\cos a^3$ to $\cos a^4$, to $\cos a^5$, &c.

198. Instead of deriving formulæ for particular cases, we shall proceed to deduce general ones, from the equations (164.)

$$(\cos x + \sqrt{-1} \sin x)^n = \cos n x + \sqrt{-1} \sin n x,$$

$$(\cos x - \sqrt{-1} \sin x)^n = \cos n x - \sqrt{-1} \sin n x.$$

By adding these two equations together, we find

$$\cos n x = \frac{(\cos x + \sqrt{-1} \sin x)^n + (\cos x - \sqrt{-1} \sin x)^n}{2};$$

and again subtracting the second from the first, we get

$$\sin n x = \frac{(\cos x + \sqrt{-1} \sin x)^n - (\cos x - \sqrt{-1} \sin x)^n}{2\sqrt{-1}}.$$

These expressions, although affected with imaginary quantities, are not on that account the less real, since all the

imaginary parts disappear by expanding the powers of the binomials contained in them. In fact, we have

$$(\cos x+\sqrt{-1}\sin x)^n=\cos x^n+\frac{n}{1}\sqrt{-1}\cos x^{n-1}\sin x$$

$$-\frac{n(n-1)}{1.2}\cos x^{n-2}\sin x^2-\&c.$$

$$(\cos x-\sqrt{-1}\sin x)^n=\cos x^n-\frac{n}{1}\sqrt{-1}\cos x^{n-1}\sin x$$

$$-\frac{n(n-1)}{1.2}\cos x^{n-2}\sin x^2+\&c.$$

and substituting these series in the above-mentioned expressions, we find

$$\cos nx=\cos x^n-\frac{n(n-1)}{1.2}\cos x^{n-2}\sin x^2$$

$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos x^{n-4}\sin x^4-\&c.$$

$$\sin nx=\frac{n}{1}\cos x^{n-1}\sin x-\frac{n(n-1)(n-2)}{1.2.3}\cos x^{n-3}\sin x^3$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\cos x^{n-5}\sin x^5-\&c.$$

199. By the formulæ in the preceding No. we develope the sines and cosines of multiple arcs, in terms which involve the powers of the sine and cosine of the simple arc; let us now proceed to the inverse problem, or that in which it is required to express the powers of the sine and cosine of the simple arc, by the sines and cosines of its multiples:

Let

$$\cos x+\sqrt{-1}\sin x=u$$

$$\cos x-\sqrt{-1}\sin x=v,$$

we have

$$\cos x = \tfrac{1}{2}(u+v), \quad \sin x = \frac{1}{2\sqrt{-1}}(u-v);$$

from which we obtain, in the first place,

$$\cos x^n = \frac{1}{2^n}(u+v)^n.$$

By expanding the second member of this equation, we sh get

$$\cos x^n = \frac{1}{2^n}\Big\{u^n + \frac{n}{1}u^{n-1}v + \frac{n(n-1)}{1.2}u^{n-2}v^2$$
$$+ \frac{n(n-1)(n-2)}{1.2.3}u^{n-3}v^3 + \&c.$$

but in the expression $(u+v)^n$, we may change v into u, a reciprocally, which will give

$$\cos x^n = \frac{1}{2^n}\Big\{v^n + \frac{n}{1}v^{n-1}u + \frac{n(n-1)}{2}v^{n-2}u^2$$
$$+ \frac{n(n-1)(n-2)}{1.2.3}v^{n-3}u^3 + \&c.\Big\}$$

and by adding together the two results, we have

$$2\cos x^n = \frac{1}{2^n}\Big\{u^n + v^n + \frac{n}{1}(u^{n-1}v + v^{n-}$$
$$\frac{n(n-1)}{1.2}(u^{n-2}v^2 + v^{n-2}u^2) + \frac{n(n-1)(n-2)}{1.2.3}(u^{n-3}v^3 + v^{n-3}u^3$$

We may give to this equation the following form:

$$2^{n+1}\cos x^n = \Big\{u^n + v^n + \frac{n}{1}uv(u^{n-2} + v^{n-}$$
$$\frac{n(n-1)}{1.2}u^2v^2(u^{n-4} + v^{n-4}) + \frac{n(n-1)(n-2)}{1.2.3}u^3v^3(u^{n-5} + v^{n-}$$

but the question

$$\cos nx = \tfrac{1}{2}(\cos x + \sqrt{-1}\sin x)^n + \tfrac{1}{2}(\cos x - \sqrt{-1}\sin$$
$$= \tfrac{1}{2}u^n + \tfrac{1}{2}$$

being true for any value of n (198.), gives us

$$u^n + v^n = 2 \cos n x,$$

and in general,

$$u^{n-m} + v^{n-m} = \cos (n-m) x;$$

besides, it is readily seen, that $u v = 1$: we shall have, therefore

$$2^{n+1} \cos x^n =$$

$$\left\{ 2 \cos n x + \frac{2 n}{1} \cos (n-2) x + \frac{2 n (n-1)}{1.2} \cos (n-4) x \right.$$
$$\left. + \frac{2 n (n-1)(n-2)}{1.2.3} \cos (n-6) x + \&c. \right\}.$$

or what is the same thing, dividing the whole by 2,

$$2^n \cos x^n = \left\{ \cos n x + \frac{n}{1} \cos (n-2) x + \frac{n(n-1)}{1.2} \cos (n-4) x \right.$$
$$\left. + \frac{n(n-1)\ (n-2)}{1.2.3} \cos (n-6) x + \&c. \right\}$$

a formula which is always applicable, whatever be the value of n.

By continuing this formula, as in that of Newton for a binomial, we shall meet with the cosines of negative arcs, whenever n is a whole number, and these are precisely the same as those of the corresponding positive arcs: we shall put, therefore, $\cos (m-n) x$ in the place of $\cos (n-m) x$; and in this case the formula will admit of abridgement.

In the developement of $(u+v)^n$, when n is a whole number, those terms, which are equidistant from the extremes, have the same coefficient; the same remark will apply likewise to the formula

$$\cos n x + \frac{n}{1} \cos (n-2) x + \frac{n(n-1)}{1.2} \cos (n-4) x$$

$$+ \frac{n(n-1)\ (n-2)}{1.2.3} \cos (n-6) x + \&c.$$

and the cosines also which are equidistant from its

extremes, are equal to each other; for the first term $\cos n x$ corresponds to the last term $\cos (n-2n)x$, or $\cos - n x$, which is the same as $\cos n x$; the term involving $\cos (n-2m)x$, which has m before it, corresponds to the term involving $\cos (-n+2m)x$, which has m after it; and since

$$\cos (-n+2m)x = \cos -(n-2m)x = \cos (n-2m)x,$$

we may omit the terms which include the cosines of negative arcs, by doubling all those which include the cosines of positive arcs.

We may therefore, by stopping at the term where the arcs become negative, write the formula thus ... $2^n \cos x^n =$

$$\left\{2\cos n x + \frac{2n}{1}\cos (n-2)x + \frac{2n(n-1)}{1.2}\cos (n-4)x + \&c.\right\}$$

We ought however to observe, that when n is an even number, the original formula involves a term equidistant from each of the extremes, and which

is represented by $\dfrac{n(n-1)\ldots\ldots\left(n-\frac{n}{2}+1\right)}{1.2\ldots\ldots\ldots\ldots\frac{n}{2}}\cos (n-n)x$

and because $\cos 0 = 1$, this reduces itself to

$$\frac{n(n-1)\ldots\ldots\left(\frac{n}{2}+1\right)}{1.2\ldots\ldots\ldots\frac{n}{2}}$$

and since this term has no other corresponding to it, it ought not to be doubled like the others, unless we previously divide it by 2, as

$$\frac{1}{2}\,\frac{n(n-1)\ldots\ldots\frac{n}{2}+1}{1.2\ldots\ldots\ldots\ldots\frac{n}{2}}.$$

By such a method we shall finally get $2^{n-1}\cos x^n =$

$$\left\{\cos x + \frac{n}{1}\cos(n-2)x + \frac{n(n-1)}{1.2}\cos(n-4)x + \frac{n(n-1)(n-2)}{1.2.3}\cos(n-6)x + \&c.\right\},$$

taking care to terminate the series when we meet with the negative arcs, and also when n is even, to take only half the coefficient of the cosine of the arc, which is evanescent. By attending to these observations, it will be very easy to construct the following table:

$\cos x = \cos^1 x$

$2\cos x^2 = \cos 2x + 1$

$4\cos x^3 = \cos 3x + 3\cos x$

$8\cos x^4 = \cos 4x + 4\cos 2x + 3$

$16\cos x^5 = \cos 5x + 5\cos 3x + 10\cos x$

$32\cos x^6 = \cos 6x + 6\cos 4x + 15\cos 2x + 10$

$64\cos x^7 = \cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x$

&c.

200. In order to determine $\sin x^n$, we shall make use of the equation

$$\sin x = \frac{1}{2\sqrt{-1}}(u-v),$$

and we shall find

$$\sin x^n = \frac{1}{(2\sqrt{-1})^n}(u-v)^n,$$

or

$$\sin x^n = \frac{1}{(2\sqrt{-1})^n}\left\{u^n - \frac{n}{1}u^{n-1}v + \frac{n(n-1)}{1.2}u^{n-2}v^2 - \frac{n(n-1)(n-2)}{1.2.3}u^{n-3}v^3 + \&c.\right\}$$

1st. Let n be an even number, or a fraction with an even numerator; in this case $(u-v)^n = (v-u)^n$; and therefore we shall also have

$$\sin x^n = \frac{1}{(2\sqrt{-1})^n}(v-u)^n.$$

By expanding the second member of this equation, and adding it to the former expression for this quantity, there will result

$$2\sin x^n = \frac{1}{(2\sqrt{-1})^n}\left\{\begin{aligned} &u^n+v^n-\frac{n}{1}(u^{n-1}v+v^{n-1}u)\\ &+\frac{n(n-1)}{1.2}(u^{n-2}v^2+v^{n-2}u^2)\\ &-\frac{n(n-1)(n-2)}{1.2.3}(u^{n-3}v^3+v^{n-3}u^3)+\&c.\end{aligned}\right\}$$

or, what is the same thing

$$2\sin x^n = \frac{1}{(2\sqrt{-1})^n}\left\{\begin{aligned} &u^n+v^n-\frac{n}{1}uv(u^{n-2}+v^{n-2})\\ &+\frac{n(n-1)}{1.2}u^2v^2(u^{n-4}+v^{n-4})\\ &-\frac{n(n-1)(n-2)}{1.2.3}u^3v^3(u^{n-6}+v^{n-6})+\&c.\end{aligned}\right\}$$

a result which is identical, except in signs, with that in the preceding No.; we may therefore write it as follows:

$$(2\sqrt{-1})^n \sin x^n =$$

$$\cos nx - \frac{n}{1}\cos(n-2)x + \frac{n(n-1)}{1.2}\cos(n-4)x$$
$$- \frac{n(n-1)(n-2)}{1.2.3}\cos(n-6)x + \&c.$$

The imaginary part disappears, since n is an even number; and we have $(2\sqrt{-1})^n = \pm 2^n$; the upper sign prevailing when n is divisible by 4, and the lower when it is merely divisible by 2.

We may apply to the second member of the equation the same reasoning as in the preceding article; and since n

is a whole number, we may from thence conclude that it is only necessary to confine ourselves to those terms which involve positive arcs, provided that we take the double of each. Likewise, since n is even, there will be a term containing $\cos (n-n)x=1$, which it will not be necessary to double; dividing the whole by 2, we shall get

$$\pm 2^{n-1} \sin x^n = \left\{ \cos nx - \frac{n}{1}\cos(n-2)x + \frac{n(n-1)}{1.2}\cos(n-4)x - \frac{n(n-1)(n-2)}{1.2.3}\cos(n-6)x + \&c. \right\}$$

taking care to stop when we come to an arc which is equal to nothing, and only to take one half of the coefficient of that term.

2d. If n be an odd number, we then have

$$(v-u)^n = -(u-v)^n,$$

consequently

$$\sin x^n = \frac{1}{(2\sqrt{-1})^n}(u-v)^n = -\frac{1}{(2\sqrt{-1})^n}(v-u)^n;$$

and the developement of the second expression gives

$$\sin x^n = \frac{1}{(2\sqrt{-1})^n}\left\{ -v^n + \frac{n}{1}v^{n-1}u - \frac{n(n-1)}{1.2}v^{n-2}u^2 + \frac{n(n-1)(n-2)}{1.2.3}v^{n-3}u^3 - \&c. \right\}$$

Adding this to the first, and making necessary reductions, we shall find

$$2\sin x^n = \frac{1}{(2\sqrt{-1})^n}\left\{ u^n - v^n - \frac{n}{1}uv(u^{n-2}-v^{n-2}) + \frac{n(n-1)}{1.2}u^2v^2(u^{n-4}-v^{n-4}) - \frac{n(n-1)(n-2)}{1.2.3}u^3v^3(u^{n-6}-v^{n-6}) + \&c. \right\}$$

[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

The imaginary factor does not affect this formula more than the preceding; for n being an odd number,

$$(2\sqrt{-1})^{n-1} = \pm 2^{n-1},$$

the upper sign applying when $n-1$ is a multiple of 4, and the under when $n-1$ is merely a multiple of 2.

We may here also limit ourselves to those terms which involve the sines of positive arcs, by taking the double of each of them; for it is obvious, from the same reasons as in the preceding articles, that the coefficients of those terms are equal, which are equidistant from the extremes; and also, that one of them involves the sign of a negative, and the other the sign of a positive arc. Likewise, since the number of terms is even, and since they are alternately negative and positive, the terms corresponding to each other will have contrary signs; but the sign of a ne-

gative arc is itself negative. The difference of the sign is thus corrected, and the two terms may be combined into one.

From these considerations, and by dividing the whole by 2, we shall get

$$\pm 2^{n-1} \sin x^n = \left\{ \sin n\,x - \frac{n}{1} \sin (n-2)\,x \right.$$

$$\left. + \frac{n(n-1)}{1.2} \sin (n-4)\,x - \frac{n(n-1)(n-2)}{1.2.3} \sin (n-6)\,x + \&c. \right\}$$

We shall be able to deduce from the two formulæ in this article the values contained in the following table:

$$\begin{aligned}
\sin x &= \sin x \\
2 \sin x^2 &= -\cos 2\,x + 1 \\
4 \sin x^3 &= -\sin 3\,x + 3 \sin x \\
8 \sin x^4 &= \cos 4\,x - 4 \cos 2\,x + 3 \\
16 \sin x^5 &= \sin 5\,x - 5 \sin 3\,x + 10 \sin x \\
32 \sin x^6 &= -\cos 6\,x + 6 \cos 4\,x - 15 \cos 2\,x + 10 \\
64 \sin x^7 &= -\sin 7\,x + 7 \sin 5\,x - 21 \sin 3\,x + 35 \sin x \\
&\&c.
\end{aligned}$$

This is the case when n is a whole number: if n be fractional, it will be necessary to have recourse to the first formula in the preceding No.; for by making $x = \frac{\pi}{2} - z$, we would have $\cos x = \sin z$; and consequently the expression for $\cos x^n$ in terms of the cosines of the multiples of x, would become that for $\sin z^n$ in terms of the cosines of the multiples of $\frac{\pi}{2} - z$, or of the compliment of the arc z.

201. Let it be required to integrate the differential $dx \cos x^4$; we shall first deduce from the formulæ in No. 199

$$\cos x^4 = \frac{1}{8} \cos 4\,x + \frac{1}{2} \cos 2\,x + \frac{3}{8},$$

and we shall get

$$\int dx \cos x^4 = \frac{1}{8}\int dx \cos 4x + \frac{1}{2}\int dx \cos 2x + \frac{3}{8}\int dx$$

$$= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3x}{8} + \text{const.}$$

This example shews very clearly the method of integrating all differentials of this kind.

212. The formulæ

$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$$

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$$

changing the functions of sines and cosines into exponentials, reduce also the integration of the one to that of the other.

We may likewise change the differential $dx \sin x^m \cos x^n$ into another, which is comprised among the binomial differentials: it is sufficient for this purpose to make $\sin x = z$, from which there results

$$\cos x = \sqrt{1-z^2}, \quad dx = \frac{dz}{\sqrt{1-z^2}} \ (35.);$$

and we finally get

$$\int dx \ \sin x^m \cos x^n = \int z^m dz\,(1-z^2)^{\frac{n-1}{2}}.$$

By applying to this last expression the reductions indicated in Nos. 170 – 172, we shall be able to deduce a complete algebraical integral, if m be an odd number; if m be even, we shall make it dependent on

$$\int dz\,(1-z^2)^{\frac{n-1}{2}},$$

and this is reducible to $\int \frac{dz}{\sqrt{1-z^2}}$, or to an arc of a

circle, if n be a whole number. In all other cases, we shall reduce the integral of the proposed formula to that of the analogous differential of the most simple form.

It is evident that we may, in a similar manner, transform differentials which involve other trigonometrical lines.

203. The formulæ (A), (B), (C), and (D), in Nos. 170, 171, 172, may be easily transformed, so as to comprise the differential $dz \sin z^m \cos z^n$; but we arrive immediately at the same result, by decomposing the differential into factors.

If we at first put it under the form $dz \sin z \cos z^n \sin z^{m-1}$, the first factor $dz \sin z \cos z^n$ may be integrated, because $dz \sin z = -d . \cos z$; and we shall find

$$\int dz \sin z^m \cos z^n = \int dz \sin z \cos z^n \sin z^{m-1} =$$

$$-\frac{1}{n+1} \cos z^{n+1} \sin z^{m-1} + \frac{m-1}{n+1} \int dz \cos z^{n+2} \sin z^{m-2};$$

and since $\cos z^{n+2} = \cos z^n . \cos z^2 = \cos z^n (1 - \sin z^2)$, we get

$$\int dz \cos z^{n+2} \sin z^{m-2} = \int dz \cos z^n \sin z^{m-2} - \int dz \cos z^n \sin z^m.$$

Substituting this in the first equation, and finding the value of $\int dz . \cos z^n \sin z^m$, there will result ($A$)

$$\int dz \sin z^m \cos z^n = -\frac{\sin z^{m-1} \cos z^{n+1}}{m+n} + \frac{m-1}{m+n} \int dz \sin z^{m-2} \cos z^n$$

We have likewise $\int dz \cos z^n \sin z^m = \int dz \cos z \sin z^m . \cos z^{n-1} =$

$$\frac{1}{m+1} \sin z^{m+1} \cos z^{n-1} + \frac{n-1}{m+1} \int dz \sin z^{m+2} \cos z^{n-2};$$

also

$$\sin z^{m+2} = \sin z^m . \sin z^2 = \sin z^m (1 - \cos z^2),$$

and consequently

$$\int dz \sin z^{m+2} \cos z^{n-2} = \int dz \sin z^m \cos z^{n-2} - \int dz \sin z^m \cos z^n.$$

This value, substituted in the expression for $\int dz \sin z^m \cos z^n$, leads to an equation from which we deduce (B)

$$\int d\,z \sin z^m \cos z^n = \frac{\sin z^{m+1} \cos z^{n-1}}{m+n} + \frac{n-1}{m+n} \int d\,z \sin z^m \cos z^n$$

By changing successively m into $m-2$, $m-4$, &c. in the first formula, n into $n-2$, $n-4$, &c. in the second, and by employing these formulæ alternately, we shall succeed in taking away from the exponents m and n beneath the sign $\int$, the greatest multiples of 2, which are respectively contained in them. We thus obtain an algebraical integral of the formula $\int d\,z \sin z^m \cos z^n$, when one of the exponents m or n is odd; but when m and n are both even, we terminate with the differential $d\,z \cos z^0 \sin z^0$, whose integral involves the arc z.

If, for example, we apply these formulæ to $\int d\,z \sin z^4 \cos z^2$, the first will give

$$\int d\,z \sin z^4 \cos z^2 = -\frac{\sin z^3 \cos z^3}{6} + \frac{3}{6} \int d\,z \sin z^2 \cos z^2;$$

and from the second we shall have

$$\int d\,z \sin z^2 \cos z^2 = \frac{\sin z^3 \cos z}{4} + \int d\,z \sin z^2 \cos z^2;$$

returning again to the first, making in it $m=2$ and $n=0$, we shall obtain

$$\int d\,z \sin z^2 = -\frac{\sin z \cos z}{2} + \frac{1}{2} \int d\,z = -\frac{\sin z \cos z}{2} + \frac{z}{2}:$$

and now ascending from this integral to that of the proposed differential, there will result

$$\int d\,z \sin z^4 \cos z^2 = -\frac{1}{6} \sin z^3 \cos z^3 + \frac{3\,.\,1}{6\,.\,4} \sin z^3 \cos z$$

$$-\frac{3\,.\,1\,.\,1}{6\,.\,4\,.\,2} \sin z \cos z + \frac{3\,.\,1\,.\,1}{6\,.\,4\,.\,2} z + \text{const.}$$

The differential $\int d\,z \sin z^4 \cos z^3$, being treated in the same way, would lead us successively to

$$\int d\,z \sin z^4 \cos z^3 = -\frac{\sin z^3 \cos z^4}{7} + \frac{3}{7} \int d\,z \sin z^2 \cos z^3$$

$$\int d z \sin z^2 \cos z^3 = \frac{\sin z^3 \cos z^2}{5} + \frac{2}{5} \int d z \sin z^2 \cos z$$

$$\int d z \sin z^2 \cos z = -\frac{\sin z \cos z^2}{3} + \frac{1}{3} \int d z \cos z;$$

and since $\int d z \cos z = -\sin z$, we should finally get

$$\int d z \sin z^4 \cos z^3 = -\frac{1}{7} \sin z^3 \cos z^4 + \frac{3 . 1}{7 . 5} \sin z^3 \cos z^2$$
$$-\frac{3 . 2 . 1}{7 . 5 . 3} \sin z \cos z^2 - \frac{3 . 2 . 1}{7 . 5 . 3} \sin z + \text{const.}$$

204. This last example, and all those in which one of the exponents m and n is an odd number, are reducible to integral algebraical functions, by observing that

$$\int d z \sin z^{2p+1} \cos z^q = \int d z \sin z . \cos z^q (\sin z^2)^p$$
$$\int d z \sin z^p \cos z^{2q+1} = \int d z \cos z . \sin z^p (\cos z^2)^q,$$

also that

$$(\sin z^2)^p = (1 - \cos z^2)^p, \quad (\cos z^2)^q = (1 - \sin z^2)^q,$$

and finally that

$$d z \sin z = -d . \cos z, \quad d z \cos z = d . \sin z.$$

We from hence arrive at

$$-\int u^q d u (1 - u^2)^p, \quad \int u^p d u (1 - u^2)^q,$$

by making $\cos z = u$, or $\sin z = u$; and the integrals of these differentials are obtained by expanding the integer powers of $1 - u^2$.

205. When $n = 0$, the formula (A) becomes

$$\int d z \sin z^m = -\frac{\sin z^{m-1} \cos z}{m} + \frac{m-1}{m} \int d z \sin z^{m-2},$$

which finally leads us to $\int d z \sin z$, or $\int d z$, according as m is odd or even.

The formula (B), when we make $m = 0$, becomes

$$\int d z \cos z^n = \frac{\sin z \cos z^{n-1}}{n} + \frac{n-1}{n} \int d z \cos z^{n-2},$$

which leads to $\int dz\cos z$, or to $\int dz$, according as n is odd or even.

206. The reductions indicated in No. 203, may be likewise applied to the two differentials

$$\frac{dz\sin z^m}{\cos z^n},\qquad \frac{dz\cos z^n}{\sin z^m}:$$

but it is proper to remark, that it is only necessary to consider one of them; for if we make $z=\frac{\pi}{2}-y$, we shall have $dz=-dy$, $\sin z=\cos y$, $\cos z=\sin y$; and the substitution of these values in the first expression will give it the form of the second, and reciprocally.

By changing $+n$ into $-n$, in the formula (A) cited in the No. above-mentioned, there will result

$$\int\frac{dz\sin z^m}{\cos z^n}=-\frac{1}{m-n}\frac{\sin z^{m-1}}{\cos z^{n-1}}+\frac{m-1}{m-n}\int\frac{dz\sin z^{m-2}}{\cos z^n}.$$

We readily see that this reduction leads to

$$\int\frac{dz\sin z}{\cos z^n},\ \text{or to}\int\frac{dz}{\cos z^n},$$

according as m is odd or even.

The first of these expressions becomes $-\int\frac{du}{u^n}$, when we make $\cos z=u$, and is easily integrated: the integral of the second is obtained by a reduction, which we shall now proceed to consider.

If we make n negative in the formula (B) of No. 203, we shall find

$$\int\frac{dz\sin z^m}{\cos z^n}=\frac{1}{m-n}\frac{\sin z^{m+1}}{\cos z^{n+1}}-\frac{(n+1)}{m-n}\int\frac{dz\sin z^m}{\cos z^{n+2}},$$

from whence we shall get

$$\int\frac{dz\sin z^m}{\cos z^{n+2}}=\frac{1}{n+1}\frac{\sin z^{m+1}}{\cos z^{n+1}}-\frac{m-n}{n+1}\int\frac{dz\sin z^m}{\cos z^n};$$

and changing n into $n-2$, there will result

$$\int \frac{dz \sin z^m}{\cos z^n} = \frac{1}{n-1} \frac{\sin z^{m+1}}{\cos z^{n-1}} - \frac{m-n+2}{n-1} \int \frac{dz \sin z^m}{\cos z^{n-2}}.$$

This formula comprehending the case in which $m=0$, may be applied to the integration of $\frac{dz}{\cos z^n}$; and in the several forms it terminates in

$$\int \frac{dz \sin z^m}{\cos z}, \text{ or} \int dz \sin z^m,$$

according as n is odd or even.

The second of these integrals has been considered in No. 205; and the first, by means of the formula (A), in No. 203, making $n=-1$, is reducible to

$$\int \frac{dz \sin z}{\cos z}, \text{ or} \int \frac{dz}{\cos z},$$

according as m is odd or even: we shall consider each of these integrals separately in the following pages.

We may also observe, that the first of these reductions becomes inapplicable, when $m=n$, and the second when $n=1$; and that this last gives the integral immediately, when $m=n-2$.

207. Let us take, for another example, $\int \frac{dz}{\sin z^m \cos z^n}$; by changing at the same time $+m$ into $-m$, and $+n$ into $-n$, in the formula given in No. 203, we shall find

$$\frac{dz}{\sin z^m \cos z^n} = \frac{1}{m+n} \frac{1}{\sin z^{m+1} \cos z^{n-1}} + \frac{m+1}{m+n} \int \frac{dz}{\sin z^{m+2} \cos z^n}$$

$$\frac{dz}{\sin z^m \cos z^n} = -\frac{1}{m+n} \frac{1}{\sin z^{m-1} \cos z^{n+1}} + \frac{n+1}{m+n} \int \frac{dz}{\sin z^m \cos z^{n+1}}$$

from whence there will result

$$\int \frac{dz}{\sin z^{m+2} \cos z^{n}} = -\frac{1}{m+1} \frac{1}{\sin z^{m+1} \cos z^{n-1}} + \frac{m+n}{m+1} \int \frac{d}{\sin z^{m}}$$

$$\int \frac{dz}{\sin z^{m} \cos z^{n+2}} = \frac{1}{n+1} \frac{1}{\sin z^{m-1} \cos z^{n+1}} + \frac{m+n}{n+1} \int \frac{d}{\sin z^{m}}$$

Changing m into $m-2$, in the first of these equations, and n into $n-2$ in the second, we shall obtain two new formulæ, (*C*) and (*D*),

$$\int \frac{dz}{\sin z^{m} \cos z^{n}} = -\frac{1}{m-1} \frac{1}{\sin z^{m-1} \cos z^{-1}} + \frac{m+n-2}{m-1} \int \frac{d}{\sin z^{m-}}$$

$$\int \frac{dz}{\sin z^{m} \cos z^{n}} = \frac{1}{n-1} \frac{1}{\sin z^{m-1} \cos z^{n-1}} + \frac{m+n-2}{n-1} \int \frac{d}{\sin z^{m} \mathrm{c}}$$

These two formulæ may be applied alternately in th same form as those in No. 203, in the example $\int d\, z \sin$ $\cos z^{2}$, and will thus successively diminish first the expone of $\sin z$, and then that of $\cos z$; and by continuing the reductions as far as possible, we shall arrive at one of th following integrals:

$$\int \frac{dz}{\sin z}, \quad \int \frac{dz}{\cos z}, \quad \int \frac{dz}{\sin z \cos z}.$$

208. We shall proceed therefore in this article, explain the method of integrating the four following d ferentials:

$$\frac{dz}{\sin z}, \quad \frac{dz}{\cos z}, \quad \frac{dz \cos z}{\sin z}, \quad \frac{dz \sin z}{\cos z}.$$

The first becomes successively

$$\frac{dz}{\sin z} = \frac{dz \sin z}{\sin z^{2}} = \frac{dz \sin z}{1-\cos z^{2}} = \frac{-dx}{1-x^{2}},$$

by making $\cos z = x$; its integral is therefore

$$\int \frac{dz}{\sin z} = -\tfrac{1}{2}\,\mathrm{l}\, \frac{1+x}{1-x} = -\tfrac{1}{2}\,\mathrm{l}\, \frac{1+\cos z}{1-\cos z} = \mathrm{l}\, \frac{\sqrt{1-\cos z}}{\sqrt{1+\cos z}} + C$$

For the second we have

$$\frac{dz}{\cos z} = \frac{dz \cos z}{\cos z^2} = \frac{dz \cos z}{1 - \sin z^2} = \frac{dx}{1-x^2},$$

by making $\sin z = x$; and consequently

$$\int \frac{dz}{\cos z} = \tfrac{1}{2} 1\left(\frac{1+x}{1-x}\right) = \tfrac{1}{2} 1 \frac{1+\sin z}{1-\sin z} = 1 \frac{\sqrt{1+\sin z}}{\sqrt{1-\sin z}} + \text{const.}$$

The third and fourth are evidently logarithmic differentials, so that we have

$$\int \frac{dz \cos z}{\sin z} = \quad 1 \sin z + \text{const.} = \int \frac{dz}{\tan z} = \int dz \cot z$$

$$\int \frac{dz \sin z}{\cos z} = -1 \cos z + \text{const.} = \int dz \tan z = \int \frac{dz}{\cot z}.$$

By adding together these two last formulæ, we shall find

$$\int \frac{dz}{\sin z \cos z} = 1 \frac{\sin z}{\cos z} + \text{const.} = 1 \tan z + \text{const.}$$

We may give to the integrals

$$\int \frac{dz}{\sin z} = 1 \frac{\sqrt{1-\cos z}}{\sqrt{1+\cos z}} + \text{const.}$$

and

$$\int \frac{dz}{\cos z} = 1 \frac{\sqrt{1+\sin z}}{\sqrt{1-\sin z}} + \text{const.}$$

a more simple form. It is known, that

$$\tan \tfrac{1}{2}(A+B) \,.\, \tan \tfrac{1}{2}(A-B) = \frac{\cos B - \cos A}{\cos B + \cos A},$$

$$\frac{\tan \tfrac{1}{2}(A+B)}{\tan \tfrac{1}{2}(A-B)} = \frac{\sin A + \sin B}{\sin A - \sin B}, \text{ (Trig. 26.).}$$

This being premised, assuming $\cos B = 1$, and $\cos A = \cos z$, or making $B = 0$, and $A = z$, the first formula will become

$$(\tan \tfrac{1}{2} z)^2 = \frac{1 - \cos z}{1 + \cos z},$$

and will consequently give

$$\int \frac{d\,z}{\sin z} = \text{l. } \tan \tfrac{1}{2} z + \text{const.}$$

If we assume, in the second formula, $\sin A = 1$, and $\sin B = \sin z$, or make $A = \frac{\pi}{2}$, and $B = z$, we shall have

$$\frac{1 + \sin z}{1 - \sin z} = \frac{\tan\left(\frac{\pi}{4} + \frac{z}{2}\right)}{\tan\left(\frac{\pi}{4} - \frac{z}{2}\right)};$$

but $\tan\left(\frac{\pi}{4} - \frac{z}{2}\right) = \cot\left(\frac{\pi}{4} + \frac{z}{2}\right) = \dfrac{1}{\tan\left(\frac{\pi}{4} + \frac{z}{2}\right)}$;

therefore $\dfrac{1 + \sin z}{1 - \sin z} = \left[\tan\left(\frac{\pi}{4} + \frac{z}{2}\right)\right]^2$;

and $\displaystyle\int \frac{d\,z}{\cos z} = \text{l}\,.\tan\left(\frac{\pi}{4} + \frac{z}{2}\right) + \text{const.}$

By carefully remarking the connection of the different formulæ constructed in the articles preceding, it will be readily seen, that the integral of $d\,z \sin z^m \cos z^n$, may be obtained in all cases in which m and n are whole numbers, whether positive or negative; this is not the case when these exponents are fractional. We must then have recourse to series, except in a very small number of cases, the integration of which is obvious and immediate.*

* See Note (K).

A general Method of obtaining Approximate Values of Integrals.

209. The developement of integrals into series does not lead to an approximation, unless where the series are convergent, which does not always happen: it is on this account that Analysts have sought for means of arriving at approximate values of integrals, whatever be the nature of the differential functions which may be proposed. The theorem of Taylor leads us in a very simple manner to the formulæ which Euler has constructed for this purpose; but before we enter upon this subject we will enumerate and explain some terms which have relation to the different points of view under which Analysts are accustomed to consider integrals.

The necessity of adding an arbitrary constant to an integral, in order to give it all the generality it implies, shews that these functions are doubly indeterminate, since we cannot assign their value by simply fixing that of the variable upon which they depend; but that it is also necessary to determine their constant, which admits of all possible values. We determine this constant in common cases, by making the integral vanish for a given value of x. We have already seen several examples of this method (164, 176, 177), which in general amounts to the same with what follows.

If $\int X\,dx = P + C$, P denoting the variable function immediately deduced by the process of integration, C the arbitrary constant, and if the integral ought to vanish for a value of $x = a$, which changes P into A; we shall then have the equation $A + C = 0$, from which we deduce

$$C = -A, \quad \text{and} \quad \int X\,dx = P - A.$$

Under this form the integral $\int X\,d\,x$ is nothing more than the difference between the value of the function P when $x=a$, and that which it acquires for every other value of the same variable. If, for example, $x=b$, changes P into B, there arises

$$\int X\,d\,x=B-A.$$

It is proper to remark, that this result may be obtained immediately, without the determination of the arbitrary constant; by simply taking the difference of the results arising from the substitution of the values $x=a$, and $x=b$, which severally change the expression $P+C$ into $A+C$ and $B+C$.

The value $x=a$, which makes the integral vanish, is called its origin; and we then say, that the *integral ought to commence when* $x=a$. If we stop at the value of $x=b$, we then say, that the *integral is complete when* $x=b$.

The two values $x=a$ and $x=b$, are designated by the common name of the *limits of the integral.*

Every integral which we express without fixing its origin or indicating its limits, is called an *indefinite integral*, and ought, in order to be *complete*, to include an *arbitrary constant.*

When we assign these limits, the integral is *definite*. If they are $x=a$ and $x=b$, for instance, we then say, that *the integral* $\int X\,d\,x$ *ought to be taken from* $x=a$ *to* $x=b$; *and this is determined by calculating the value of the variable part of the integral, both when* $x=a$ *and* $x=b$; *and then subtracting the first result from the second.* In this case it is quite unnecessary to connect with the integral an arbitrary constant, since it would disappear by the subtraction.

It is important to make ourselves familiar with these expressions, which are often used, and which the considerations immediately following will render still more significant.

210. This being premised, the series of Taylor giving, when x becomes $x+h$,

$$y+\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\&c.$$

is not sufficient to determine the value which a function assumes under these circumstances, when we only know its differential coefficients; since the primitive value y remains indeterminate, which consequently corresponds to the arbitrary constant: but the difference between this value, and that which answers to $x+h$, being only dependent on this series:

$$\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\&c.$$

is entirely known.

If we make $\int X\,dx=y$, we shall have

$$\frac{dy}{dx}=X,\ \frac{d^2y}{dx^2}=\frac{dX}{dx},\ \frac{d^3y}{dx^3}=\frac{d^2X}{dx^2},\ \&c.$$

the differential coefficients of the given functions x will be all determined, and the series will become

$$X\frac{h}{1}+\frac{dX}{dx}\frac{h^2}{1.2}+\frac{d^2X}{dx^2}\frac{h^3}{1.2.3}+\&c.$$

To deduce from this formula the value of $\int X\,dx$ from $x=a$ to $x=b$, it will be sufficient to assume $h=b-a$, and to replace x by a, in the function X, and its differential coefficients, which we will represent in this case by A, A', A'', &c.: we shall find, between the limits $x=a$ and $x=b$,

$$\int X\,dx=A\frac{(b-a)}{1}+A'\frac{(b-a)^2}{1.2}+A''\frac{(b-a)^3}{1.2.3}+\&c.$$

The preceding series is in general so much the more convergent the smaller the interval $b-a$; but when it has too large a value we divide it into a number of parts suffi-

ciently great to make the intervals themselves sufficiently small; and we calculate separately the value of the integral, relative to each of these intervals. We suppose, in order to simplify the formulæ, that the interval $b-a$ is divided into n parts, each equal to α; and that the quantities A, A', A'', &c. are severally changed into A_1, A'_1, A_1'', &c. A_2, A_2', A_2'', &c. when we substitute in them $a+\alpha$, $a+2\alpha$, &c. in the place of a; we shall have at first, between a and $a+\alpha$,

$$\frac{A\,\alpha}{1}+\frac{A'\,\alpha^2}{1\,.\,2}+\frac{A''\,\alpha^3}{1\,.\,2\,.\,3}+\text{\&c.}$$

between $a+\alpha$ and $a+2\,\alpha$,

$$\frac{A_1\,\alpha}{1}+\frac{A_1'\alpha^2}{1.2}+\frac{A_1''\alpha^3}{1\,.\,2\,.\,3}+\text{\&c.}$$

between $a+2\,\alpha$, and $a+3\,\alpha$,

$$\frac{A_2\,\alpha}{1}+\frac{A_2'\,\alpha^2}{1\,.\,2}+\frac{A_2''\,\alpha^3}{1.2.3}+\text{\&c.}$$

The sum of all these series, which are n in number, will compose the total value of $\int X\,d\,x$, between the limits $x=a$ and $x=b$, which will consequently be ...(I)

$$\int X\,d\,x=\left\{\begin{array}{l} \frac{\alpha}{1}(A+A_1+A_2\,\ldots\ldots\ldots+A_{n-1}) \\ +\ \frac{\alpha^2}{1\,.\,2}\ (A'+A_1'+A_2'\,\ldots\ldots\ldots+A'_{n-1}) \\ +\ \frac{\alpha^3}{1\,.\,2\,.\,3}(A''+A_1''+A_2''\ldots\ldots\ldots+A''_{n-1}) \end{array}\right.$$

+ &c.

211. If we were to take α so small that we might confine ourselves to its first power, the preceding result would be reduced to

$$\int X\,d\,x=A\,\alpha+A_1\alpha+A_2\alpha\ldots\ldots\ldots+A_{n-1}\alpha,$$

a series whose different terms are nothing more than th successive values of the quantity $X\,d\,x$, when we substitu

in it a, $a+\alpha$, $a+2\alpha$, &c. in the place of x; and make $dx=\alpha$. It is under this point of view that we conceive the integral $\int X\,dx$ to be the sum of an infinite number of elements, severally equal to the consecutive values of the differential, corresponding to the different changes which are experienced by the variable x. (See the note at page 179).

It yet remains to be observed, that the sum of this series, whatever be the number of its terms, provided that they all have the same sign, will be less than $n\,\alpha\,A_m$, if A_m denotes the greatest of the quantities A, A_1, A_2, ... A_{n-1}, and that the contrary will be the case if A_m be the least of them. We infer from thence, that if the function x does not change its sign between the limits $x=a$ and $x=b$, and if M and m be its greatest and least values in this interval, the integral $\int X\,dx$, taken between these limits, will be $< M(b-a)$ and $> m(b-a)$.

212. The difference between the two values of y, corresponding to $x=a$ and $x=b$, may also be obtained by commencing from the last of these values, by means of the formula

$$y_{,}=y-\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\frac{h^2}{1.2}-\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\text{\&c.}$$

in which $y_{,}$ answers to $x-h$, which also gives

$$y-y_{,}=\frac{dy}{dx}\frac{h}{1}-\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}-\text{\&c.}$$

To apply this last series to $\int X\,dx$, it will be necessary in X and its differential coefficients, to change x into b; and supposing that from this substitution we get the quantities B, B', B'', &c. we shall find between the limits $x=a$ and $x=b$,

$$\int X\,dx=\frac{B(b-a)}{1}-\frac{B'(b-a)^2}{1.2}+\frac{B''(b-a)^3}{1.2.3}-\text{\&c.}$$

When we divide the interval $b-a$ into n parts, each equal to α, we obtain from the above formula, between the limits $a+\alpha$ and a,

$$\frac{A_1\,\alpha}{1}-\frac{A'_1\,\alpha^2}{1.2}+\frac{A''_1\,\alpha^3}{1.2.3}-\&c.$$

between $a+2\,\alpha$ and $a+\alpha$

$$\frac{A_2\,\alpha}{1}-\frac{A'_2\,\alpha^2}{1.2}+\frac{A''_2\,\alpha^3}{1.2.3}-\&c.$$

between $a+3\,\alpha$ and $a+2\,\alpha$

$$\frac{A_3\,\alpha}{1}-\frac{A'_3\,\alpha^2}{1.2}+\frac{A''_3\,\alpha^3}{1.2.3}-\&c.$$

the sum of these series which are likewise n in number, gives, between the limits $x=b$, $x=a$.........(II.)

$$\int X\,d\,x=\left\{\begin{array}{l} \frac{\alpha}{1}\;(A_1+A_2+A_3\ldots\ldots\ldots\ldots+A_n) \\ -\frac{\alpha^2}{1.2}\;(A'_1+A'_2+A'_3\ldots\ldots\ldots\ldots+A'_n) \\ +\frac{\alpha^3}{1.2.3}\;(A''_1+A''_2+A''_3\ldots\ldots+A''_n) \\ -\&c. \end{array}\right.$$

213. By reducing this last series to the forms affected with the first power of α, we should have

$$\int X\,d\,x=A_1\,\alpha+A_2\,\alpha+A_3\,\alpha\ldots\ldots+A_n\,\alpha,$$

an expression whose difference from the true value would be $+$, if that in the corresponding expression in the preceding No. was $-$, and *vice versâ*, always supposing that the quantities A, A_1, A_2, &c. are all of the same sign, and compose a series either constantly increasing, or constantly decreasing.

The same may be proved of the series (I.) and (II.); but we shall not in this place stop to demonstrate it; we shall

merely observe, that in consequence of this remark, we may take for greater accuracy one half of the sum of these last-mentioned expressions, and consequently between the limits $x=a$ and $x=b$, we shall have the formula (III.)

$$\int X dx = \begin{cases} \frac{\alpha}{1} \quad [A_1+A_2+A_3\ldots\ldots+A_{n-1}+\frac{1}{2}(A+A_n)] \\ +\frac{\alpha^2}{1.2} \cdot \frac{1}{2}(A'-A'_n) \\ +\frac{\alpha^3}{1.2.3} \; [A''_1+A''_2+A''_3\ldots\ldots+A''_{n-1}+\frac{1}{2}(A''+A''_n)] \\ +\frac{\alpha^4}{1.2.3.4} \cdot (A'''-A'''_n) \\ + \&c. \end{cases}$$

214. The consideration of curve lines conducts us likewise in a very simple manner to the principle conclusions, established in the preceding articles.

Since we express the area of a curve whose ordinate is X (76.), by $\int X\, d\, x$, if $B\,C\,Z$, fig. 35, represents this curve, and be the origin of the abscissa, and $X = P\,M$, the expression $X\,d\,x$ will be equally the differential of the segments $B\,M\,P$, $D\,E\,M\,P$, and of the segment $A\,C\,M\,P$, which commences at the origin of the abscissa; thus the ordinate which limits the segment on this part, will be absolutely indeterminate. The ordinate $M\,P$, which forms the other limit, is likewise indeterminate, as long as we assign no particular value to the abscissæ $A\,P$; but when we assign the abscissa of the first and of the last ordinate, the segment will be completely determined. Fig. 35.

If the variable function P of the integral $\int X\,d\,x = P + C$, be supposed to vanish at the point B, it will express immediately the value of the areas $B\,C\,A$, $B\,E\,D$, $B\,M\,P$; consequently if we should wish that the segments should commence from the ordinate $A\,C$, we must subtract from the above-mentioned areas the space $B\,C\,A$: this space re-

presents the constant, determined upon the condition that the quantity $P+C$ should vanish at the point A; but when we consider at the same time the two limits of a segment, it is of no use to trouble ourselves with the constant; for whether we suppose the areas to commence from the point B, or from the point A, upon the axis of the abscissæ, the segment $DEMP$ for example, is equally obtained by the difference of the segments BMP, BED, or by that of the segments $ACMP$, and $ACED$.

215. The inspection of fig. 35, will readily shew that the area of the segment of any curve is always comprised between the sum of a series of inscribed rectangles, PR, $P'R'$, $P''R''$, &c. and that of a series of rectangles circumscribed $P'S$, $P''S'$, $P'''S''$, &c. the first series constructed upon the least ordinates of each of the curvilinear trapeziums $P'M'$, $P''M''$, $P'''M'''$, &c. and the second on the greatest. It is obvious, that if we take

$$AP=a,\quad PP'=P'P''=P''P''', \text{\&c.}=\alpha,$$

we shall have

$$PM=A,\quad P'M'=A_1,\quad P''M''=A_2,\quad P'''M'''=A_3, \text{\&c.}$$

the sum of the inscribed rectangles will be

$$A\alpha+A_1\alpha+A_2\alpha+A_3\alpha+\text{\&c.} \qquad (1.)$$

and that of the circumscribed rectangles will be

$$A_1\alpha+A_2\alpha+A_3\alpha+\text{\&c.} \qquad (2.)$$

We easily see that the difference between the rectangles inscribed and circumscribed, is equal to the rectangle $MRQN$, which is equal to the sum of the rectangles MM', $M'M''$, $M''M'''$, &c. and that consequently this difference may be made as small as we chuse, by diminishing the distance of the ordinates.

In fig. 35, where the ordinates form increasing series, the inscribed rectangles are formed upon the first ordinate of each curvilinear trapezium; and the circumscribed rectangles upon the last; but if they should pass through a

maximum, as in the figure 36, this would be the case in the part CM'', which precedes the *maximum*, whilst the contrary would take place in the part $M''Z$, which follows it; in this case the series (1), which is at first less than the curvilinear space, would become greater, and the series (2), which is at first greater than this space, would then become smaller. FIG. 36.

216. We shall approximate still more to the true value of the segment of the curve proposed, by taking instead of the rectangles inscribed and circumscribed, the sum of the trapeziums terminated by the chords of the arcs MM', $M'M''$, $M''M'''$, &c.

These trapeziums having the same altitude PP', and each ordinate, except the first, being common to two trapeziums, their sum will be reciprocally equal to the series

$$\alpha\,[A_1+A_2+A_3+\ldots\ldots A_{n-1}+\tfrac{1}{2}(A+A_n)],$$

the value of which is intermediate to those of the series (1) and (2).

Finally, it is evident, by the figure 37, that the curvilinear area $PMNQ$ is $<$ than the rectangle QE, and $>$ than the rectangle PF, the one constructed on the greatest, and the other on the least of the ordinates which are comprised between the limits AP and AQ of this segment. FIG. 37.

217. The application of the formulæ (III.) of No. 213, may present some difficulties. It cannot be applied when the function X becomes infinite; and near those values of the abscissa which make X infinite, it is not sufficient to diminish the interval α, or the distance of the ordinates, in order to compensate the effect of their rapid increase: we must still have recourse to transformations by which this rapid variation may be counteracted.

Let us take for example $X = \frac{1}{\sqrt{(1-x)}}$; it is evident that when x becomes nearly equal to unity a very slight change in the value of this variable produces a very considerable one in that of X; if therefore we should wish to find the value of the integral $\int \frac{dx}{\sqrt{1=x}}$, from $x=0$ to $x=1-\delta$, δ being a very small quantity, it would be necessary near the second limit, to increase very considerably the number of the intermediate values which are given to x.

The same integral cannot be calculated immediately as far as $x=1$; for in that case X becomes infinite, whilst the value of $\int X\,dx$ is not so, since

$$\int \frac{dx}{\sqrt{1-x}} = -2\sqrt{1-2} + \text{const.}$$

This difficulty arises from the passage of the quantity $\sqrt{1-x^2}$ from the denominator to the numerator, in the course of the integration; and will occur in general whenever X is of the form $\frac{V}{(a-x)^{\frac{p}{q}}}$, and when we have $p<q$. To remove it, we must make $a-x=z^q$, which will give $x=a-z^q$, $dx=-q\,z^{q-1}dz$, and $X\,dx=-q\,V z^{q-p-1}dz$, a quantity which is no longer infinite when $x=a$ or $z=0$, if the function V continues finite under these circumstances. We must then calculate the value of the integral $\int V z^{q-p-1}dz$, from $z=0$ to $z=\delta$, δ being a very small quantity, and we shall thus have the part of the value of $\int \frac{V\,dx}{(a-x)^{\frac{p}{q}}}$, which corresponds to the interval comprised between $x=a$ and $x=a-\delta$.

We may also obtain the integral $\int \frac{V\,dx}{(a-z)^{\frac{p}{q}}}$ from $x=a$

to $x = a - \delta$, by simply making $x = a - z$; since the smallness of the variable z comprised between the very narrow limits 0 and δ, allows us to simplify very much the differential coefficient. If we had for example $\int \frac{x^2\,dx}{\sqrt{(a^4 - x^4)}}$, the differential to integrate after the above-mentioned transformation, would be

$$\frac{-(a-z)^2\,dz}{\sqrt{4a^3z - 6a^2z^2 + 4az^3 - z^4}} = \frac{-(a^2 - 2az + z^2)\,dz}{\sqrt{z} \cdot \sqrt{4a^3 - 6a^2z + 4az^2 - z^3}}.$$

By reducing the fraction

$$\frac{a^2 - 2az + z^2}{\sqrt{4a^3 - 6a^2z + 4az^2 - z^3}}$$

into a series arranged according to the powers of z, we should get by stopping at the squares of that variable,

$$-\int \frac{dz\sqrt{a}}{2\sqrt{z}}\left(1 - \frac{5z}{4a} - \frac{5z^2}{32a^2}\right) = -\frac{\sqrt{az}}{2}\left(2 - \frac{5z}{6a} - \frac{1}{16}\frac{z^2}{a^2}\right).$$

This result, which vanishes when $z = 0$, will give by the substitution of δ for z the value of the integral sought for from $x = a$ to $x = a - \delta$, the other part of the integral may be calculated by means of the series in No. 213.

In general, different transformations which long practice and great readiness in analysis can alone suggest, will render these series applicable in many cases which at first sight do not appear to be reducible by any of the methods enumerated.

218. The integral $\int \frac{e^{-\frac{1}{x}}dx}{x}$ not being obtainable by the reduction of $e^{-\frac{1}{x}}$ into a series, except in the case in which x is very large, we shall proceed to shew in what manner Euler has calculated its value from $x = 0$ to $x = 1$, by means of the formula III in No. 213.

We may at once change

$$\int \frac{e^{-\frac{1}{x}}dx}{x} \text{ into} \int x\frac{e^{-\frac{1}{x}}dx}{x^2} = e^{-\frac{1}{x}}x - \int e^{-\frac{1}{x}}dx;$$

the part $e^{-\frac{1}{x}}$ vanishes when $x=0$, which is likewise the case with the second part $e^{-\frac{1}{x}}$, as we shall proceed to shew. We have for this integral

$$X = e^{-\frac{1}{x}},\quad \frac{dX}{dx} = e^{-\frac{1}{x}}\frac{1}{x^2},\quad \frac{d^2X}{dx^2} = e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right)$$

$$\frac{d^3X}{dx^3} = e^{-\frac{1}{x}}\left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right), \text{ \&c.}$$

If we make $x=0$, these expressions vanish (58), and consequently the quantities A, A', A'', &c. will be evanescent likewise: then substituting α, 2α, 3α, &c. in the place of x, we shall obtain the values of A_1, A'_1, A''_1, &c. A_2, A'_2, A''_2, &c. and from $x=0$ to $x=\alpha$, we shall have

$$\int e^{-\frac{1}{x}}dx =$$

$$\frac{\alpha}{1}\left[e^{-\frac{1}{\alpha}} + e^{-\frac{1}{2\alpha}} \ldots\ldots + e^{-\frac{1}{(n-1)\alpha}}\right] + \frac{1}{2}\frac{\alpha e^{-\frac{1}{n\alpha}}}{1}$$

$$-\frac{1}{2}\frac{\alpha^2 e^{-\frac{1}{n\alpha}}}{1.2}\frac{1}{n^2\alpha^2}$$

$$+\frac{\alpha^3}{1.2.3}\left[e^{-\frac{1}{\alpha}}\left(\frac{1}{\alpha^4} - \frac{2}{\alpha^3}\right) + e^{-\frac{1}{2\alpha}}\left(\frac{1}{16\alpha^4} - \frac{2}{8\alpha^3}\right) + \ldots\right.$$

$$\left. + e^{-\frac{1}{(n-1)\alpha}}\left(\frac{1}{(n-1)^4\alpha^4} - \frac{2}{(n-1)^3\alpha^3}\right)\right] + \frac{1}{2}\frac{\alpha^3 e}{1.2.3}^{-\frac{1}{n\alpha}}\left(\frac{1}{n^4\alpha^4} - \frac{2}{n^3\alpha^3}\right)$$

$$-\frac{1}{2}\frac{\alpha^4 e}{1.2.3.4}^{-\frac{1}{n\alpha}}\left(\frac{1}{n^6\alpha^6} - \frac{6}{n^5\alpha^5} + \frac{6}{n^4\alpha^4}\right)$$

+ &c.

When we wish to stop at the limit $x=1$, we must make $\alpha = \frac{1}{n}$, and there results for that case $\int e^{-\frac{1}{x}}dx =$

$$\frac{1}{n}\left[e^{-\frac{n}{1}}+e^{-\frac{n}{2}}+e^{-\frac{n}{3}}\ldots+e^{-\frac{n}{n-1}}\right]+\frac{1}{2ne}-\frac{1}{4n^2e}$$

$$+\frac{1}{6}\left[\frac{(n-2)}{1}e^{-\frac{n}{1}}+\frac{(n-4)}{16}e^{-\frac{n}{2}}+\frac{(n-6)}{81}e^{-\frac{n}{3}}\ldots\ldots\ldots\right.$$

$$\left.+\frac{n-2n+2}{(n-1)^4}e^{-\frac{n}{n-1}}\right]$$

$$-\frac{1}{12n^3e}-\frac{1}{48n^4e}+\text{\&c.}$$

By limiting ourselves to the terms above-written, and making $n=10$, we shall find, according to Euler, the value of $\int e^{-\frac{1}{x}}dx$ to a millioneth part of unity nearly, and we shall obtain its value with an accuracy twenty times as great, if we make $n=20$.

The details contained in this and the preceding article are sufficient to shew how, by the assistance of transformations, and by calculating the value of an integral in several, we may approximate to it, even when the series which express it are only convergent for very narrow limits.

219. The series mentioned in Nos. 19. and 21. give likewise two general developements of the integral $\int X\,dx$. If we designate by C the value of this integral when $x=0$, and represent by A, A', A'', &c. the values of X, $\frac{dX}{dx}$, $\frac{d^2X}{dx^2}$, &c. under these circumstances, we shall have

$$\int X\,dx=C+A\frac{x}{1}+A'\frac{x^2}{1.2}+A''\frac{x^3}{1.2.3}+\text{\&c.}$$

a series in which C occupies the place of the arbitrary constant.

If we commence with the greatest value of $\int X\,dx$, which is represented by y, to arrive at that which corres-

ponds to $x=0$, and which is denoted by C, it is evident that we must make $h=-x$, in the formula in No. 21, which will give

$$C=y-\frac{dy}{dx}\frac{x}{1}+\frac{d^2y}{dx^2}\frac{x^2}{1.2}-\frac{d^3y}{dx^3}\frac{x^3}{1.2.3}+\&c.$$

substituting in this equation for y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. their values, and disengaging $\int Xdx$, we shall have

$$\int Xdx=C+X\frac{x}{1}-\frac{dX}{dx}\frac{x^2}{1.2}+\frac{d^2X}{dx^2}\frac{x^3}{1.2.3}-\&c.$$

C denoting, in this case also, the arbitrary constant.

The process of integration also conducts us to this developement: for if we divide the differential $X\,dx$ into the two factors X and dx, and integrate the second, we shall have $\int X\,dx=Xx-\int x\,dX$; but

$$\int x\,dX=\int\frac{dX}{dx}.\,x\,dx=\frac{1}{2}x^2\frac{dX}{dx}-\frac{1}{2}\int x^2\frac{d^2X}{dx},$$

$$\int x^2\frac{d^2X}{dx}=\int\frac{d^2X}{dx}.\,x^2\,dx=\frac{1}{3}x^3\frac{d^2X}{dx^2}-\frac{1}{3}\int x^3\frac{d^3X}{dx^2},$$

$$\int x^3\frac{d^3X}{dx^2}=\int\frac{d^3X}{dx^3}.\,x^3\,dx=\frac{1}{4}x^4\frac{d^3X}{dx^3}-\frac{1}{4}\int x^4\frac{d^4X}{dx^3},$$

&c.

putting successively for $\int x\,dX$, $\int x^2\frac{d^2X}{dx}$, &c. their several values, there will result

$$\int X\,dx=X\frac{x}{1}-\frac{dX}{dx}\frac{x^2}{1.2}+\frac{d^2X}{dx^2}\frac{x^3}{1.2.3}-\&c.$$

and in order that the expression for the integral may be complete, we must add an arbitrary constant to this developement, by which means it will become similar to the preceding. This series was first given by John Bernouilli, whose name it bears; and it has the same relation to the

Integral Calculus, which that of Taylor has to the Differential.

220. Hitherto we have only considered the differential coefficient of the first order; but if we knew the differential coefficient of the second order only, it would then require two successive integrations to obtain the primitive function from which it is derived. Let X be the differential coefficient of the second order of the function y, we shall have $\frac{d^2 y}{d x^2} = X$, and multiplying both sides by $d x$, there will result $\frac{d^2 y}{d x} = X d x$; now $\frac{d^2 y}{d x}$ is the differential of $\frac{d y}{d x}$, taken upon the supposition that $d x$ is constant: we shall have therefore $\frac{d y}{d x} = \int X d x$. If P represents that primitive function of x which is equal to $\int X d x$, and C the arbitrary constant, there will arise $\frac{d y}{d x} = P + C$; multiplying the two members of this equation by $d x$, we shall find $d y = P d x + C d x$, and by integrating we shall get $y = \int P d x + C x + C'$, C' denoting a second arbitrary constant. If we replace P by $\int X d x$, there will result $y = \int d x \int X d x + C x + C'$, an expression which indicates two successive operations.

We may reduce this expression to two simple integrals, by the method of integration by parts; for restoring P in the place of $\int X d x$, we shall have

$$\int P d x = P x - \int x d P = x \int X d y - \int X x d x,$$

and consequently

$$y = x \int X d x - \int X d x + C x + C'.$$

We now proceed to differentials of the third order. Let

X be the differential coefficient of the function y, relative to this order: we shall have $\frac{d^3 y}{d x^3} = X$, from whence $\frac{d^3 y}{d x^2} = X d x$; but $\frac{d^3 y}{d x^2} = d \cdot \frac{d^2 y}{d x^2}$; therefore $\frac{d^2 y}{d x^2} = \int X d x + C$, which gives $\frac{d^2 y}{d x} = d x \int X d x + C d x$. Integrating again there will result $\frac{d y}{d x} = \int d x \int X d x + C x + C'$, or from what we have remarked above

$$\frac{d y}{d x} = x \int X d x - \int X x d x + C x + C'.$$

From this we deduce

$$d y = x d x \int X d x - d x \int X x d x + C x d x + C' d x,$$

and by integrating we get

$$y = \int x d x \int X d x - \int d x \int X x d x + \tfrac{1}{2} C x^2 + C' x + C'',$$

C'' being the constant introduced by this last integration. It is easily seen that

$$\int x d x \int X d x = \tfrac{1}{2} x^2 \int X d x - \tfrac{1}{2} \int X x^2 d x$$

$$\int d x \int X x d x = x \int X x d x - \int X x^2 d x;$$

substituting these values, and consolidating all the similar terms, we shall find

$$y = \tfrac{1}{2}(x^2 \int X x d x - 2 x \int X x d x + \int X x^2 d x) + \tfrac{1}{2}(C x^2 + 2 C' x + C''$$

We will now shew in what manner successive integrals are denoted: when X designates the differential coefficient of the second order, we have $d^2 y = X d x^2$, and by taking the integral of each member we find $d y = \int X d x^2$, and now integrating a second time, there arises $y = \int\int X d x^2 = \int^2 X d$ We have in the same manner, when X denotes the differential of the third order,

$d^2y=\int X\,d\,x^3$, $d\,y=\int\int X\,d\,x^3$, $y=\int\int\int X\,d\,x^3=\int^3 X\,d\,x^3$, and so on for the higher orders.

Each differentiation introducing but one power of $d\,x$, we may leave this power only under the different signs $\int$, which will furnish us with the following relations:

$$\int X\,d\,x^2=d\,x\int X\,d\,x,\quad \int\int X\,d\,x^2=\int d\,x\int X\,d\,x$$

$$\int X d\,x^3=d\,x^2\int X d\,x,\quad \int\int X d\,x^3=\int d x^2\int X d\,x=d\,x\int d\,x\int X\,d\,x$$

$$\int\int\int X\,d\,x^3=\int d\,x\int d\,x\int X\,d\,x,\ \&c.$$

where it is necessary to observe, that each sign $\int$ embraces all those which follow it.

This being premised, by neglecting the arbitrary constant, and integrating by resolution into parts, as above, we shall find

$$\int X\,d\,x=\int X\,d\,x$$

$$\int^2 X d\,x^2 = \frac{1}{1}\left[x\int X\,d\,x-\int X x\,d\,x\right]$$

$$\int^3 X d\,x^3=\frac{1}{1.2}\left[x^2\int X\,d\,x-2\,x\int X\,x\,d\,x+\int X\,x^2\,d\,x\right]$$

$$\int^4 X\,d\,x^4=\frac{1}{1.2.3}\left[x^3\int X d\,x-3\,x^2\int X x\,d\,x+3\,x\int X x^2\,d\,x-\int X x^3\,d\,x\right],$$

&c.

The numerical coefficients of these expressions are the same as those of the powers of the binomial $a-b$; and whilst the exponent of x without the signs diminishes by unity in each term, in proceeding from the left to the right, its exponent under that sign increases by the same quantity.

We shall restore the arbitrary constants which are omitted in this formula, by writing $\int X\,d\,x+C$ for $\int X\,d\,x$, $\int X\,x\,d\,x+C'$ for $\int X\,x\,d\,x$, $\int X\,x^2\,d\,x+C''$ for $\int X\,x^2\,d\,x$, and so on for the others: for the constants C, C', C'', &c. being multiplied into the different powers of x, admit not of further reduction.

221. The differentials of which we have hitherto treated are taken upon the supposition that $d x$ is constant, since these are the only differentials which include but one differential coefficient. In fact, if we suppose $d x$ variable likewise, we have (116.) $d^2 y = q\,d x^2 + p\,d^2 x$; if therefore the differential $U d x^2 + V d^2 x$ was proposed, it is necessary in order that this expression may have any meaning, that $V = p$ and $U = q$, from which there results $U = \frac{d V}{d x}$; and this condition being satisfied, we have only to integrate $V d x$. It is easy to extend this remark to differentials of any order whatever.

On the Application of the Integral Calculus to the Rectification of Curves, to the Quadrature of Curves, and Curve Surfaces, and to the Curvature of Solids.

On the Quadrature of Curves.

222. The general problem of the Quadrature of Curves reduces itself to the integration of the differential $X d x$, where X represents that function of x which is equal to the ordinate y of the proposed curve (76.). We have already explained in the preceding pages the principal analytical methods which have been hitherto discovered, of effecting this integration, whether accurately, or by approximation; and our only object in the present chapter is the application of these methods to curves which are most commonly known.

The curves which have the most simple integration are the parabolas of different orders, represented by the equation $y^n = p x^m$: from this we get $y = p^{\frac{1}{n}} x^{\frac{m}{n}}$, and consequently

$$\int X\,dx = \int p^{\frac{1}{n}} x^{\frac{m}{n}}\,dx = \frac{n p^{\frac{1}{n}}}{m+n}\, x^{\frac{m+n}{n}} + \text{const.}$$

All these curves, as we see, are *quadrable*; that is to say, we have a finite algebraical expression for the surface of the segment comprised by their arc, the axis of the abscissa, and the ordinate. It is very easy, when we have the expression of this segment, to calculate that of any other space included between a portion of the curve and right lines, which forms with the abscissæ and the ordinates, polygonal figures whose areas are determinable by Elementary Geometry; we shall see some examples of this a little further on (229. 230).

The curves proposed pass through the origin of the abscissa, since we have at the same time $x=0$ and $y=0$; if we wish to express their area, commencing from this point, we must suppress the arbitrary constant, since the expression $\frac{n p^{\frac{1}{n}}}{m+n} x^{\frac{m+n}{n}}$ vanishes when we make $x=0$. To obtain the value of the area $B\,C\,M\,P$, fig. 38, comprehended between the ordinates $B\,C$ and $M\,P$, which correspond to the abscissæ $A\,B=a$ and $A\,P=x$, it will be sufficient to subtract from $\frac{n p^{\frac{1}{n}}}{m+n} x^{\frac{m+n}{n}}$, which expresses the area $A\,C\,M\,P$, the quantity $\frac{n p^{\frac{1}{n}}}{m+n} a^{\frac{m+n}{n}}$ which is equal to the area $A\,C\,B$; and we shall thus get FIG. 38.

$$B\,C\,M\,P = \frac{n p^{\frac{1}{n}}}{m+n}\left(x^{\frac{m+n}{n}} - a^{\frac{m+n}{n}}\right).$$

When the expónent n is even, the expression $\frac{n p^{\frac{1}{n}}}{m+n} x^{\frac{m+n}{n}}$

is susceptible of the double sign $\pm$, and since in that case the same abscissæ $A\,P$ belong to two branches of the curves $A\,C\,M$ and $A\,c\,m$, we have two segments $A\,C\,M\,P$ and $A\,c\,m\,P$; that which comprises the positive ordinates has a positive value, and the other has a negative value.

When the exponents m and n are both odd, the quantity $x^{\frac{m+n}{n}}$ has but one sign, and continues always positive, whatever be the value of x; but it is easily seen, that in this case one of the two branches of the proposed curve has its abscissæ and ordinates negative at the same time; it follows therefore from this, that the areas corresponding to negative ordinates and abscissæ ought to be considered as positive.

If n only is odd, then the quantity $x^{\frac{m+n}{n}}$ becomes negative at the same time with x; but in this case the two branches of the proposed curve are on the same side of the abscissæ, and the ordinates continue always positive.

From these remarks we may conclude, that *the area of a curve is positive when the abscissa and ordinate have the same sign, and negative when their signs are different.*

All parabolic segments have a constant ratio to the rectangle $A\,D\,M\,P$, constructed upon the abscissa and the ordinate; for the expression

$$\frac{n}{m+n}\,p^{\frac{1}{n}}\,x^{\frac{m+n}{n}} = \frac{n}{m+n}\,x\,.\,p^{\frac{1}{n}}\,x^{\frac{m}{n}}$$

which is equal to $\frac{n}{m+n}\,x\,y$, since $y=p^{\frac{1}{n}}\,x^{\frac{m}{n}}$.

When $n=m$, the parabola becomes a right line, since in this case we have $y=p^{\frac{1}{n}}\,x$; the segment $A\,C\,M\,P$ becomes the triangle $A\,M\,P$, whose area by the formula above given, is equal to $\frac{1}{2}\,x\,y$; which is also known from Elementary Geometry.

By making $n=2$ and $m=1$, we have the common parabola, and we find $\frac{2}{3}xy$ for the value of the segment $ACMP$.

223. We will now proceed to find the value of this segment in curves, represented by the equation $x^m y^n = p$. This equation is deducible from $y^n = p x^m$, by changing m into $-m$; we have $y = p^{\frac{1}{n}} x^{-\frac{m}{n}}$, and consequently

$$\int X d x = \frac{n p^{\frac{1}{n}}}{n-m} x^{\frac{n-m}{n}} + \text{const.}$$

The curves proposed are hyperbolas of different orders referred to their asymptotes, and are composed of several branches, such as UMV, fig. 39, inscribed within the angles which the asymptotes form with each other. If we reckon the segments from the origin of the abscissæ, they will comprehend the indefinite space which is included between the part CV of the curve and its asymptote AY; the value of this space is infinite, or finite, according as m is greater or less than n. In fact, to get the value of the space $BCMP$, taken from the abscissa $AB=a$ to the abscissa $AB=b$, we must (209.) successively make $x=a$ and $x=b$ in the expression $\frac{n p^{\frac{1}{n}}}{n-m} x^{\frac{n-m}{n}}$, and then subtract the first result from the second; we shall get $BCMP = \frac{n p^{\frac{1}{n}}}{n-m}\left(b^{\frac{n-m}{n}} - a^{\frac{n-m}{n}}\right)$. If we now suppose $a=0$; the point B will coincide with the point A, and the space $BCMP$ will be changed into $YAMP$; now the quantity $a^{\frac{n-m}{n}}$ will be infinite or nothing, according as we have $m >$ or $< n$: in the first case,

FIG. 39.

$$YAPM = \frac{n p^{\frac{1}{n}}}{m-n}\left(\frac{1}{0} - b^{\frac{n-m}{n}}\right),$$

and in the second

$$YAPM = \frac{np^{\frac{1}{n}}}{n-m}\left(b^{\frac{n-m}{n}} - 0\right) = \frac{np^{\frac{1}{n}}}{n-m} b^{\frac{n-m}{n}}.$$

Supposing a to be of a determinate magnitude, and making b infinite, we shall then get the indefinite space $XBCU$, which will be infinite if m be less than n, or finite and equal to $\frac{np^{\frac{1}{n}}}{m-n} a^{\frac{n-m}{n}}$ if m be greater than n. It follows from hence, that when m and n are unequal, one of the asymptotic spaces is finite, and the other infinite.

The reason of this difference is founded on the greater or less rapidity with which the curve approaches to its asymptote; and since $y = \frac{p^{\frac{1}{n}}}{x^{\frac{m}{n}}}$, and $x = \frac{p^{\frac{1}{m}}}{y^{\frac{n}{m}}}$, it is easily seen that when we have $m > n$, y decreases much more rapidly than x, and that consequently the curve approaches much more rapidly to the asymptote upon which the abscissæ are taken, than to that which is parallel to the ordinates and *vice versa.*

By putting y in the place of $p^{\frac{1}{n}} x^{-\frac{m}{n}}$, in the expression

$$\frac{np^{\frac{1}{n}}}{n-m} x^{\frac{n-m}{n}} = \frac{n}{n-m} x \cdot p^{\frac{1}{n}} x^{-\frac{m}{n}},$$

it will become $\frac{n}{n-m} xy$, and the value of the area $YAPMV$ will be $\frac{nxy}{n-m}$ + const. It might seem that the expression $\frac{nxy}{n-m}$ ought to vanish when $x = 0$; but what we have just proved shews the necessity of making

no inference of this kind before we have substituted, in the place of y, its value in terms of x.

224. When $n=m$ we have $x\,y=p^{\frac{1}{n}}$, or $x\,y=p$, if we change $p^{\frac{1}{n}}$ into p, which we are at liberty to do, the curve in question is the common hyperbola, and is also equilateral if the angle of the co-ordinates be a right one. The general expression for the area found in the preceding article, presents itself in this case under an infinite form, whatever be the value of x, and the differential of this expression being $\frac{p\,d\,x}{x}$, has for its integral $p\,1\,x+\text{const}$. The asymptotic spaces are both infinite in this case; for x becomes so both by the supposition of $x=0$, and by that of x being equal to an infinite quantity.

Let $p=a^2$ and $U\,M\,V$, fig. 40, one of the branches of the equilateral hyperbola, whose *power* is equal to a^2, and $A\,C$ its axis; if we draw $B\,C$ a perpendicular to the asymptote from the vertex C, we shall have $AB=a$; and since the area $B\,C\,M\,P=a^2\,1\,A\,P-a^2\,1\,.\,A\,B=a^2\,1\,.\frac{A\,P}{A\,B}$, if we assume $A\,B$ equal to unity, there will result, since $1\,.\,1=0$, $B\,C\,M\,P=1\,.\,A\,P$. We shall have in the same manner $1\,.\,A\,P'=B\,C\,M'P'$, $1\,.\,A\,P''=B\,C\,M''\,P''$, &c. from which it follows, that if the abscissæ $A\,P$, $A\,P'$, $A\,P''$, &c. are taken in geometrical progression, the corresponding areas $B\,C\,M\,P$, $B\,C\,M'P'$, $B\,C\,M''P''$, &c. will be in arithmetical progression. FIG. 40.

225. The hyperbola which we have just considered, being equilateral, has only furnished us with Naperian logarithms; but by varying the angle of the asymptotes, and always taking $A\,B=1$, we may obtain an infinite number of other systems of logarithms. Let $U\,M\,V$, fig. 41, be any hyperbola whatever; drawing the ordinates $P\,M$, $P_1\,M_1$, &c. parallel to the asymptote $A\,Y$, we shall FIG. 41.

be able to prove, by a process of reasoning analogous to that in No. 76. that the parallelogram $PMRP'$ is the differential of $BCMP$. Now if we draw $P'Q$ perpendicular to PM, we shall find $P'Q = PP'.\sin P'PQ = PP'.\sin XAY$; representing by ω the angle of the asymptotes, we shall have $P'Q = dx \sin \omega$, and consequently $PMRP' = y\,dx \sin \omega$. If we substitute for y its value $\frac{1}{x}$, there will result $\frac{dx}{x}\sin \omega$ for the differential of the area $BCMP$, and consequently $BCMP = \text{l}.x = \text{l}.AP$, taking $\sin \omega$ for the modulus (27.)

The modulus of common logarithms being 0,4342945 (29), will give $\sin \omega = 0{,}4342945$, from which it follows that the asymptotes of the hyperbola, whose areas are expressed by the common tabular logarithms, make with each other an angle of 25°. 55′. 16″. 19‴, nearly.*

226. By making $AC = a$, $AP = x$, and $PN = y$, fig.
Fig. 42. 42. the equation of the circle ANE will be $y^2 = 2ax - x^2$; and the differential of its segment ANP will have for its expression $dx\sqrt{2ax - x^2}$, which is transformed into $-du\sqrt{a^2 - u^2}$, by making $x = a - u$, and which is also reducible to $du\,(a^2 - u^2)^{-\frac{1}{2}}$, or $\frac{du}{\sqrt{a^2 - u^2}}$, by the formula ($B$) in No. 171, and the complete integral of it is

$$-\tfrac{1}{2}(a - x)\sqrt{2ax - x^2} + \tfrac{1}{2}a^2.\text{arc}\left(\cos = \frac{a - x}{a}\right),$$

when we substitute for u its value, a result which vanishes when $x = 0$.

We easily recognise in the part

$$\tfrac{1}{2}(a - x)\sqrt{2ax - x^2},$$

the expression for the surface of the triangle PCN, and consequently find that

* See Note (I.)

$$\tfrac{1}{2}a^2.\text{arc}\left(\cos = \frac{a-x}{a}\right) \text{ or } \tfrac{1}{2}\,AC.\text{arc}\,AN$$

is the value of the sector ACN.

By supposing $x=2a$, in the expression for ANP, it becomes $\frac{1}{2}a^2.\text{arc}\ (\cos=-1)=\frac{1}{2}a^2\pi$, designating by π the semi-circumference of the circle whose radius is 1, and it then becomes the expression for a semi-circle: we shall have therefore for the whole circle $a^2\pi=\frac{1}{2}a.2a\pi$, as we have already proved in the Elements of Geometry.

The developement of $\int dx\sqrt{2ax-x^2}$, which has been found in No. 179, gives approximate values of the area APN.

227. The ordinate of the ellipse being $\frac{b}{a}\sqrt{2ax-x^2}$,

the elliptic segment AMP will be equal to $\frac{b}{a}\int dx\sqrt{2ax-x^2}$;

and as it commences at the same time with the circular segment ANP, we have $ANP : AMP :: a : b$; for it may readily be inferred from No. 209, that when two differentials are in a constant ratio, the integrals are likewise in the same ratio, when these integrals vanish simultaneously.

It appears from the above, that since the area of a circle described on the axis major of an ellipse as a diameter, is to the area of the ellipse, or the major axis to the minor, the area of the latter must be equal to that of a circle whose radius is a mean proportional between its semi-axes; for it follows from the above-mentioned proportion that the area of the ellipse is $\pi a^2\times\frac{b}{a}$, or πab, and this last quantity evidently represents the area of a circle whose radius is equal to $\sqrt{ab}$.

228. The hyperbola referred to its major axis is expressed by the equation

$$y^2 = \frac{b^2}{a^2}(2ax + x^2),$$

from whence we conclude that

$$AQR = \frac{b}{a}\int dx\sqrt{2ax + x^2}.$$

This integral may be found by means of logarithms (161.) or it may be expanded into a series; but instead of stopping to calculate these results, we shall proceed to the consideration of elliptic and hyperbolic sectors, whose differential expressions are of very frequent occurrence.

FIG. 42. 229. Let $ABab$, fig. 42. be an ellipse whose semi-axis major $AC = a$, and whose semi-axis minor $BC = b$, making $CP = x$, there results

$$PM = y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

It is evident that the sector

$$ACM = CMP + AMP,$$

and that

$$d.\,ACM = d.\,CMP + d.\,AMP,$$

$$CMP = \frac{1}{2}CP \times PM = \frac{1}{2}\frac{bx}{a}\sqrt{a^2 - x^2},$$

$$d.\,CMP = \frac{1}{2}\frac{b}{a}\left(dx\sqrt{a^2 - x^2} - \frac{x^2\,dx}{\sqrt{a^2 - x^2}}\right),$$

$$d.\,AMP = -\frac{b}{a}dx\sqrt{a^2 - x^2}.$$

The last of these differentials is affected with the sign $-$, because the area AMP decreases when x increases; and they give

$$d.\,ACM = -\frac{1}{2}\frac{b}{a}\frac{a^2\,dx}{\sqrt{a^2 - x^2}}.$$

If we make $\frac{b}{a} = 1$, the elliptic sector ACM will be changed into the sector ACN, which belongs to the circle $AEae$ described upon the axis major as a diameter: we shall have therefore

$$d.ACN = -\frac{1}{2}\frac{a^2dx}{\sqrt{a^2-x^2}} = \frac{1}{2}a \times -\frac{adx}{\sqrt{a^2-x^2}};$$

but $-\frac{adx}{\sqrt{a^2-x^2}}$ being the differential of the arc AN, we obtain the same result as from the Elements of Geometry, which is

$$ACN = \frac{1}{2}a \times AN = \frac{1}{2}AC \times AN,$$

and since the sectors ACN and ACM have their common origin in the point A, we may then conclude (228.) that the elliptic sector

$$ACM = \frac{b}{a}ACN = \frac{1}{2}BC \times AN.$$

230. In the hyperbola XAx corresponding to the same axes as those of the ellipse $ABab$, and whose equation therefore is

$$y = \frac{b}{a}\sqrt{x^2 - a^2},$$

the sector $ACR = CQR - AQR$, which gives

$$d.ACR = d.CQR - dAQR;$$

and since

$$CQR = \frac{1}{2}CQ \times QR = \frac{1}{2}\frac{bx}{a}\sqrt{x^2 - a^2},$$

$$d.AQR = \frac{b}{a}dx\sqrt{x^2 - a^2},$$

we shall have

$$d.ACR = \frac{1}{2}\frac{b}{a}\frac{a^2dx}{\sqrt{x^2 - a^2}};$$

from whence we see that the differentials of the hyperbolic and elliptic sectors are identical, with the exception of the signs of the quantities involved.

FIG. 41. 231. The hyperbolic sector ACM, fig. 41, is equal to the asymptotic space $BCMP$; for

$$ACM = BCMP + ABC - AMP,$$

and

$$ABC = \frac{AB \times BC \times \sin B}{2} = \frac{AP \times PM \times \sin B}{2} = AMP.$$

232. The preceding examples are sufficient to shew in what manner the Integral Calculus is applicable to the theory of curves; nevertheless we cannot quit this subject without giving some of the very interesting results which Geometers have obtained, on the subject of transcendental curves.

In the logarithmic curve, whose equation is $y = \mathrm{l}\, x$, we have $\int y\,dx = \int dx\,\mathrm{l}\,x = x\,\mathrm{l}x - x + \text{const.}$ (182). The variable part of this expression becomes equal to nothing when $x = 0$; for by making $x = \frac{1}{m}$, it takes the form $-\frac{\mathrm{l}\,m}{m} - \frac{1}{m}$, which is evanescent when m is infinite (58); it from hence appears to be unnecessary to add a constant to the integral, when we wish to express the value of the segments com-

FIG. 43. mencing from the point A, fig. 43.

By making $x = AE = 1$, we obtain the expression for the asymptotic space $cAEx$, which is finite and equal to -1.

If we consider the axis of the ordinates as forming the line of the abscissæ, we shall have $\int x\,dy = \int dx = x$, for the space $COMX$, terminated by the axis AC, for which the expression is algebraical: we have added a constant to it, since it vanishes at the same time with x. The space $cAEx$, which corresponds to $x = AE = 1$, has by this formula the same expression as by the preceding, the sign not being considered.

We have hitherto supposed the modulus equal to unity; if it be denoted by M, we should have

$\int dx \log x = x \log x - \int M dx = x \log x - Mx$ and $\int x dy = M x$.

233. The equation of the cycloid being

$$dx = \frac{y\,dy}{\sqrt{2ay - y^2}} \quad (102),$$

there results

$$\int y\,dx = \frac{y^2\,dy}{\sqrt{2ay - y^2}};$$

it would be easy to integrate this expression by means of arcs of circles; but we may obtain a more simple result by making $z = 2a - y$, which gives

$$dy = -dz, \; dx = -\frac{(2a - z)\,dz}{\sqrt{2az - z^2}}.$$

In fact, z representing the ordinate QM, fig. 44, referred to the line CK, the differential of the area $ACQM$ will be expressed by FIG. 44.

$$z\,dx = -\frac{(2az - z^2)\,dz}{\sqrt{2az - z^2}} = -dz\sqrt{2az - z^2};$$

and consequently

$$ACQM = -\int dz\sqrt{2az - z^2} + \text{const.}$$

At C, where $z = 2a$, the integral $\int dz\sqrt{2az - z^2}$ is equal to the area gmq of the generating circle, and it is evanescent at the point K, where $z = 0$; consequently the space ACK is equal to the semi-circle $gmqg$. For any point Q, $\int dz\sqrt{2az - z^2}$ will give the area of the segment gmn corresponding to $gn = QM$, and we shall have

$ACQM = gmqg - gmn$, $KMQ = ACK - ACQM = gmn$.

The rectangle AK, having its altitude $IK = gq$, and its base $AI = gmq$, will be quadruple of the semi-circle $gmqg$, and if we subtract from this rectangle, the space $ACK = gmqg$, there will remain $AMKI = 3\,gmqg$. It

follows from this that the space $AKLA$, comprised between the curve of the cycloid and its axis, is triple of the generating circle.

234. It yet remains to speak of spirals; and we shall first consider those which are represented by the equation $u = at^n$ (104), in which t is equal to the arc ON, fig. 45,
Fig. 45. and $u = AM$. Since this is a polar equation, the differential of the area will be $\frac{u^2dt}{2}$ (111); putting for u its value, and integrating, there will result $\frac{a^2t^{2n+1}}{4n+2}$ + const.; but the constant may be suppressed if we reckon the areas from the line AO, where $t=0$, and consequently the area $ACM = \frac{a^2t^{2n+1}}{4n+2}$. After one revolution of the radius vector, we shall have the space $ACMB = \frac{a^2(2\pi)^{2n+1}}{4n+2}$, π being the semi-circumference of the circle ON; when the radius vector returns to the position AM, we shall have the space $ACMBC'M' = a^2\frac{(2\pi+ON)^{2n+1}}{4n+2}$, and so on for other positions of AM.

In the spiral of Archimedes (104), $a = \frac{1}{2\pi}$, $n = 1$, and $ACM = \frac{t^3}{24\pi^2}$, a result, which becomes, when $t = 2\pi$, equal to $\frac{\pi}{3}$.

In the hyperbolic spiral, where $n = -1$, we find

$$ACM = -\frac{a^2}{2t} + \text{const.}$$

The area of this curve, which makes an infinite number of revolutions round the point A, is infinite when $t = 0$; we

must proceed therefore in this case in the hyperbolas, and the areas comprehended between the two distances corresponding to $t=b$ and $t=c$, will be found to be

$$\frac{a^2}{2}\left(\frac{1}{b}-\frac{1}{c}\right).$$

Finally in the logarithmic spiral (114), $t=1u$, $dt=\frac{du}{u}$, and the differential $\frac{u^2dt}{2}$ becoming $\frac{u\,du}{2}$, gives $ACM=\frac{u^2}{4}$. This area is nothing when $u=0$, in which case t is infinite; for this curve, as well as the preceding makes an infinite number of revolutions round the pole A.

235. The differential of the arc of a curve when referred to rectangular co-ordinates, is expressed by $\sqrt{dx^2+dy^2}$ (75); if we substitute in this expression, instead of dy^2, its value deduced from the differential equation of the proposed curve, it will assume the form $X\,dx$, and its integral will give the length of the arc of this curve. To demand the length of the arc of a curve, is to demand its *rectification*, since the accurate solution of this problem, affords us the means of assigning a right line whose length is equal to that of the arc in question.

236. We shall take for our first example the parabolas of different species, which are represented by the equation $y=px^n$, n being any number either whole or fractional: there results

$$dy=npx^{n-1}dx,\ \sqrt{dx^2+dy^2}=dx\sqrt{1+n^2p^2x^{2n-2}};$$

the parabolic arc will therefore be expressed by

$$\int(1+n^2p^2x^{2n-2})^{\frac{1}{2}}\,dx.$$

This integral may be obtained under a finite and algebraical form, when the exponent $2n-2$ is equal to unity, or is contained in it any number of times exactly (169).

In the first place let $2n-2=1$, from which it follows that $n=\frac{3}{2}$, and

$$\int(1+n^2p^2x^{2n-2})^{\frac{1}{2}}dx=\frac{8}{27p^2}\left(1+\frac{9}{4}p^2x\right)^{\frac{3}{2}}+\text{const.}$$

the curve proposed will be determined by the equation $y=px^{\frac{3}{2}}$, or $y^2=p^2x^3$, and consequently is the parabola of the third order, which is the evolute of the common parabola (99). If we reckon the arc from the point where $x=0$, we shall have

$$\frac{8}{27p^2}\left[\left(1+\frac{9}{4}p^2x\right)^{\frac{3}{2}}-1\right].$$

By making successively $2n-2=\frac{1}{2},=\frac{1}{3}$, &c. there will result $n=\frac{5}{4}$, $\frac{7}{6}$, &c. which shews that the parabolas represented by the equations $y^4=p^4x^5$, $y^6=p^6x^7$, are rectifiable: the rectification of all other parabolas can only be effected by approximation.

For the common parabola, in which $n=2$, we have $\int dx\,(1+4p^2x^2)^{\frac{1}{2}}$: by the formula (B) of No. 171, we find $\int dx\,(1+4p^2x^2)^{\frac{1}{2}}=\frac{1}{2}x\,(1+4p^2x^2)^{\frac{1}{2}}+\frac{1}{2}\int\frac{dx}{\sqrt{1+4p^2x^2}}$;

and since

$$\int\frac{dx}{\sqrt{1+4p^2x^2}}=\frac{1}{2p}\,(\text{l}2px+\sqrt{1+4p^2x^2})+\text{ const. }(162),$$

there will result

$$\int dx\,(1+4p^2x^2)^{\frac{1}{2}}=\frac{1}{2}x\,(1+4p^2x^2)^{\frac{1}{2}}+\frac{1}{2p}\,\text{l}\,(2px+\sqrt{1+4p^2x^2})+\text{c}$$

This is the value of any arc of the common parabola; we may suppress the constant in this expression, if we suppose the integral to commence when $x=0$.

The arc of the hyperbolas determined by the equation $y=px^{-n}$, is expressed by $\int x^{-n-1}dx\,(x^{2n+2}+n^2p^2)^{\frac{1}{2}}$, which cannot be integrated unless by approximation.

237. The differential of the arc of a circle is $\frac{adx}{\sqrt{a^2-x^2}}$, or $\frac{adx}{\sqrt{2ax-x^2}}$ (75), according as we employ the equation $y^2=a^2-x^2$ or $y^2=2ax-x^2$; neither of these expressions admits of integration except by approximation, for which purpose we have already given several series (179).

238. We now proceed to the ellipse, and we shall assume for its equation $y^2=\frac{b^2}{a^2}(a^2-x^2)$; the differential of its arc will be $\frac{dx\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}$; making for greater simplicity, the major axis $a=1$, and the square of the excentricity $a^2-b^2=1-b^2=e^2$, the arc will become

$\int\frac{dx\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}$. We have already, in No. 180, deduced a series which gives an approximate value of this integral, when e is very small, and which will apply to all ellipses of inconsiderable eccentricity.

If in this series we suppose $x=1$, and put $\frac{\pi}{2}$ in the place of A which in that case becomes a quadrant, there results

$$\frac{1}{2}\pi\left(1-\frac{1}{2}e^2-\frac{1.1.1.3}{2.2.4.4.}e^4-\frac{1.1.1.3.3.5}{2.2.4.4.6.6}e^6-\&c.\right),$$

a series which converges with great rapidity when e is a small fraction.

239. The differential of an elliptic arc is expressed in a very simple manner, by means of the arc corresponding to it in a circle described on the axis major of the ellipse as a diameter. Let. $EN=\phi$, fig. 42, we shall have FIG. 42.

$$CP=x=\sin\phi,\quad \frac{dx}{\sqrt{1-x^2}}=d\phi,$$

and consequently

$$d.BM = d\phi\sqrt{1-e^2\sin\phi^2}.$$

240. The equation of the hyperbola being

$$y^2 = \frac{b^2}{a^2}(x^2-a^2),$$

we have $\dfrac{dx\sqrt{(a^2+b^2)x^2-a^4}}{a\sqrt{x^2-a^2}}$ for the differential of its arc; making $a=1$, $a^2+b^2=1+b^2=e^2$, this arc will be expressed by $\int \dfrac{dx\sqrt{e^2x^2-1}}{\sqrt{x^2-1}}$ and may, when e is nearly equal to unity, be developed in a series by a process analogous to that in No. 180.

241. It now only remains to make a few remarks concerning the rectification of transcendental curves. The equation of the cycloid being

$$\frac{dx}{dy} = \frac{y}{\sqrt{2ay-y^2}} \quad (102),$$

we thence deduce

$$\sqrt{dx^2+dy^2} = \frac{dy\sqrt{2a}}{\sqrt{2a-y}},$$

a differential whose integral is

$$= -2\sqrt{2a(2a-y)} + \text{const.}$$

But it is evident that $\sqrt{2a(2a-y)}$ is the expression for
FIG. 44. the chord mg, fig 44, of the generating circle; and as the variable part of the integral vanishes at the point K where $y=2a$, it follows consequently that it expresses the arc MK: we have therefore $MK=2mg$, $AK=2qg$, and therefore $AM=AK-MK=2(gq-mg)$: these results agree with that in No. 103.

242. To give an example of the application of the formula $\sqrt{u^2dt^2+du^2}$, which expresses the differential of the arc of a curve referred to polar co-ordinates,

(110), we shall take the case of the spirals which are represented by the equation $u=at^n$; and we shall have to integrate the differential

$$dt\sqrt{a^2t^{2n}+n^2a^2t^{2n-2}}=at^{n-1}dt(t^2+n^2)^{\frac{1}{2}}.$$

When $n=1$, we have simply $adt\,(t^2+1)^{\frac{1}{2}}$, a differential of the same form as that for the arc of the common parabola (236); from which it follows that it is the rectification of this curve upon which depends that of the spiral of Archimedes.

In the logarithmic spiral, we have $t=\mathrm{l}u$, which gives $\sqrt{u^2dt^2+du^2}=du\sqrt{2}$; the arc of this curve has therefore for its expression $u\sqrt{2}$ + const., or simply $u\sqrt{2}$, commencing from the origin of the radius vectors; and we see, that though there is between this origin and any point of the curve at an infinite distance from it, an infinite number of revolutions, yet they include an arc of finite length, which is equal to the diagonal of the square described on the radius vector.

On the Cubature of Solids terminated by Curve Surfaces, and on the Quadrature of their Surfaces; on the Rectification of Curves of double Curvature.

243. The curve surfaces which Geometers first considered, were those of revolution, since the differentials of their areas and of their solid contents have a more simple expression, than those which correspond to curve surfaces in general.*

If u represent the solid content of the body generated by the segment AMP, fig. 46, of any curve AZ revolving round the axis AB taken in its plane, it is evident FIG. 46.

---

* See Note (L.)

that its volume, which is terminated by the circular base described by the ordinate PM, is a function of the abscissa $AP=x$. If we take another abscissa AP', and if we draw a second ordinate $M'P'$ and the right lines MR and SM', parallel to PP', we shall see that the volume u is increased by the whole volume generated by the curvilinear trapezium $PMM'P'$, whilst it revolves round PP', and that this last body, which is comprized between the cylinders generated by the rectangles MP' and $M'P$, differs so much the less from either of these bodies, the nearer the points M and M' are to each other, so that the limit of the ratios of these three bodies to each other is united; we may therefore, when the limits alone are concerned, take the cylinder described by the rectangle MP' for the body generated by $PMM'P'$. The base of this cylinder being the circle described by the radius $PM=y$, its volume will be $\pi y^2 \times PP'$, π representing the ratio of the circumference of a circle to its diameter; and we shall find, by the same reasoning as in No. 76, that $\frac{du}{dx}=\pi y^2$, whence $u=\pi\int y^2dx$. When therefore we have the equation of the curve AMZ, we may substitute for y its value in terms of x, and by integration we shall obtain the value of the volume of any segment of the body which is generated by this curve.

244. To find the differential of the area of the surface of the same body, it is necessary to observe that its increment, or the area described by the arc MOM' which approximates perpetually to the chord MM', will itself approximate to the area of the curve surface of the truncated cone which is described by this chord; and in reasoning of limits, we may assume one for the other. But the area of the curve surface of the truncated cone described by MM', will be expressed by

$$\tfrac{1}{2}MM'(2\pi MP+2\pi M'P')$$
$$=\pi MM'(MP+M'P');$$

and if we compare it with the increment of the abscissa PP', we shall get

$$= \frac{MM'}{PP'}(MP + M'P');$$

or, taking the limits, $M'P'$ becomes identical with MP or y, and $\frac{MM'}{PP'} = \sqrt{1 + \frac{dy^2}{dx^2}}$ (75): consequently the differential coefficient of the area described by the arc AM is equal to

$$2\pi y \sqrt{1 + \frac{dy^2}{dx^2}},$$

and therefore $2\pi y\sqrt{dx^2 + dy^2}$ is the differential of this area.

We may obtain immediately both this expression and that in the preceding No., if we consider the curve AMZ as a polygon; for then the element of the volume is the cylinder generated by the rectangle MP', and that of the area of the surface is the area of the truncated cone described by the chord MM'.

245. We shall not dwell long on the application of these expressions, which in themselves present but little difficulty. If we take the equation to the ellipse $y^2 = \frac{b^2}{a^2}(2ax - x^2)$, we shall find that the volume of the body which it generates by revolving round its axis major, or the *elongated ellipsoid** is equal to $\frac{4\pi ab^2}{3}$, since a segment of this body has for its expression

$$\int \frac{\pi b^2}{a^2}(2ax - x^2)dx = \frac{\pi b^2}{a^2}\left(ax^2 - \frac{x^3}{3}\right) + \text{const.} \quad (243).$$

When $a = b$, the proposed body becomes a sphere, and the expression for its volume becomes $\frac{4\pi a^3}{3}$, the same that is found by the Element of Geometry.

---

* Prolate Spheroid.

If the abscissæ of the ellipse commenced at its centre, or if we employed the equation $y^2=\frac{b^2}{a^2}(a^2-x^2)$, we should have the same result, observing that in order to comprehend the entire body, it would be necessary to take the integral from $x=a$ to $x=-a$. The volume of the segment would then become

$$\int\frac{\pi b^2}{a^2}(a^2-x^2)\,dx=\frac{\pi b^2}{3a^2}(3a^2x-x^3)+\text{const.}$$

and we should also have

$$\int\frac{2\pi b dx\sqrt{a^4-(a^2-b^2)x^2}}{a^2}$$

for the expression for the area of its surface. This integral is easily shewn to be dependent on the area of a circular segment whose abscissa is x and whose radius is $\frac{a^2}{\sqrt{a^2-b^2}}$; when we suppose $a=b$, it becomes

$$\int 2\pi a dx=2\pi ax+\text{const.}$$

and gives, by taking it from $x=a$, to $x=-a$, $4\pi a^2$ for the value of the whole surface of the sphere.

246. We now proceed to consider curve surfaces in general, by referring them to three planes perpendicular to each other, by means of the three co-ordinates $AP=x$,

FIG. 47. $PM'=y$, $M'M+z$, fig. 47.

The segment $APGMM'Q$, having its base $APM'Q$ upon the plane of the x, y, and which is terminated by the two planes $PM'MG$, $QM'MH$, respectively parallel to that of the y, z, and that of the x, z, and by the curve surface proposed, is necessarily a function of the two independent variables x and y; it may be extended successively in the direction of each, or vary with respect to them both simultaneously. In fact, if we suppose y to remain constant, and x to be changed into $AP+Pp$, this segment will be increased by the segment $PGMM'm'mpg$, and also by the segment $QHMM'n'nqh$, if we suppose y alone to

vary by the quantity Qq: finally, if x and y become simultaneously $AP+Pp$, $AQ+Qq$, the same segment will then be terminated by the planes $pN'Ng$, $qN'Nh$, and will differ from its primitive value by the two segments already mentioned, and by the prism $M'm'N'n'nMmN$, which is nothing but the increment of the first segment when we make y alone to vary, and that of the second when in this last we suppose x alone to vary.

In order to abridge, we shall represent by Pm, Qn, and $M'N$, the two segments and the prism of which we have been just speaking; we shall also represent by u the function of x and y which expresses the volume of the segment $APGMM'Q$. This being premised, the differential coefficient $\frac{du}{dx}$, relative to the variable x, expressing the limit of the ratio of the increment of the function to that of this variable, will be equal to the limit of the ratio of the segment Pm to the increment Pp of the abscissa. If we afterwards make y or AQ alone to vary, which will change Pm into $Pm+M'N$, the ratio $\frac{Pm}{Pp}$ will be increased by the quantity $\frac{M'N}{Pp}$; and the ratio of this last with the increment Qq of y, will have evidently for a limit $\frac{d\frac{du}{dx}}{dy}$ or $\frac{d^2u}{dxdy}$; we shall know therefore this differential coefficient, if we succeed in determining in a function of the variables x and y the limit of $\frac{M'N}{Pp\times Qq}$. Now the prism $M'N$ approximates to the parallelopiped formed on the base $M'm'N'n'$ and the ordinate MM', and may be made to differ from it by as small a quantity as we please; but, if we take one for the other, since we are now reasoning of limits, we may substitute $\overline{M'm'}\times\overline{M'n'}\times\overline{M'M}$, for the prism $M'N$;

and since $M'm'=Pp'$, $M'n'=Qq'$, the ratio $\frac{M'N}{Pp\times Qq}$ is reduced to $M'M=z$. It follows from this that $\frac{d^2u}{dxdy}=z$; and that in order to obtain the value of the segment $APGMM'Q$, it is necessary, by means of integration, to remount from the differential coefficient $\frac{d^2u}{dxdy}$ to the function u.

247. Although the differential coefficient $\frac{d^2u}{dxdy}$ is relative to two variables, we may nevertheless arrive at its primitive function, by the methods given for the integration of functions of one variable only, inasmuch as each of these is considered as constant in its turn. In fact, since $\frac{d^2u}{dxdy}=\frac{d\frac{du}{dx}}{dy}$, we shall have $\frac{d\frac{du}{dx}}{dy}dy=z\,dy$; and taking the integral of each member, considering y alone as variable, there will result $\frac{du}{dx}=\int z\,dy$, from whence we shall deduce

$$\frac{du}{dx}dx=dx\int z\,dy:$$

integrating again, but with reference to x alone, we shall find

$$u=\int dx\int z\,dy.$$

If we only consider this investigation in a light purely analytical, it is evident that the constant which it will be necessary to add to complete the first integral, may involve x in any manner whatever: and that which we annex to the second integral, ought to be considered as any function whatever of y; and this is the case, because every function

of x ought to disappear as a constant, when we differentiate with reference to y only, and the same must happen to every function of y, when we differentiate solely with reference to x.

The order in which the integrations succeed each other is indifferent (122). If we had first considered x as variable, we should have had $\frac{d^2 u}{dx dy} = \frac{d \frac{du}{dy}}{dx}$; and from this we should have successively deduced

$$\frac{du}{dy} = \int z\,dx, \text{ and } u = \int dy \int z\,dx.$$

This result and the preceding may be written as follows;

$$u = \iint z\,dy\,dx, \text{ and } u = \iint z\,dx\,dy,$$

making the two differentials to succeed the last sign $\int$, of integration, which is allowable if we keep in mind that each sign is relative to one of the variables only.

To illustrate and confirm the principles preceding, let us suppose $z = \frac{1}{x^2+y^2}$; there will result

$$u = \iint \frac{dx\,dy}{x^2+y^2} = \int dx \int \frac{dy}{x^2+y^2} = \int dy \int \frac{dx}{x^2+y^2}.$$

The first succession of integrals gives

$$\int \frac{dy}{x^2+y^2} = \frac{1}{x} \text{ arc } \left(\tan = \frac{y}{x}\right) + X',$$

a result in which X' represents an arbitrary function of x, added in order to complete the integral; integrating again with reference to x; and making $\int X'\,dx = X$, we find

$$\int dx \int \frac{dy}{x^2+y^2} = \int dx \left[\frac{1}{x} \text{ arc } \left(\tan = \frac{y}{x}\right) + X'\right]$$

$$= \int \frac{dx}{x} \text{ arc } \left(\tan = \frac{y}{x}\right) + X.$$

The integral $\int \frac{dx}{x}$ arc $\left(\tan = \frac{y}{x}\right)$ is obtainable in series, substituting in the place of arc $\left(\tan = \frac{y}{x}\right)$ its developement $\frac{y}{x} - \frac{y^3}{3x^3} + \frac{y^5}{5x^5} - \&c.$ (176); and as it is necessary, after this integration, to add a function of y, which being represented by Y, we shall finally get

$$\int\int \frac{dx\,dy}{x^2+y^2} = X + Y - \frac{y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \&c.$$

If we proceed in an inverse order, taking the second succession of integrals, we shall find

$$\int \frac{dx}{x^2+y^2} = \frac{1}{y} \text{ arc } \left(\tan = \frac{x}{y}\right) + Y',$$

$$\int dy \int \frac{dx}{x^2+y^2} = \int dy \left[\frac{1}{y} \text{ arc } \left(\tan = \frac{x}{y}\right) + Y'\right]$$

$$= \int \frac{dy}{y} \text{ arc } \left(\tan = \frac{x}{y}\right) + Y;$$

but if we observe that

$$\text{arc } \left(\tan = \frac{x}{y}\right) = \frac{\pi}{2} - \text{arc } \left(\tan = \frac{y}{x}\right),$$

we shall have after the final integration and the addition of an arbitrary function of x,

$$\int\int \frac{dx\,dy}{x^2+y^2} = \frac{\pi}{2} \,\mathrm{l}\, y - \int \frac{dy}{y} \text{ arc } \left(\tan = \frac{y}{x}\right) + Y + X;$$

and as we may comprehend the term $\frac{\pi}{2} \,\mathrm{l}\, y$ in the arbitrary function Y, this result, which will be thus changed into

$$\int\int \frac{dx\,dy}{x^2+y^2} = X + Y - \int \frac{dy}{y} \text{ arc } \left(\tan = \frac{y}{x}\right),$$

will be identical with the preceding, as we may easily con-

vince ourselves by substituting for the arc $\left(\tan = \frac{y}{x}\right)$ the series by which it is expressed.

248. When we consider $\int\int z\, d\,x\, dy$ as expressing the volume of a body, it is necessary to pay attention to the limits between which each integral ought to be taken, and which depend upon the nature of the surfaces by which the proposed body is terminated laterally.

The most simple case is that in which the body is connected by four planes, parallel two and two respectively to the co-ordinate planes CAD, BAD. If we suppose that the first two correspond to the abscissæ $x=a$, and $x=a'$, and the second to the abscissæ $y=b$, and $y=b'$, we must take the integral $\int z\, d\,x$, from $x=a$ to $x=a'$, considering y as constant; and if we call the result P, it will remain to take the integral $\int P\, dy$, from $y=b$ to $y=b'$.

When the proposed body is terminated laterally by curve surfaces, the extreme values of one of these variables are connected with those of the other, as we shall see in the following example, in which it is proposed to find the volume of a sphere whose center is in A, and whose radius is equal to r.

We have $x^2+y^2+z^2=r^2$, and consequently

$$\int\int z\, d\,x\, dy = \int\int dx\, dy\, \sqrt{r^2-x^2-y^2};$$

we find at first, supposing y constant (161, and 173.)

$$\int z\, d\,x = \int dx\, \sqrt{r^2-x^2-y^2} = \frac{1}{2}\, x\, \sqrt{r^2-x^2-y^2}$$

$$+\frac{1}{2}(r^2-y^2)\,\text{arc}\left(\sin = \frac{x}{\sqrt{r^2-y^2}}\right).$$

In this case the extreme value of x is represented by QF, the ordinate of the circle $BFEC$, which limits the intersections of the sphere with the plane BAC; and if the

volume which it is required to find, be terminated by the plane CAD, it is evident that the integral above-mentioned ought to be taken from $x=0$ to $x=QF$. But QF is dependent on AQ, for if $z=0$, we find $x^2+y^2=r^2$ for the equation of the circle $BFEC$, from which it follows that $QF=\sqrt{r^2-\overline{AQ}^2}$; and consequently for any given value of y, the extreme values of x are $x=0$ and $x=\sqrt{r^2-y^2}$.

The result obtained above reduces itself to $\frac{\pi}{4}(r^2-y^2)$, since the arc $(\sin=1)=\frac{\pi}{2}$, and the integral $\int dy\int z\,dx$ becomes

$$\frac{\pi}{4}\int dy\,(r^2-y^2)=\frac{\pi}{4}\left(r^2-\frac{y^3}{3}\right).$$

This last integral ought to be taken between the greatest value of y, which in the case before us is $AC=r$, and the least, which we will assume equal to nothing, supposing the body bounded on this side by the plane: the volume therefore of the segment $ABCD$, which is the eighth part of the sphere, is $\frac{\pi r^3}{6}$. It is proper to remark that we may obtain immediately the volume of the whole hemisphere above the plane BAC, by taking the first integral from $x=+\sqrt{r^2-y^2}$, to $x=-\sqrt{r^2-y^2}$; for in this case the extreme values of x are terminated by the circumference of the circle $BFEC$, whose radius is AC, and we have thus the complete value of $\int z\,dx=\frac{\pi}{2}(r^2-y^2)$. Taking afterwards

$$\int dy\int z\,dx=\frac{\pi}{2}\left(r^2y-\frac{y^3}{3}\right),$$

from $y=r$, to $y=-r$, that is to say, from the extremity C of the diameter of the circle $BFEC$, to the other extremity

which falls behind the plane BAD, we obtain $\frac{2\pi r^3}{3}$, and doubling this, we have, as above, $\frac{4\pi r^3}{3}$ for the whole sphere.

249. In considering differentials as the indefinitely small increments of the variables or of the functions on which they depend, it is evident that the complete value of $dy\int z\,dx$ is the expression for the segment $FHQqhf$ comprised between two planes parallel to the plane ABD of the x and z; but $\int z\,dx$ being the area of the section FHQ, it follows that the indefinitely small segment $FHQqhf$ may be considered as equal to $\overline{FHQ} \times \overline{Qq}$, that is to say, to the area of the curve which forms the base of the segment, multiplied by the thickness Qq. We finally observe that $\int dy\int z\,dx$ is the sum of all the corresponding segments which are comprised in the whole volume of the body.

250. In general, if it was necessary to determine the portion of the proposed body, terminated laterally by the cylindrical figure raised perpendicularly to the plane ABC, fig. 48, upon the given curve $E'N'G'$, we must take the given integral $\int z\,dx$, from $x=AP$, to $x=Ap$, so that the expression $dy\int z\,dx$ may become that of the segment $MM'N'Nnn'm'm$. The lines AP and Ap, respectively equal to QM', and QN', will be given in a function of $AQ=y$, by the equation of the curve $E'N'G'$, of which they are the abscissæ; if we represent them by $F(y)$ and $f(y)$, we ought to take $\int z\,dx$, from $x=F(y)$ to $x=f(y)$, which will, as we see, introduce new functions of y which z would not comprehend, and which may increase or diminish the difficulty of the second integration. In order to obtain in this last integral the total value of the volume sought for, or the sum of the segments for which we have already found a general expression, it will be necessary to

Fig. 48.

take $\int dy \int z\,dx$, from $y=AF$ to $y=AH$, values which answer to the limits E' and G', of the curve $E'N'G'$, in the direction of the y (80).

It may happen that the *contour* $E'N'G'$, instead of being a continuous curve, may be an assemblage of several portions of different curves; the application of the principles given above to this case is too easy to render it necessary to detain ourselves with it.

251. We arrive at the general expression for the differential of the area of a curve surface, by supposing this surface to be divided into zones, such as $EGge$, fig. 47, by means of planes parallel to one of the co-ordinate planes, and by considering each of these zones as separated into quadrangular portions $MmNn$, by planes parallel to another co-ordinate plane. By the inspection of the figure, we see that the area $DGMH$ which we will represent by s, is increased by the quadrilateral figure $GMmg$, when x is increased by Pp, which is also increased by $MmNn$, when y is afterwards augmented by Qq. By a process of reasoning, similar to that in No. 246, we shall see that the limit of the ratio $\frac{MmNn}{Pp \times Qq}$ is equal to the differential coefficient $\frac{d^2 s}{dx\,dy}$.

FIG. 47.

To obtain this limit, we at first observe the four planes

$$m'M \text{ and } N'n,\ n'M \text{ and } N'm,$$

parallel two and two to the planes of the x, z, and the y, z, which determine the quadrilateral curve surface $MmNn$, determine likewise upon the tangent plane at the point M, fig. 49, a parallelogram $MXYZ$, in which all the lines drawn from the point M, would be tangents to the different sections which would be made in the quadrilateral curve surface by planes passing through the ordinate MM', and would also have with the arcs of these sections a ratio

FIG. 49.

constantly approximating to unity (74); we may, therefore, in the limit which is required to be found, substitute instead of the quadrilateral curve surface $MmNn$, the parallelogram $MXYZ$, which is equal to the square root of the sum of the squares of its projections upon the three co-ordinate planes *; now these projections being formed by lines which are parallel to each other, are necessarily parallelograms: that which is formed upon the plane of the x, y, is the rectangle $M'm'N'n'$ which is expressed by $dxdy$. By drawing YY'' and ZZ'' parallel to $M'n'$ and $m'N'$, we should form the projection $MXZ''Y''$ upon the plane of the x, z, equal to $MY'' \times M'm'$; and since

$$MY'' = n'Y - M'M = \frac{dz}{dy}dy,$$

we shall have

$$MXZ''Y'' = \frac{dz}{dy}dxdy.$$

We shall find in a similar manner that the projection upon the plane of the y, z, is $\frac{dz}{dx}dxdy$: we shall have therefore

$$MXZY = dxdy\sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}} = dxdy\sqrt{1 + p^2 + q^2},$$

making $\frac{dz}{dx} = p$, and $\frac{dz}{dy} = q$; and hence there results

$$\frac{MXYZ}{\overline{Pp} \times \overline{Qq}} = \frac{d^2s}{dxdy}\sqrt{1 + p^2 + q^2}.$$

This shews that the area of the surface is determined, as well as the volume, by a differential coefficient of the second order, and that we obtain both the one and the other by the same mode of integration; so that $dy\int dx$

---

* This proposition is demonstrated in No. 61. of my Complément des Elémens de Géometre.*

* See Note (M.)

$\sqrt{1+p^2+q^2}$ represents the area of the zone $FHhf$, fig. 47, and that we have

$$s=\int dy\int dx\sqrt{1+p^2+q^2}=\iint dx\,dy\sqrt{1+p^2+q^2}.$$

252. The application of Analysis to Mechanical questions often conducts us to integrals of the form $\iiint V\,dx\,dy\,dz$, which we call *triple integrals*, from their analogy to those of the form $\iint V\,dx\,dy$, designated by the name of *double integrals*. In the first, the function V may involve three variables x, y, z, each being considered as independent of the other two, so that each sign of integration may apply to one of the variables only. It is easily seen, that these integrals arise from the determination of a function u, involving three variables x, y, z, and of which we only know the differential coefficient $\frac{d^3u}{dx\,dy\,dz}$, given by the equation $\frac{d^3u}{dx\,dy\,dz}=V$; for we deduce from this, by operating in the same manner as in No. 247, 1st, by considering x and y as constant,

$$\frac{d^3u}{dx\,dy\,dz}dz=d.\frac{d^2u}{dx\,dy}=V\,dz,\ \frac{d^2u}{dx\,dy}=\int V\,dz+T'',$$

T'' being an arbitrary function of x and y; 2d, by assuming x and z as constant, we get

$$\frac{d^2u}{dx\,dy}dy=d.\frac{du}{dx}=dy\int V\,dz+T''dy,\ \frac{du}{dx}=\int dy\int V\,dz+T'+S',$$

T' representing an arbitrary function of x and y, resulting from $\int T''\,dy$, and S' an arbitrary function of x and z. 3dly, by considering y and z as constant,

$$\frac{du}{dx}dx=du=dx\int dy\int V\,dz+T'dx+S'dx$$

$$=\int dx\int dy\int V\,dz+T+S+R,$$

T and S representing the arbitrary functions severally resulting from $\int T'\,dx$ and $\int S'\,dx$, and R being an arbitrary function of y and z: the complete integral therefore includes three arbitrary functions; namely, one of x and y, one of x and z, and one of y and z. By representing the

differentials under the last sign of integration, $\int dz \int dy \int V dx$ becomes $\int\int\int V dx\, dy\, dz$, and has, under this last form, the same signification as under the preceding.

This example is sufficient to shew in what manner we may revert from the differential coefficient of any order of a function of several variables, to the function itself. The arbitrary functions introduced here only refer, as in No. 247, to those cases in which the integrals are taken between the limits within which the variables $x, y, z,$ are independent of each other; but it most frequently happens that the integral relative to z ought to be taken from $z = F(x, y)$ to $z = f(x, y)$, F and f being given functions, the integral relative to y, from $y = F_1(x)$ to $y = f_1(x)$, and lastly the integral relative to x, from $x = a$ to $x = a'$.

On the Integration of Differential Equations of two Variables.

On the Separation of the Variables in Differential Equations of the first Order.

253. In all that we have hitherto said on the subject of the Integral Calculus we have supposed that the differential coefficients were expressed directly by means of the variable on which their primitive function depended; but most commonly we have merely a differential equation in which these different quantities are involved. For the first order, the differential equation when it is of the first degree with respect to dx and dy, has necessarily the form $Mdx + Ndy = 0$, and it expresses, as we have shewn in No. 31, the relation which subsists between the variable x, the function y, and its differential coefficient $\frac{dy}{dx}$.

The method, which first presented itself to Analysts, of discovering the primitive equation from which

this derives its origin, was that of endeavouring to separate the variables, that is to say, of reducing the equation $M\,dx + N\,dy = 0$ to the form $X\,dx + Y\,dy = 0$, X being a function of x alone, and Y a function of y. In fact, when we have arrived at this equation, the terms $X\,dx$ and $Y\,dy$ are integrated by the methods taught in the preceeding pages; and we have $\int X\,dx + \int Y\,dy = C$, where C is an arbitrary constant.

254. In order to give an example of those cases in which the differential equation presents itself immediately under the above form, let us take $x^m\,dx + y^n\,dy = 0$; we immediately find $\frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{n+1} = C$.

If the proposed equation was $y\,dx - x\,dy = 0$, the separation would be very easily effected; for, dividing by xy, we should find $\frac{dx}{x} - \frac{dy}{y} = 0$; taking separately the integral of each term of this last equation, we should have $\mathrm{l}x - \mathrm{l}y = C$, or $\mathrm{l}\frac{x}{y} = C$: since we may consider the arbitrary constant as a logarithm, we may therefore assume $\mathrm{l}\frac{x}{y} = \mathrm{l}c$. Passing from logarithms to numbers, there would result $\frac{x}{y} = c$, or $x = cy$.

After this example, we readily observe that the separation of the variables will be effected in the same manner in the equations $Y\,dx - X\,dy = 0$, $XY_1\,dx - YX_1\,dy = 0$; for the first gives

$$\frac{dx}{X} - \frac{dy}{Y} = 0,$$

and the second

$$\frac{X\,dx}{X_1} - \frac{Y\,dy}{Y_1} = 0.$$

In general, when we determine the value of $\frac{dy}{dx}$ in the proposed equation, and find $\frac{dy}{dx} = XY$, we easily deduce

$$Xdx - \frac{dy}{Y} = 0,$$

and consequently

$$\int Xdx - \int \frac{dy}{Y} = C.$$

255. There still remains a very general class of differential equations, in which we may easily separate the variables, including all those in which M and N are homogeneous functions of x and y. The principle upon which this separation depends is this, *that if in an algebraical function of the quantities* x, y, z, *where the sum of the exponents of each of these letters is the same in all the terms, and equal to* m, *we substitute* Px *for* y, Qx *for* z, *&c. the result will be divisible by* x^m. In fact, any term of this function which is of the form $Ax^n y^p z^q$, &c. will become by the substitution above-mentioned $AP^pQ^q \ldots\ldots x^{n+p+q+\&c.}$, but by the hypothesis we have in every term $n+p+q+$ &c. $\ldots = m$, and therefore x^m will be a common factor. It follows from hence that if the proposed function was equal to zero, or even if it was a fraction having for its numerator and denominator two homogeneous polynomist functions of the same degree, the quantity x would disappear entirely from the result.

From what has preceded, it appears that it is sufficient to make $y=xz$, in order to separate the variables in the homogeneous equation $Mdx+Ndy=0$; in fact, the functions M and N take the form Zx^m, Z_1x^m, Z and Z_1 being functions of the new variable z only, and since $dy=zdx+xdz$, there arises, by dividing by x^m, $Zdx+Z_1$

$(xdz+zdx)=0$, a result which we may put under the form

$$\frac{dx}{x}+\frac{Z_1 dz}{Z+zZ_1}=0,$$

from which we get

$$\int\frac{dx}{x}+\int\frac{Z_1 dz}{Z+zZ_1}=C.$$

We shall first apply this transformation to the equation

$$x\,dx+y\,dy=ny\,dx,$$

which becomes

$$(x-ny)dx+y\,dy=0;$$

by transposing all the terms to one side, we shall get $Z=1-nz$, $Z_1=z$, and $\int\frac{dx}{x}+\int\frac{zdz}{1-nz+z^2}=C$, or $lx+\int\frac{zdz}{1-nz+z^2}=C$. The integral $\int\frac{zdz}{1-nz+z^2}$ may be simplified by observing that

$$\frac{zdz}{1-nz+z^2}=\frac{1}{2}\frac{2zdz-ndz}{1-nz+z^2}+\frac{1}{2}\frac{ndz}{1-nz+z^2};$$

for it then becomes

$$lx+\frac{1}{2}l(1-nz+z^2)+\frac{1}{2}\int\frac{ndz}{1-nz+z^2}=C.$$

The integral, which it yet remains to find, will depend on logarithms if $\frac{n}{2}>1$, on arcs of circles if $\frac{n}{2}<1$, and will be algebraical if $\frac{n}{2}=1$. We shall only state the result which belongs to this last case: $\int\frac{ndz}{1-nz+z^2}$ then becomes

$$\int\frac{2dz}{(1-z)^2}=\frac{2}{1-z},\ \tfrac{1}{2}l(1-nz+z^2)=l(1-z);$$

and we therefore have $\mathrm{l}\,x + \mathrm{l}\,(1-z) + \frac{1}{1-z} = C$, or $\mathrm{l}\,(x-y) + \frac{x}{x-y} = C$, if we replace z by its value $\frac{y}{x}$.

The term $\frac{x}{x-y}$ may be changed into a logarithm, by simply observing, from the definition of Naperian logarithms, that any quantity u is the logarithm of the number e^u; we may consequently write the preceding equation under the form

$$\mathrm{l}\,(x-y) + \mathrm{l}\,e^{\frac{x}{x-y}} = \mathrm{l}\,c,$$

from which we successively deduce

$$\mathrm{l}.\,(x-y)e^{\frac{x}{x-y}} = \mathrm{l}\,c, \text{ and } (x-y)\,e^{\frac{x}{x-y}} = c.$$

It is proper to pay attention to this method of passing from logarithms to numbers, since it is very frequently used.

Let it be proposed to integrate the equation

$$x\,dy - y\,dx = dx\sqrt{x^2+y^2}.$$

By making $y = xz$, transposing all the terms to one side of the equation, and dividing by x, we shall find

$$dx\sqrt{1+z^2} - x\,dz = 0,$$

which will give

$$\frac{dx}{x} - \frac{dz}{\sqrt{1+z^2}} = 0.$$

We shall also obtain, by the separate integration of each term,

$$\mathrm{l}\,x - \mathrm{l}(z+\sqrt{1+z^2}) = \mathrm{l}\,c, \text{ or } \frac{x}{z+\sqrt{1+z^2}} = c;$$

and replacing z by its value $\frac{y}{x}$, there will result

$$\frac{x^2}{y+\sqrt{x^2+y^2}} = c, \text{ or } -y + \sqrt{x^2+y^2} = c,$$

by multiplying the numerator and denominator of the fraction by $y-\sqrt{x^2+y^2}$: and by exterminating the radical, we shall finally get $x^2 = c^2 + 2cy$.

(256.) The equation

$$(a+mx+ny)\,dx + (b+px+qy)\,dy=0$$

may be easily made homogeneous. Substituting $t+\alpha$, in the place of x, and $u+\beta$ in that of y, we shall have $dx=dt$, $dy=du$, and

$$(a+m\alpha+n\beta+mt+nu)\,dt+(b+p\alpha+q\beta+pt+qu)\,du=0;$$

we may make the constant terms disappear, by assuming $a+m\alpha+n\beta=0$, $b+p\alpha+q\beta=0$, from which equations we may determine the quantities α and β; and there then remains the differential equation

$$(mt+nu)\,dt+(pt+qu)\,du=0,$$

which is homogeneous with respect to the new variables t and u.

The preceding transformation is the same as that which we make use of, in order to change the origin of the coordinates upon a plane (Trig. 116.); it gives no result when $mq-np=0$, a case in which α and β become infinite; but then we have $q=\frac{np}{m}$, and consequently

$$px+qy=\frac{p}{m}(mx+ny):$$

the proposed equation being changed into

$$a\,dx+b\,dy+(mx+ny)\left(dx+\frac{p}{m}\,dy\right)=0,$$

it is sufficient to make $mx+ny=z$, in order to separate the variables.

Substituting this value, as well as that of dy, which results from it, and disengaging dx, we find

$$dx + \frac{(bm + pz)dz}{amn - bm^2 + (mn - pm)z} = 0;$$

the integral of this equation will involve logarithms except in the case where $mn - pm = 0$, when it will be

$$x + \frac{2bmz + pz^2}{2(amn - bm^2)} = C.$$

The transformation employed in this last case has changed the equation into another which contains but one of the variables; and it is readily seen that whatever be the equation which we have thus treated, we shall be able to give it the form $dx + Zdz = 0$, Z being a function of z only, and that we may thence deduce $x + \int Zdz = C$.

257. The separation of the variables may be effected in a very simple manner in the equation $dy + Pydx = Qdx$, where P and Q denote any functions whatever of x. Substituting Xz and $zdX + Xdz$ in the place of y and dy, it becomes

$$zdX + Xdz + PXzdx = Qdx.$$

The quantity X being considered as an indeterminate function of x, we are at liberty to assume it in such a manner that the preceding equation may be divided into two others in which the variables may be separated. Now it is easily seen that this condition will be satisfied, if we make $Xdz + PXzdx = 0$, which gives also $zdX = Qdx$. If we divide the first of these equations by X, it becomes $dz + Pzdx = 0$; from which we deduce $\frac{dz}{z} + Pdx = 0$, $\text{l}z + \int Pdx = 0$, and passing from logarithms to numbers $z = e^{-\int Pdx}$: we neglect here the arbitrary constant, as it will be sufficient to add it at the end of the operation. Afterwards deducing the value of dX from the second equation, and substituting in it the value of z which we have first found, we shall have

$$dX = e^{\int P dx} Q\, dx, \quad X = \int e^{\int P dx} Q\, dx + C,$$

and consequently

$$y = e^{-\int P dx} \left(\int e^{\int P dx} Q\, dx + C\right).$$

The equation $dy + P y\, dx = Q dx$ is remarkable, from the circumstance of the variable y and its differential not exceeding the first degree; and we call it from thence, a *linear equation* of the first order, though it would be more properly termed *an equation of the first degree and of the first order**.

258. The first Analysts who wrote on the subject of the Integral Calculus, classed differential equations by the number of their terms. In those equations which have but two terms, and whose form is consequently $\beta u^g z^h dz = \alpha u^e z^f du$, the variables are separated immediately, since we thence deduce $\beta z^{h-f} dz = \alpha u^{e-g} du$; but this is not the case with equations of three terms, which are comprised in the formula

$$\gamma u^i z^k\, dz + \beta u^g z^h du = \alpha u^e z^f du.$$

We may give it a more simple form, by dividing all the terms by $\gamma u^i z^f$; it will become

$$z^{k-f} dz + \frac{\beta}{\gamma} u^{g-i} z^{h-f} du = \frac{\alpha}{\gamma} u^{e-i} du;$$

then supposing

$$z^{k-f} dz = \frac{dy}{k-f+1}, \qquad u^{g-i} du = \frac{dx}{g-i+1},$$

we shall have

$$z^{k-f+1} = y, \qquad u^{g-i+1} = x,$$

---

* The term *linear* is very improper: it has relation to Geometry, and in applying it to equations, we have had in view the straight line, in the equation for which the ordinate and the abscissa do not exceed the first degree: we cannot however, properly regard as linear, equations of the form $dy + Py dx = Q\, dx$, which most commonly belong to transcendental curves.

and

$$dy + \frac{(k-f+1)\beta}{(g-i+1)\gamma} y^{\frac{h-f}{k-f+1}} dx = \frac{(k-f+1)\alpha}{(g-i+1)\gamma} x^{\frac{e-g}{g-i+1}} dx:$$

by making for greater brevity

$$\frac{(k-f+1)\beta}{(g-i+1)\gamma} = b, \qquad \frac{(k-f+1)\alpha}{(g-i+1)\gamma} = a,$$

$$\frac{h-f}{k-f+1} = n, \qquad \frac{e-g}{g-i+1} = m,$$

there will result the equation

$$dy + by^n dx = ax^m dx.$$

(259.) The most simple case, after that in which the equation with respect to y is of the first degree, is that which arises from making $n=2$. We then get the equation $dy + by^2 dx = ax^m dx$, first considered by Riccati, an Italian Geometer, whose name it bears.

The variables are separated immediately in this equation, when $m=0$; it becomes $dy + by^2 dx = adx$, and gives

$$dx = \frac{dy}{a-by^2} = \frac{1}{2\sqrt{a}}\left[\frac{dy}{\sqrt{a}+y\sqrt{b}} + \frac{dy}{\sqrt{a}-y\sqrt{b}}\right].$$

We find, by integrating,

$$x = \frac{1}{2\sqrt{ab}} \, l\left(\frac{\sqrt{a}+y\sqrt{b}}{\sqrt{a}-y\sqrt{b}}\right) + C.$$

In order to discover the means of making this equation homogenous, we make $y = z^k$; it is then changed into

$$kz^{k-1}dz + bz^{2k}dx = ax^m dx,$$

and will assume the form required, if $k-1 = 2k = m$, which gives $k = -1$, and also supposes $m = -2$; there thence arises

$$-\frac{dz}{z^2} + \frac{bdx}{z^2} = \frac{adx}{x^2}.$$

260. We shall not detain ourselves with the integration of this last equation; but we shall proceed to a transformation of greater generality, namely, that which results from making $y=Ax^p+x^q z$. We find upon this hypothesis

$$dy=(pAx^{p-1}+qx^{q-1}z)dx+x^q dz$$
$$y^2dx=(A^2x^{2p}+2Ax^{p+q}z+x^{2q}z^2)dx,$$

and consequently

$$x^q dz+(qx^{q-1}+2bAx^{p+q}+bx^{2q}z)z\,dx$$
$$+(pAx^{p-1}+bA^2x^{2p})dx=ax^m dx.$$

This equation will reduce itself to three terms, if we have $p-1=2p$, $pA+bA^2=0$, $q-1=p+q$, $q+2bA=0$. The first and the third agree in giving $p=-1$, we deduce from the second and the fourth $A=\frac{1}{b}$, $q=-2$, values which lead us to that of $y=\frac{1}{bx}+\frac{z}{x^2}$,

$$x^{-2}dz+bx^{-4}z^2dx=ax^m dx,$$
$$\text{or } dz+bz^2\frac{dx}{x^2}=ax^{m+2}dx.$$

By this means the proposed equation will be reduced to homogeneity, if $m=-2$; and it likewise shews that we shall be able to separate the variables of $m=-4$, since we shall have in this case

$$dz+(bz^2-a)\frac{dx}{x^2}=0,\text{ or }\frac{dz}{bz^2-a}+\frac{dx}{x^2}=0.$$

If in the equation $dz+bz^2\frac{dx}{x^2}=ax^{m+2}dx$, we make $z=\frac{1}{y'}$, there will result

$$-dy'+b\frac{dx}{x^2}=ay'^2x^{m+2}dx,\text{ or } dy'+ay'^2x^{m+2}dx=b\frac{dx}{x^2};$$

afterwards making $x^{m+3}dx = \frac{dx'}{m+3}$, we shall find

$$x^{m+3} = x',\ dx = \frac{1}{m+3} x'^{-\frac{m+2}{m+3}} dx',$$

$$dy' + \frac{a}{m+3} y'^2 dx' = \frac{b}{m+3} x'^{-\frac{m+4}{m+3}} dx';$$

then, making for greater brevity

$$\frac{a}{m+3} = b',\quad \frac{b}{m+3} = a' \text{ and } -\frac{m+4}{m+3} = m',$$

we shall get the equation

$$dy' + b'y'^2 dx' = a'x'^{m'} dx'$$

which is similar to the proposed equation, and therefore susceptible of the same transformations: the separation of the variables will be consequently possible, after the substitution of $y' = \frac{1}{b'x'} + \frac{z'}{x'^2}$, if $m' = -4$.

If this condition was not satisfied, we might still make in the transformed equation

$$z' = \frac{1}{y''},\ x'^{m'+3} = x'',$$

$$\frac{a'}{m+3} = b'',\ \frac{b'}{m'+3} = a'',\ -\frac{m'+4}{m'+3} = m'';$$

the similarity of these expressions with those preceding, leads us necessarily to the equation

$$dy'' + b''y''^2 dx'' = a''x''^{m''} dx'',$$

which is still similar to the proposed equation, and susceptible of the separation of its variables if $m'' = -4$.

By pursuing this process, we should arrive at an equation where the variables may be separated, if in this series of expressions

$$m,\ m' = -\frac{m+4}{m+3},\ m'' = -\frac{m'+4}{m'+3},$$

$$m''' = -\frac{m''+4}{m''+3},\ \&c.$$

we should find one equal to -4. By supposing successively that this was the case with m, m', m'', m''', &c. we get for the value of m, the numbers -4, $-\frac{8}{3}$, $-\frac{12}{5}$, $-\frac{16}{7}$, &c. which are comprehended in the formula

$$m = -\frac{4i}{2i-1},$$

i being any whole positive number.

These cases however do not include all those which can be derived from the preceding transformations. To find a new series of them, it is sufficient to begin with making $y = \frac{1}{y'}$, in the proposed equation, which will give

$$dy' + a y'^2 x^m dx = b dx,$$

and putting

$$x^{m+1} = x',\ \frac{a}{m+1} = b',\ \frac{b}{m+1} = a',\ -\frac{m}{m+1} = m',$$

there will result

$$dy' + b' y'^2 dx' = a' x'^{m'} dx'.$$

This new equation being similar to the one proposed is also susceptible of the same operations, that is to say, if in it we make

$$y' = \frac{1}{b x'} + \frac{z'}{z'^2},$$

and pursue the same process as in the preceding page, we shall arrive at a transformed equation in which the variables are separable, if the number m' be any one of those which are comprehended in the formula $-\frac{4i}{2i-1}$, and

consequently, if we had

$$-\frac{m}{m+1}=-\frac{4i}{2i+1}.$$

We deduce from this

$$m=-\frac{4i}{2i+1};$$

and giving to i the values 1, 2, 3, &c. there results the series of numbers

$$-\tfrac{4}{3},\ -\tfrac{8}{5},\ -\tfrac{12}{7},\ -\tfrac{16}{9},\ \&c.$$

It follows, therefore, from what precedes, that the variables in Riccati's equation may be separated, when the exponent $m=-2$, and also when it is any one of the numbers comprehended in the formula $\frac{-4i}{2i\mp 1}$.

We might multiply examples still more; but all these particular equations, which are most commonly of a very singular form, and which hardly ever occur in the application of analysis to physical questions, present little that is interesting; we shall proceed therefore to explain another method, discovered by Euler.

Investigation of a Factor necessary to render integrable a Differential Equation of the first Order.

261. It is necessary to keep in mind, that a differential equation is not always the immediate result of the differentiation of a function of two variables; but that it most commonly arises from the elimination of an arbitrary constant, between the primitive equation from which it derives its origin, and the immediate differential of this equation (43).

The elimination is effected immediately, when the primitive equation is under the form $u=c$, u representing any function whatever of x and y; for by differentiating we have $du=0$. If the function du has no factor by which it can be divided, it will always preserve the form of a complete differential of two variables, and the equation $\frac{d^2u}{dx\,dy}=\frac{d^2u}{dy\,dx}$ (122) furnishes the means of discovering this form; for when we have

$$du=Mdx+Ndy,$$

there results from it

$$\frac{du}{dx}=M,\ \frac{du}{dy}=N,\ \frac{d^2u}{dx\,dy}=\frac{dM}{dy}=\frac{dN}{dx},$$

and consequently

$$\frac{dM}{dy}=\frac{dN}{dx}.$$

It is necessary therefore that every function $Mdx+Ndy$, which is a complete differential, should satisfy this equation; and when this condition is fulfilled, it will be easy to ascend to its integral, since then we shall have $M=\frac{du}{dx}$, $N=\frac{du}{dy}$, which give us the values of the partial differentials.

If we take the value of the partial differential, relative to x, for example, there will result $\frac{du}{dx}dx=Mdx$, and consequently $u=\int Mdx+Y$. We add in this case, as in that in No. 247, an arbitrary function of y, since the integration has only taken place with respect to one of the variables; but here this function may be determined, since the function of u must satisfy the equation $N=\frac{du}{dy}$.

The equation $u=\int M\,dx+Y$ gives

$$\frac{du}{dy}=\frac{d\int M\,dx}{dy}+\frac{dY}{dy};$$

representing $\int M\,dx$ by v, we have

$$\frac{du}{dy}=\frac{dv}{dy}+\frac{dY}{dy}=N,$$

from whence we deduce

$$\frac{dY}{dy}=N-\frac{dv}{dy},$$

and by integrating,

$$Y=\int\left(N-\frac{dv}{dy}\right)dy;$$

we shall find therefore

$$u=\int M\,dx+\int\left(N-\frac{dv}{dy}\right)dy:$$

which is the integral of the function proposed.

This result shews that the function $N-\frac{dv}{dy}$ ought to involve the variable y only, otherwise it would not be true, as we have supposed, that $M\,dx$ and $N\,dy$ were the partial differentials of the same function u; and by developing this function we are conducted to the equation of condition $\frac{dM}{dy}=\frac{dN}{dx}$, which was before found by considerations of an inverse nature.

It is evident, that in order to obtain $\frac{dv}{dy}=\frac{d\int M\,dx}{dy}$, we must substitute $y+dy$, for y, in the function $\int M\,dx$, which will then become

$$\int\left(M+\frac{dM}{dy}dy+\text{\&c.}\right)dx=\int M\,dx+$$

$$\int \frac{dM}{dy}\, dx dy + \&c. = \int M\, dx + dy \int \frac{dM}{dy}\, dx + \&c.$$

since the sine $\int$ is relative to the variable x only; we shall have therefore

$$\frac{d\int M\,dx}{dy} = \int \frac{dM}{dy}\, dx:$$

substituting this value of $\frac{dv}{dy}$, in $Y = \int \left(N - \frac{dv}{dy}\right) dy$, there will result

$$Y = \int \left(N - \int \frac{dM}{dy}\, dx\right) dy.$$

By taking the differentials, at first with respect to y, we shall find

$$\frac{dY}{dy} = N - \int \frac{dM}{dy}\, dx;$$

and afterwards differentiating with reference to x, we shall finally get

$$\frac{dN}{dx} - \frac{dM}{dy} = 0. *$$

---

* The equation $\frac{d\int M dx}{dy} = \int \frac{dM}{dy}\, dx$, includes the theorem given by Leibnitz, for the purpose of differentiating under the sign $\int$. He calls this process *differentiatio de curvâ in curvam*, because in the question which he proposed to solve, he passed from one curve to another of the same kind, by making the constant to vary.

The theorem of Leibnitz may be deduced immediately from the equation $\frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx}$, since if we make $u = \int M dx$, we shall have

$$\frac{du}{dx} = M, \quad \frac{d^2u}{dx\,dy} = \frac{dM}{dy};$$

and by integrating with respect to x, we shall find

$$\int \frac{d^2u}{dx\,dy}\, dx = \int \frac{d^2u}{dy\,dx}\, dx = \frac{du}{dy} = \int \frac{dM}{dy}\, dx.$$

262. The function $\frac{y\,dx - x\,dy}{x^2+y^2}$ being written thus:

$$du = \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy$$

gives successively

$$M = \frac{y}{x^2+y^2},\ N = -\frac{x}{x^2+y^2}$$

$$\frac{dM}{dy} = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{dN}{dx}$$

$$\int M\,dx = \int \frac{y\,dx}{x^2+y^2} = \int \frac{\frac{dx}{y}}{1+\frac{x^2}{y^2}}$$

$$= \text{arc}\left(\tan = \frac{x}{y}\right),$$

whence $$u = \text{arc}\left(\tan = \frac{x}{y}\right) + Y.$$

Differentiating and considering the whole as variable, we shall find

$$du = \frac{y\,dx - x\,dy}{x^2+y^2} + dY;$$

comparing this with the proposed function, we shall have

$$dY = 0, \text{ whence } Y = \text{const.}$$

and consequently

$$\int \frac{y\,dx - x\,dy}{x^2+y^2} = \text{arc}\left(\tan.\frac{x}{y}\right) + \text{const.}\ *$$

---

* We have particularly selected this function for integration, because it serves as the basis of a very elegant demonstration of the principle of the composition of forces, which is given by Laplace in his *Mécanique Céleste*.

Again, let us take the equation

$$\frac{dx}{x}+\frac{y^2\,dx}{x^3}-\frac{y\,dy}{x^2}+\frac{(y\,dx-x\,dy)\sqrt{x^2+y^2}}{x^3}+\frac{dy}{2y}=0$$

By comparing it with the formula $M\,dx+N\,dy=0$ we have

$$M=\frac{x^2+y^2+y\sqrt{x^2+y^2}}{x^3},\quad N=\frac{-y-\sqrt{x^2+y^2}}{x^2}+\frac{1}{2y},$$

and we find

$$\frac{dM}{dy}=\frac{2y+\sqrt{x^2+y^2}}{x^3}+\frac{y^2}{x^3\sqrt{x^2+y^2}},$$

$$\frac{dN}{dx}=\frac{2y+2\sqrt{x^2+y^2}}{x^3}-\frac{x}{x^2\sqrt{x^2+y^2}};$$

these values being equated and reduced, give

$$\frac{dM}{dy}=\frac{dN}{dx}=\frac{2y}{x^3}+\frac{x^2+2y^2}{x^3\sqrt{x^2+y^2}},$$

and consequently the proposed equation may be integrated immediately. We first obtain

$$\int M\,dx=\mathrm{l}\,x-\frac{y^2}{2x^2}+y\int\frac{dx}{x^3}\sqrt{x^2+y^2};$$

but $$\int\frac{y\,dx}{x^3}\sqrt{x^2+y^2}=-\frac{y\sqrt{x^2+y^2}}{2x^2}$$
$$+\tfrac{1}{2}\,\mathrm{l}\,\frac{(-y+\sqrt{x^2+y^2})}{x}:$$

therefore $$\int M\,dx=\mathrm{l}\,x-\frac{y^2}{2x^2}-\frac{y\sqrt{x^2+y^2}}{2x^2}$$
$$+\tfrac{1}{2}\,\mathrm{l}\,\frac{(-y+\sqrt{x^2+y^2})}{x}=v.$$

We afterwards find

$$N - \frac{dv}{dy} = - \frac{\sqrt{x^2 + y^2}}{2x^2} + \frac{y^2}{2x^2\sqrt{x^2 + y^2}}$$

$$+ \frac{1}{2\sqrt{x^2+y^2}} + \frac{1}{2y} = \frac{1}{2y},$$

and finally $Y = \frac{1}{2} \mathrm{l}\, y$, from whence there results

$$u = \mathrm{l}\, x - \frac{y^2}{2x^2} - \frac{y\sqrt{x^2+y^2}}{2x}$$

$$+ \tfrac{1}{2} \mathrm{l} \left(\frac{-y^2 + y\sqrt{x^2+y^2}}{x}\right) = c.$$

The form of this example was too complicated, to make it possible to recognize, by inspection alone, whether it was a complete differential or not; and in all similar cases it will be necessary to commence by ascertaining whether the proposed equation satisfies the condition of integrability.

263 When the primitive equation is not of the form $u=c$, or when the differential $du=0$ includes factors which afterwards disappear, the resulting differential equation of the first order no longer admits of immediate integration. If we had, for example, $u=y-cx=0$, we should find $du=dy-c\,dx=0$, and eliminating c, there would result $x\,dy-y\,dx=0$, an equation which does not satisfy the condition of integrability, since it gives

$$M = -y, \; N = x, \; \frac{dM}{dy} = -1, \; \frac{dN}{dy} = 1.$$

But if we disengage the constant c, we shall have $c = \frac{y}{x}$, and by differentiating, $\frac{x\,dy - y\,dx}{x^2} = 0$; under this form

$$M = -\frac{y}{x^2}, \quad N = \frac{1}{x}, \quad \frac{dM}{dy} = -\frac{1}{x^2} = \frac{dN}{dx};$$

we see therefore, that the integrability of the equation $x\,dy-y\,dx=0$ depends on the restitution of the factor

$\frac{1}{x^2}$, which has disappeared after the differentiation and the elimination of the arbitrary constant.

In general, every differential equation, in which dx and dy do not exceed the first degree, must have resulted from the elimination of the constant c, in an equation of the form $P+cQ=0$, P and Q representing any functions whatever of x and y. We find by this elimination $QdP - PdQ = 0$, whilst by differentiating the equation $\frac{P}{Q} = -c$, we should have obtained $\frac{QdP-PdQ}{Q^2}=0$: the first method of proceeding causes the factor $\frac{1}{Q^2}$ to disappear; and with it all the factors may disappear likewise which are common to the two quantities QdP and PdQ.

It follows from what we have just observed, that when the equation

$$Mdx + Ndy = 0$$

does not satisfy the condition of integrability, it is because differentiation and subsequent elimination of the arbitrary constant contained in the primitive equation, have caused a factor to disappear, which if it were known and restored, would render the first member of the proposed equation a complete differential of two variables. Let z be this factor; we shall consequently have $zMdx + zNdy = du$, u being the primitive function of x and y; and therefore

$$\frac{d.zM}{dy} = \frac{d.zN}{dx}.$$

By developing this last equation, we shall find

$$M\frac{dz}{dy} + z\frac{dM}{dy} = N\frac{dz}{dx} + z\frac{dN}{dx},$$

$$\text{or} \qquad M\frac{dz}{dy} - N\frac{dz}{dx} + \left(\frac{dM}{dy} - \frac{dN}{dx}\right)z = 0 \ldots (A).$$

If we were able, in general, to deduce from this equation a value of z, the integration of any differential equations whatever of the first order, would be effected by the process in No 261; but the equation (A) in almost all cases present greater difficulties than the one proposed, since the function z which it involves depends on two variables, and has two differential coefficients, and is consequently of the same species with those whose formation has been indicated in No. 261. We cannot in this place undertake its resolution, which, as we shall see hereafter, brings us to the point from which we started; but we shall proceed to make known some of the properties of the factor z.

264. It is always easy to find the factor z, when we know the primitive equation corresponding to the proposed differential equation, which we may put under the form $u=c$, c being the arbitrary constant. By differentiating, we find

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy = 0,$$

and comparing this with $zMdx + zNdy = du$, there results

$$z = \frac{\frac{du}{dx}dx + \frac{du}{dy}dy}{Mdx + Ndy};$$

the quotient is independent of the differentials dx and dy. We likewise obtain the factor z, by equating together the values of $\frac{dy}{dx}$, deduced from the equations

$$\frac{du}{dx}dx + \frac{dx}{dy}dy = 0, \quad Mdx + Ndy = 0,$$

which gives $\frac{\frac{du}{dx}}{\frac{du}{dy}} = \frac{M}{N}$, and consequently shews that if M and N have no common factor, z will be the greatest common divisor of the differential coefficients $\frac{du}{dx}$, and $\frac{du}{dy}$.

If we know a value of z, we may deduce from it an infinite number of others, by observing that if we multiply the two members of the equation $zMdx + zNdy = du$, by any function whatever of u, which we will represent by $\phi(u)$, the two members of the result

$$z\phi(u)Mdx + z\phi(u)Ndy = \phi(u)du,$$

will be complete differentials; that z being a factor proper to render the equation $Mdx + Ndy = 0$ integrable, the product $z\phi(u)$ will possess the same property.

265. There are some cases where the factor z need only involve one of the variables x and y, and then it is easy to obtain an expression for it by means of the equation (A). If, in this equation, we suppose, $\frac{dz}{dy} = 0$, it will become

$$-N\frac{dz}{dx} + z\left(\frac{dM}{dy} - \frac{dN}{dx}\right) = 0,$$

from which we shall deduce

$$\frac{dz}{z} = \frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right)dx,$$

an equation which is true if the quantity

$$\frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right)$$

reduces itself to a function of x. Representing this function by X, and integrating, we shall find

$$lz = \int X\,dx, \text{ or } z = e^{\int X dx}.$$

This formula is applicable to the equation

$$dy + Pydx = Q\,dx:$$

we observe that

$$M = Py - Q, \quad N = 1, \qquad \frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right) = P,$$

and consequently $z = e^{\int P dx}$. If we multiply the equation $dy + Pydx - Qdx = 0$ by $e^{\int P dx}$, we find $e^{\int P dx}\,dy + (Py - Q)\,e^{\int P dx}\,dx = 0$; integrating the term $e^{\int P dx} dy$, with reference to y, we obtain $u = y e^{\int P dx} + X$, X being a function of x, determined by the equation

$$\frac{d\,.\,y e^{\int P dx}}{dx} + \frac{dX}{dx} = (Py - Q)\,e^{\int P dx},$$

from which we deduce

$$\frac{dX}{dx} = -\,e^{\int P dx} Q, \qquad X = -\int e^{\int P dx} Q\,dx,$$

and consequently

$$y e^{\int P dx} - \int e^{\int P dx} Q\,dx = C,$$

or, as in No. 257,

$$y = e^{-\int P dx}\left(\int e^{\int P dx} Q\,dx + C\right).$$

We will not stop to consider the case where the factor need only involve the variable y; we readily see that the expression for it will then be $e^{\int Y dy}$, by making

$$Y = \frac{1}{M}\left(\frac{dN}{dx} - \frac{dM}{dy}\right),$$

and that this commonly applies when Y is absolutely independent of x.

266. There exist, between a homogeneous function and its differential coefficients, some peculiar relations, which materially facilitate their integration.

If V represent an homogeneous function of x, y, &c. and if in it we substitute tx, ty, &c. in the place of x, y, &c. it will necessarily take the form $t^m V$, m being the sum of the exponents of the variables in each term (255). If we now suppose that t becomes $t+g$, we shall have $(t+g)^m V$ instead of V, and $(1+g)^m V$, if we make $t=1$. Upon the same hypothesis x, y, &c. will be respectively changed into

$$x+gx,\ y+gy,\ \&c.$$

and putting gx for h, gy for k, in the formula in No. 121, we shall arrive at this equation

$$\left.\begin{array}{l} V+\dfrac{dV}{dx}gx+\dfrac{dV}{dy}gy+\ \&c. \\ +\dfrac{1}{2}\left\{\dfrac{d^2V}{dx^2}g^2x^2+2\dfrac{d^2V}{dxdy}g^2xy+\dfrac{d^2V}{dy^2}g^2y^2+\&c.\right\} \\ \quad +\ \&c. \end{array}\right\}=(1+g)^m V$$

By developing the second member, and comparing together those terms which are affected with the same powers of the indeterminate quantity g, we shall have

$$\left.\begin{array}{l} \dfrac{dV}{dx}x+\dfrac{dV}{dy}y+\ \&c. = mV \\ \dfrac{d^2V}{dx^2}x^2+2\dfrac{d^2V}{dxdy}xy+\dfrac{d^2V}{dy^2}y^2=m(m-1)V \\ \&c. \end{array}\right\}$$

267. By means of these relations, the factor z immediately presents itself in all homogeneous differential equations. If $M\,dx+N\,dy=0$ is an equation of this kind, and if the sum of the exponents of x and y in M and N be equal to m, by supposing that z is likewise an homogeneous function of the n^{th} degree, and making $zM\,dx+zN\,dy=du$, it follows, from the theorem demonstrated in the preceding, No., that

$$zMx+zNy=(m+n+1)u,$$

since the degree of the function u will be necessarily higher by unity than that of the functions zM and zN. By dividing the first equation by the second, there will result

$$\frac{M\,dx + N\,dy}{Mx + Ny} = \frac{1}{m+n+1} \cdot \frac{du}{u};$$

and since the second member of this result is a complete differential, it is necessary that the first member should be a complete differential likewise; from whence it follows that $\frac{1}{Mx + Ny}$ will be one of the factors proper to render integrable the equation $M\,dx + N\,dy = 0$.

On Equations of the first Order, in which the Differentials exceed the first Degree.

268. By the generation of differential equations, of which we have given several examples, No. 43, we see that there are some in which the differentials surpass the first degree. The general formula for these equations is

$$dy^n + P\,dy^{n-1}\,dx + Q\,dy^{n-2}\,dx^2 \ldots + T\,dy\,dx^{n-1} + U\,dx^n = 0;$$

if we divide it by the highest power of dx, it will become

$$\left(\frac{dy}{dx}\right)^n + P\left(\frac{dy}{dx}\right)^{n-1} + Q\left(\frac{dy}{dx}\right)^{n-2} \ldots + T\,\frac{dy}{dx} + U = 0:$$

by resolving it with respect to the differential coefficient $\frac{dy}{dx}$, and representing its roots by p, p', p'', &c. we shall have

$$\frac{dy}{dx} - p = 0, \quad \frac{dy}{dx} - p' = 0, \quad \frac{dy}{dx} - p'' = 0, \text{ \&c.}$$

equations which may all be treated by the preceding methods, since the differentials in each do not exceed the first

degree. The integral of each of these will be likewise the integral of the proposed equation, which will be also satisfied by the values deduced from the equation formed by the product of all these integrals.

In fact, the proposed equation being equivalent to

$$\left(\frac{dy}{dx}-p\right)\left(\frac{dy}{dx}-p'\right)\left(\frac{dy}{dx}-p''\right)\ldots=0,$$

will be verified by all the equations which will render one of its factors equal to nothing. Besides, if we consider that an equation of the form

$$MNP\ldots\ldots=0,$$

can only be true by the successive evanescence of each of its factors, we shall thence conclude that the immediate differential of its first members, namely

$$dM\,.\,NP\ldots\ldots+dN\,.\,MP\ldots\ldots+\text{\&c}=0,$$

will always reduce itself to one term; for if we take, for example, $M=0$, there will only remain $dM\,.\,NP\ldots=0$, or merely $dM=0$: the equation $MNP\ldots=0$ will verify therefore the differential equation which would be satisfied by the equation $M=0$.

The two following examples, although very simple, will remove all the difficulties which are connected with the preceding statement.

1st. Let $dy^2-a^2dx^2=0$; this equation is decomposable into $dy+a\,dx=0$, $dy-a\,dx=0$, whose integrals are $y+ax=c$, $y-ax=c'$; we readily see that each of these results satisfies the proposed equation. The equation $(y+ax-c)\,(y-ax-c')=0$ satisfies it likewise, for it gives

$$(y+ax-c)\,(dy-a\,dx)+(y-ax-c')\,(dy+a\,dx)=0;$$

from whence

$$dy=\frac{[(y+ax-c)-(y-ax-c')]\,a\,dx}{2y-(c+c')};$$

putting successively, instead of y, its values $c-a\,x$, $c'+a\,x$, we find

$$d\,y=-a\,d\,x, \quad d\,y=+a\,d\,x.$$

The integral $(y+a\,x-c)\,(y-a\,x+c')=0$, involving two arbitrary and irreducible constants, might appear more general than those of the other equations of the first degree, which only involve one constant; but we must keep in mind that each of its factors ought to be considered separately, and that we deduce from it no other lines than those which would result from an integral including one constant only, of which this equation is likewise susceptible. This last integral is obtained by making $d\,y=m\,d\,x$ in the differential equation $d\,y^2-a^2\,d\,x^2=0$, which is thus changed into $m^2-a^2=0$, by which the quantity m is determined, whose value we ought afterwards to substitute in the integral of $d\,y=m\,d\,x$, which is $y=m\,x+c$. It follows from this that the integral of the proposed equation is the result of the elimination of m between the equations

$$y=m\,x+c, \qquad m^2-a^2=0\,;$$

if we effect this operation, there will arise

$$\left(\frac{y-c}{x}\right)^2-a^2=0.$$

This primitive equation being of the second degree, gives for each particular value of the constant, two straight lines, inclined in different directions with respect to the axis of the x, which is also the whole that is furnished by the other integral $(y+a\,x-c)\,(y-a\,x-c')=0$, excepting that each factor only represents lines inclined in the same direction; but since by giving separately to c and c' all possible values, these quantities will necessarily pass through the same degrees of magnitude, by collecting together those straight lines which correspond to the same values of the constants c and c', we shall fall in with the solutions com-

prised in the integral $\left(\frac{y-c}{x}\right)^2 - a^2 = 0$, which is limited to the single constant c.

We ought to observe, that every equation which involves only dy, dx, and constant quantities, may be integrated by making in it, as above, $dy = m\,dx$.

2d. Let us now consider the equation $dy^2 - ax\,dx^2 = 0$; we deduce from it $dy + dx\sqrt{ax} = 0$, $dy - dx\sqrt{ax} = 0$, and by integrating we shall have

$$y + \tfrac{2}{3} a^{\frac{1}{2}} x^{\frac{3}{2}} - c = 0, \qquad y - \tfrac{2}{3} a^{\frac{1}{2}} x^{\frac{3}{2}} - c' = 0.$$

These equations, as well as their product, may be separately considered as the integrals of the proposed equation; but this case is different from the preceding, since the radicals which the integrals just obtained involve, have with each other a connection which gives the means of comprehending them both in the same equation, and with one constant only. In fact, if we make the radical disappear in the equation

$$y + \tfrac{2}{3}\sqrt{ax^3} - c = 0,$$

we obtain $(y-c)^2 = \frac{4}{9} a x^3$. This result is still the integral of the proposed equation, to which it will immediately conduct us by the elimination of c. It belongs to a species of parabolas, each of whose irrational equations represent but one branch; and the product of these equations will correspond to groups of branches belonging to different curves, but which being collected together by two and two for the same values of the constants, would give nothing more than the rational integral.

269. Although the preceding statement makes the integration of equations in which the differentials exceed the first degree, depend upon the resolution of algebraical equations only, we shall now mention some cases in which

the integration is effected more easily by the assistance of peculiar analytical artifices, and which elude, at least in part, the difficulties which are presented by the resolution of the proposed differential equation, with respect to $\frac{dy}{dx}$.

When this equation involves but one of the two variables, x for instance, we immediately deduce from it $\frac{dy}{dx}=X$, from whence $y=\int X\,dx$; but if the equation be more easily resolvible with respect to x, than with respect to the coefficient $\frac{dy}{dx}$, which we will represent by p, and if we also have $x=P$, where P is a function of p, we shall observe that the equation $\frac{dy}{dx}=p$, or $dy=p\,dx$, gives $y=px-\int x\,dp$; putting for x its value P, there will result $y=Pp-\int P\,dp$: the integral sought for will therefore be the result of the elimination of p, between the two equations

$$x=P, \quad y=Pp-\int P\,dp.$$

Let us take as an example, $x\,dx+a\,dy=b\sqrt{dx^2+dy^2}$, or $x+ap=b\sqrt{1+p^2}$, by writing p in the place of $\frac{dy}{dx}$. This last equation gives immediately

$$x=-ap+b\sqrt{1+p^2}, \quad P=-ap+b\sqrt{1+p^2},$$

and consequently

$$y=bp\sqrt{1+p^2}-\tfrac{1}{2}ap^2-b\int dp\sqrt{1+p^2}.$$

270. We will now consider the cases in which the two variables enter simultaneously into the proposed equation; supposing however that one of them, y for instance, does not exceed the first degree, we then get y equal to a function of x and of p, so that

$$dy = R\,dx + S\,dp,$$

and consequently

$$p\,dx = R\,dx + S\,dp, \text{ or } (R-p)\,dx + S\,dp = 0.$$

If we should succeed in integrating this last equation, we should have between p, x, and an arbitrary constant, a relation, by means of which, exterminating p from the equation proposed, we should obtain a primitive equation, which would involve an arbitrary constant, and which would also be the integral sought for.

The following formula, which is involved in the general case, is very remarkable, and of very extensive application, and its integration is very easily effected.

If the proposed equation could be put under the form $y = px + P$, and if P involved the coefficient p only, we should then have $dy = p\,dx + \left(x + \frac{dP}{dp}\right)dp$; and since $dy = p\,dx$, there will remain the equation $\left(x + \frac{dP}{dp}\right)dp = 0$, which may be decomposed into the two factors $x + \frac{dP}{dp} = 0$, and $dp = 0$. Eliminating p between the first of these and the proposed equation, we shall get a primitive equation, which will satisfy the one proposed; but which involving no arbitrary constant, will only be a particular solution. The second factor being integrated, gives $p = c$, or $dy = c\,dx$, and $y = cx + c'$. The constants c and c' are not both arbitrary; for, by making in the proposed equation $p = c$, we have $y = cx + C$, C being what P becomes by this substitution, and from which we infer that $c' = C$: the integral of the proposed equation is therefore $y = cx + C$, and is found by changing p into c.

Let us take for an example the equation

$$y\,dx - x\,dy = n\sqrt{dx^2 + dy^2}.$$

It may be put at once under the form

$$y=px+n\sqrt{1+p^2};$$

and by differentiating, we find

$$dy=pdx+xpd+\frac{npdp}{\sqrt{1+p^2}}, \qquad xdp$$

and since $dy=pdx$, there will remain

$$xdp+\frac{npdp}{\sqrt{1+p^2}}=0.$$

This equation may be decomposed into two factors

$$x+\frac{np}{\sqrt{1+p^2}}=0, \text{ and } dp=0;$$

the second factor leads to $p=c$, and the integral sought for is

$$y=cx+n\sqrt{1+c^2}.$$

The first factor gives

$$y=\pm\frac{x}{\sqrt{n^2-x^2}},\ \sqrt{1+p^2}=-\frac{np}{x}=\mp\frac{n}{\sqrt{n^2-x^2}};$$

substituting in the proposed equation, we have $y^2+x^2=n^2$, an equation involving no arbitrary constant, and which is not included in the integral

$$y=cx+n\sqrt{1+c^2},$$

and which is nevertheless of such a kind, that the values of y and dy, which are deduced from it, satisfy the proposed differential equation, of which it consequently offers a *particular solution*. We shall return hereafter to the particular consideration of solutions of this nature.

On the Integration of Differential Equations of the second and higher Orders.

271. The difficulty of the integration of equations becomes so much the greater the higher the order of the differential coefficients which they involve, and we only succeed in effecting it in a very small number of very limited equations. We have already seen in No. 220, in what manner we ought to treat equations of the form $\frac{d^n y}{dx^n}=X$, X designating a function of x; we shall therefore pass directly to those which involve only two differential coefficients.

In the second order these equations involve only $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$; and making for greater brevity $\frac{dy}{dx}=p$, which gives $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, we shall get the equation $\frac{dp}{dx}=P$, P being a function of p. We deduce from this $dx=\frac{dp}{P}$, and consequently $x=\int\frac{dp}{P}$; and putting for dx its value in the equation $dy=p\,dx$, we find also $y=\int\frac{p\,dp}{P}$: then it only remains to eliminate p between the two equations $x=C+\int\frac{dp}{P}$, and $y=C'+\int\frac{p\,dp}{P}$, in order to get the integral required in terms of x and y. It will likewise be complete, since it involves two arbitrary constants; and we have seen in No. 44, that this is the greatest number which the integral of any equation of the second degree can possibly contain. The elimination of p cannot be effected unless we shall have previously effected the integrations

indicated; but by means of quadratures we shall be able to construct the curve which may be sought for.

Let us take for example the equation $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y}=a.$

By putting $p\,dx$ for dy, and $dp\,dx$ for d^2y, we shall change this equation into $\frac{(1+p^2)^{\frac{3}{2}}dx}{dp}=a$, from which we deduce

$$dx=\frac{a\,dp}{(1+p^2)^{\frac{3}{2}}},\ dy=p\,dx=\frac{a\,p\,dp}{(1+p^2)^{\frac{3}{2}}}.$$

The integration will give

$$x=C+\frac{ap}{\sqrt{1+p^2}},\quad y=C'-\frac{a}{\sqrt{1+p^2}};$$

eliminating p, we shall get $(x-C)^2+(y-C')^2=a^2$.

The proposed differential equation is nothing more than the general expression of the radius of curvature, made equal to a constant quantity a (99); and as we ought to expect, the integral is the equation to a circle of which this constant quantity is the radius.

272. It is proper to remark, that the equations $x=C+\int\frac{dp}{P}$ and $y=C'+\int\frac{p\,dp}{P}$, severally satisfy the differential equation $\frac{dp}{dx}=P$, and that by supposing their second members integrated, they will be only of the first order; there are consequently two equations of the first order, which satisfy the proposed equation of the second, and both of which are therefore its integrals, whilst an equation of the first order has one integral only. It is easy to discover the reason of this difference.

Let $U=0$ be a primitive equation between x, y, and two constants C and C': if we differentiate this equation

twice successively, we shall be able to eliminate between $U=0$, $dU=0$, $d^2U=0$, the two constants, and thus arrive at an equation of the second order, which will be independent of them; but the combination of the equations $U=0$, and $dU=0$, will lead to two different equations of the first order: the one will result from the elimination of the constant C', and the other from that of C. We will represent them by $V=0$, $V'=0$; it is evident that we shall arrive at an equation of the second order by eliminating C between $V=0$ and $dV=0$, as well as C' between $V'=0$ and $dV'=0$: each of the equations $V=0$, $V'=0$, is therefore the integral of that of the second order. We call them *first integrals*, to distinguish them from the primitive equations $U=0$, which is the *second integral*.

We readily see that we shall be able to deduce the equation $U=0$, from the two equations $V=0$ and $V'=0$, by eliminating between them the differential coefficient $\frac{dy}{dx}=0$, and that consequently we shall have the second integral, or the primitive equation of an equation of the second order, when we shall have determined its two first integrals, and when we shall be able to eliminate between them the coefficient $\frac{dy}{dx}$.

These remarks may be extended to equations of any order. For the third, for example, the primitive equations ought to contain three arbitrary constants (44); and we arrive at the differential equation of this order by eliminating these constants between the equations

$$U=0, \quad dU=0, \quad d^2U=0, \quad d^3U=0;$$

but if we merely exterminate two constants, we shall have three differential equations of the second order, since we may preserve each of the three constants in its turn. The equations which we thus obtain are the *first integrals* of

the differential equation of the third order; which must necessarily result from the elimination of the constant, which they severally involve. The *second integrals* are in this case the equations of the first order, which are given by the elimination of each of the constants, between the equations $U=0$ and $d\,u=0$, and the primitive equation $U=0$ is the *third integral.* Without extending these considerations any further, we may conclude from them, *that a differential equation of the* n$^{\text{th}}$ *order, has* n *first integrals*, and as these integrals are of the $n-1$th order, they involve only $n-1$ coefficients

$$\frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \frac{d^3y}{dx^3},\ \ldots\ldots\ \frac{d^{n-1}y}{dx^{n-1}};$$

if therefore we can eliminate them, we shall have the n^{th} integral, or the primitive equation which answers to the differential equation proposed.

273. We reduce in general to the integration of functions of one variable, those equations of whatever order, in which a differential coefficient is expressed in terms of one of the next inferior order. If we had, for example, $\frac{d^3y}{dx^3}$ expressed by a function of $\frac{d^2y}{dx^2}$, we should make $\frac{d^2y}{dx^2}=q$, from which there would result $\frac{d^3y}{dx^3}=\frac{dq}{dx}$; and consequently the proposed equation would be transformed in $\frac{dq}{dx}=Q$, Q representing a given function of q. We should deduce from this last equation

$$dx=\frac{dq}{Q},\qquad x=\int\frac{dq}{Q}+C;$$

since $\frac{d^2y}{dx^2}=q$, we should deduce successively

$$\frac{dy}{dx}=\int q\,dx=\int\frac{q\,dq}{Q}+C',$$

$$y=\int dx\int q\,dx=\int\frac{dq}{Q}\left(\int\frac{q\,dq}{Q}+C'\right)+C'';$$

the integral sought for would then be the result of the elimination of q between the two equations

$$x = \int \frac{dq}{Q} + C, \quad y = \int \frac{dq}{Q} \left(\int \frac{q\,dq}{Q} + C' \right) + C'',$$

and there would be three arbitrary constants in the result. We might extend this method of reduction to any order whatever.

274. We shall employ ourselves in this article with the equation $\frac{d^2y}{dx^2} = Y$, Y representing any function whatever of y. If we make $dy = p\,dx$, we may deduce from it $dx = \frac{dy}{p}$, which gives $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{p\,dp}{dy}$: substituting this in the proposed equation, there results from it $p\,dp = Y\,dy$; and by integrating we find $p^2 = 2\int Y\,dy + C$, whence there arises

$$p = \frac{dy}{dx} = \sqrt{C + 2\int Y\,dy}, \text{ and } x = \int \frac{dy}{\sqrt{C + 2\int Y\,dy}} + C'.$$

It is proper to observe that the above integration may be effected by multiplying the proposed equation by dy; for there thence arises $\frac{dy}{dx} \cdot \frac{d^2y}{dx} = Y\,dy$, and since $\frac{d^2y}{dx} = d \cdot \frac{dy}{dx}$, we have $\frac{1}{2}\frac{dy^2}{dx^2} = \int Y\,dy + C$, or $\frac{dy}{dx} = \sqrt{2C + 2\int Y\,dy}$.

If we apply this method to the equation

$$d^2y\sqrt{ay} = dx^2,$$

we shall have

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{ay}}, \quad \frac{dy}{dx} \cdot \frac{d^2y}{dx} = \frac{dy}{\sqrt{ay}},$$

and by integrating $\frac{1}{2}\frac{dy^2}{dx^2} = \frac{2}{a}\sqrt{ay} + C$,

changing C into $\frac{2c}{\sqrt{a}}$, we shall deduce from thence

$$\frac{dy^2}{dx^2} = \frac{4}{\sqrt{a}}(\sqrt{y} + c),\quad \frac{2dx}{\sqrt[4]{a}} = \frac{dy}{\sqrt{c+\sqrt{y}}};$$

now making $c + \sqrt{y} = z$, there will result

$$\frac{dx}{\sqrt[4]{a}} = \frac{(z-c)\,dz}{\sqrt{z}} = (z^{\frac{1}{2}} - cz^{-\frac{1}{2}})\,dz,$$

and finally

$$\frac{x}{\sqrt[4]{a}} = \frac{2}{3}z^{\frac{3}{2}} - 2cz^{\frac{1}{2}} + c' = \frac{2}{3}(\sqrt{y} - 2c)\sqrt{c+\sqrt{y}} + c'.$$

275. The method pursued in the preceding article reduces in general to the integration of functions of one variable, all those equations of whatever order, in which a differential coefficient is given in terms of one of an order inferior to it by two. If we had for instance, $\frac{d^4y}{dx^4}$ given by a function of $\frac{d^2y}{dx^2}$, we should represent $\frac{d^2y}{dx^2}$ by q, from which there would follow

$$\frac{d^3y}{dx^3} = \frac{dq}{dx},\quad \frac{d^4y}{dx^4} = \frac{d^2q}{dx^2};$$

and the proposed equation would be transformed into

$$\frac{d^2q}{dx^2} = Q,$$

Q representing a given function of q. Then multiplying the two members by dq, there would arise

$$\frac{dq}{dx}\cdot\frac{d^2q}{dx} = Q\,dq;$$

whence we should deduce, as in the preceding No.,

$$\frac{1}{2}\frac{dq^2}{dx^2} = \int Q\,dq + C,\ \frac{dq}{dx} = \sqrt{2\int Q\,dq + C},$$

$$dx = \frac{dq}{\sqrt{2\int Q\,dq + C}},\quad x = \int \frac{dq}{\sqrt{2\int Q\,dq + C}} + C$$

but from $\frac{d^2y}{dx^2} = q$, we conclude successively that

$$\frac{dy}{dx} = \int q\,dx = \int \frac{q\,dq}{\sqrt{2\int Q\,dq + C}} + C'',$$

$$y = \int dx \int q\,dx = \int \frac{dq}{\sqrt{2\int Q dq + C}}\left(\int \frac{q\,dq}{\sqrt{2\int Q dq + C}} + C''\right) + C'''$$

the integral would consequently be the result of the elimination of q between the two equations

$$x = \int \frac{dq}{\sqrt{2\int Q\,dq + C}} + C',$$

$$y = \int \frac{dq}{\sqrt{2\int Q\,dq + C}}\left(\int \frac{q\,dq}{\sqrt{2\int Q\,dq + C}} + C''\right) + C''',$$

involving four arbitrary constants. We should treat in the same way analogous equations of higher orders.

276. We have seen in the Differential Calculus (115), that beyond the first order, the form of the differential equations would be changed, according as we assumed x or y, or even a function of these quantities, for the *independent variable*, and that this amounts to the same thing as assuming a constant dx, or dy, or a given function of these differentials and of their variables; it is therefore necessary, when we propose to integrate an equation which exceeds the first degree, to know upon which of these hypotheses it has been calculated. The preceding examples would all correspond to the case of y being a function of x, and consequently of dx being constant; but it will be easy to discover among equations deduced relative to other hypotheses, to which of them they may be referred.

It is immediately evident, that if we represent by Q any function whatever of $\frac{dx}{dy}$, every equation of the form $\frac{d^2x}{dy^2} = Q$, and in which dy is considered as constant, may be treated in the same way as that in No. 271, by making $\frac{dx}{dy} = q$, and $\frac{d^2x}{dy^2} = \frac{dq}{dy}$. We may also reduce it immediately to the form $\frac{d^2y}{dx^2} = P$, by passing, by the process in No. 116, to the hypothesis of dx being constant, which will be effected by the substitution of $-\frac{dx\,d^2y}{dy^3}$ in the place of $\frac{d^2x}{dy^2}$.

If the proposed equation had been taken upon the hythesis of $\sqrt{dx^2 + dy^2}$ being constant, and if it involved only dx, dy, d^2y, or dy, dx, d^2x, it might still be treated in the same manner as that in No. 271, after transforming it into one in which dx was constant.

277. We now proceed to the equations which involve the two differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and the independent variable x. It is evident that these equations are reduced immediately to the first order, by the substitution of $p\,dx$ and of $dp\,dx$, in the place of dy and d^2y. If we could integrate the transformed equation, and if we were also able to deduce from the integral an expression for p in terms of x, we should obtain y, by the equation $y = \int p\,dx$; and if this transformed equation should give x in terms of p, we might make use of the formula

$$y = px - \int x\,dp \text{ (269)}.$$

We shall not stop to enumerate the different cases susceptible of integration which are presented by the equations proposed; they will be easily discovered by the application of the various processes considered in the preceding pages for the purpose of integrating equations of the first order.

If the proposed equations were between $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and y, we might reduce them to the preceding case, by assuming dy constant, in the place of dx, or else by exterminating dx by means of its value $\frac{dy}{p}$, deduced from the equation $dy = p\,dx$; and we should thus have

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{p\,dp}{dy}:$$

the transformed equation would then only include p, dp and dy. If it was possible to be integrated, and if it gave p in terms of y, we should find x by means of the formula $x = \int \frac{dy}{p}$, or by the formula $x = \frac{y}{p} + \int \frac{y\,dp}{p^2}$, when we had y in terms of p.

Let there be, for example, the differential equation

$$\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,d^2y} = X,$$

X representing a function of x only; this equation may be transformed into

$$\frac{(1 + p^2)^{\frac{3}{2}}\,dx}{dp} = X,$$

or

$$\frac{dx}{X} = \frac{dp}{(1 + p^2)^{\frac{3}{2}}}.$$

By integrating there arises

$$\int \frac{dx}{X} + C = \frac{p}{\sqrt{1+p^2}};$$

if we represent $\int \frac{dx}{X} + C$ by V, there results from it

$$p = \frac{V}{\sqrt{1-V^2}}, \; y = \int p\,dx + C' = \int \frac{V\,dx}{\sqrt{1-V^2}} + C'.$$

There will be no difficulty in observing that the processes in this and the preceding article, would reduce to an inferior order, every equation of whatever order, in which were only involved one of the variables and the differential coefficients of either.

278. There are also some other forms of equations of the second order, whose primitive equations may be obtained, or at least their reduction to the first order; but they are of a nature too limited to merit a place here. What is most remarkable in this and the succeeding orders, are the properties of equations of the first degree, which are formed in the manner of those in No. 257, and whose essential character is to contain only the function sought for and its differential coefficients of the first degree. The form of these equations is, in the second order

$$d^2y + P\,dy\,dx + Qy\,dx^2 = R\,dx^2,$$

in the third

$$dy^3 + P\,d^2y\,dx + Q\,dy\,dx^2 + Ry\,dx^3 = S\,dx^3,$$

and in general

$$d^n y + P\,d^{n-1}y\,dx + Q\,d^{n-2}y\,dx^2 \ldots + Uy\,dx^n = V\,dx^n$$

the letters P, Q,......U and V, designating given functions of x

The equation of the first degree and of the second order

$$d^2y + Pdydx + Qydx^2 = Rdx^2,$$

is reducible to the equation

$$d^2z + Pdzdx + Qzdx^2 = 0,$$

by the same transformation which was used in No. 257 to make the equation $dy + Pydx = Qdx$, dependent on

$$dz + Pzdx = 0.$$

In fact, the hypothesis of $y = Xz$, gives

$$dy = Xdz + zdX,$$
$$d^2y = Xd^2z + 2dXdz + zd^2X,$$

and changes the proposed equation into

$$X(d^2z + Pdzdx + Qzdx^2)$$
$$+ 2dXdz + PzdXdx + zd^2X = Rdx^2.$$

If we make

$$d^2z + Pdzdx + Qzdx^2 = 0,$$

and if we succeed in deducing from this equation the value of z in terms of x, we shall have for the purpose of determining the function X, the equation

$$2dXdz + PzdXdx + zd^2X = Rdx^2,$$

which with respect to the variables x and X, is of the same kind with those in No. 277; and by making $dX = X'dx$, it will be changed into

$$2X'dz + PzX'dx + zdX' = Rdx,$$

or $$dX' + \left(P + \frac{2dz}{zdx}\right)X'dx = \frac{Rdx}{z}.$$

This equation being only of the first degree and of the first order with respect to X', leads (257) to

$$X' = e^{-\int\left(Pdx + \frac{2dz}{z}\right)}\left[\int e^{\int\left(Pdx + \frac{2dz}{z}\right)}\frac{Rdx}{z} + C\right],$$

a result which becomes

$$X' = \frac{e^{-\int P dx}}{z^2}\left[\int e^{\int P dx} R z dx + C\right],$$

by observing that

$$e^{-\int \frac{2dz}{z}} = \frac{1}{z^2}, \quad e^{\int \frac{2dz}{z}} = z^2;$$

we shall finally get

$$X = \int X' dx + C, \qquad y = z\int X' dx + C'z.$$

279. It is of very great consequence to remark, that it is not necessary to have the complete integral of the equation

$$d^2z + P dz dx + Q z dx^2 = 0,$$

but merely a particular value of z, which satisfies it; for the arbitrary constants are involved implicitly in the expression of y.

The preceding calculus also shews in what manner we might deduce the complete integral of the equation in terms of z; for if we made $R = 0$, the equation in y will be similar to this, and we shall have

$$X' = \frac{Ce^{-\int P dx}}{z^2}, \quad X = \int X' dx + C;$$

and $$y = z\int X' dx + C'z$$

will be the complete integral of the equation

$$d^2y + P dy dx + Q y dx^2 = 0,$$

z being in this case a particular value of y.

280. The equation

$$d^2z + P dz dx + Q z dx^2 = 0$$

is reducible to the first order, by making $z = e^{\int t dx}$ t designating a new variable; for we thus get

$$dz = e^{\int t dx} t dx, \quad d^2z = e^{\int t dx}(t^2 dx^2 + dt dx);$$

the function $e^{\int t dx}$ becomes a common factor of the proposed equation, which is reduced to

$$t^2 dx^2 + dt dx + Pt dx^2 + Q dx^2 = 0,$$

or to $$dt + (t^2 + Pt + Q) dx = 0. *$$

When the coefficients P and Q are constants, we shall represent them by A and B; the equation

$$dt + (t^2 + Pt + Q) dx = 0$$

becomes

$$dt + (t^2 + At + B) dx = 0,$$

in which the variables are separated, if we give it the form

$$\frac{dt}{(t^2 + At + B)} + dx = 0;$$

but as it is merely necessary to satisfy this equation, we readily observe, that if we make $t = m$, m being a constant, we shall have $dt = 0$, and

$$m^2 + Am + B = 0.$$

This last equation gives in general two values of m; if we represent them by m' and m'', we shall likewise have for $e^{\int t dx}$ two values, which are

$$e^{\int m' dx} = e^{m'x}, \qquad e^{\int m'' dx} = e^{m''x};$$

we shall have therefore at the same time two particular values of z, which are

$$z = e^{m'x} \text{ and } z = e^{m''x}.$$

We might with one of these values, as we have shewn above, find a complete value of z; but in equations of the second order, of the form of those above-mentioned,

* It may be proper to remark, that the above transformation will reduce in general to the first order every equation of the second, which is homogeneous with respect to the quantities z, dz, and $d^2 z$, considered as distinct variables.

we obtain immediately the complete value of the function, when we have two particular values, z' and z'', by assuming

$$z = C z' + C' z'',$$

C and C' representing two arbitrary constants; for if we substitute this value and its differentials, and then collect together the terms multiplied by the same constant, we shall find

$$C(d^2z + Pd\,z'd\,x + Q\,z'd\,x^2)$$
$$+ C'(d^2z'' + Pd\,z''d\,x + Q\,z''d\,x^2),$$

a result which is equal to nothing, independently of the values of C and C', since the quantities which multiply these constants become equal to nothing at the same time with the first member of the proposed equation.

281. When the values of m are imaginary, and consequently of the form

$$m' = \alpha + \beta\sqrt{-1}, \qquad m'' = \alpha - \beta\sqrt{-1},$$

we have

$$z = C e^{\alpha x + \beta x\sqrt{-1}} + C' e^{\alpha x - \beta x\sqrt{-1}}$$
$$= e^{\alpha x}(C e^{\beta x\sqrt{-1}} + C' e^{-\beta x\sqrt{-1}});$$

we render this result real, by suppressing the imaginary exponentials, by means of sines and cosines. For we have (164)

$$e^{\beta x\sqrt{-1}} = \cos\beta x + \sqrt{-1}\sin\beta x,$$
$$e^{-\beta x\sqrt{-1}} = \cos\beta x - \sqrt{-1}\sin\beta x,$$
$$z = e^{\alpha x}[(C + C')\cos\beta x + (C - C')\sqrt{-1}\sin\beta x];$$

and making

$$C + C' = c, \qquad (C - C')\sqrt{-1} = c',$$

we have

$$z = e^{\alpha x}(c\cos\beta x + c'\sin\beta x);$$

or otherwise

$$z = p\,e^{\alpha x}\sin(\beta x + q),$$

by making

$$c = p \sin q, \quad c' = p \cos q.$$

When the roots m' and m'' are equal, the value of z being reduced to

$$Ce^{m'x} + C'e^{m'x} = (C + C')e^{m'x},$$

becomes incomplete, it will be necessary in this case to make use of the particular value $z = e^{m'x}$, in order to obtain the complete integral, following the process in No. 279; but we arrive at it more easily by considerations analogous to those in No. 56, by supposing that m and m' differ from each other by a very small quantity.

Let us suppose $m'' = m' + k$; there thence results

$$z = Ce^{m'x} + C'e^{m'x + kx} = e^{m'x}(C + C'e^{kx});$$

developing e^{kx} according to the powers of k, we have

$$z = e^{m'x}\left(C + C' + C'kx + C'\frac{k^2x^2}{2} + \&c.\right)$$

$$= e^{m'x}\left(c + c'x + c'\frac{kx^2}{2} + \&c.\right)$$

by putting

$$C + C' = c, \quad C'k = c'.$$

This last expression, which satisfies the proposed equation for all values of k, agrees with it likewise, if $k = 0$, or $m'' = m'$, and in that case it becomes

$$z = e^{m'x}(c + c'x).$$

282. Let us take the more general equation

$$d^2y + A\,dy\,dx + By\,dx^2 = R\,dx^2.$$

We have, for the equation

$$d^2y + P\,dy\,dx + Qy\,dx^2 = R\,dx^2,$$

from the formulæ in No. 278,

$$y = z\int X dx + Cz$$

$$= Cz + z\int \frac{e^{-\int P dx}}{z^2} dx \left(\int e^{\int P dx} R z dx + C\right);$$

This expression, involving two arbitrary constants, is complete, so that it is merely necessary to substitute in it a particular value of z. The proposed equation depends on

$$d^2z + Adzdx + Bzdx^2 = 0;$$

and since the coefficients A and B are constant, we may satisfy this last equation by simply supposing $z = e^{mx}$, which gives

$$m^2 + Am + B = 0:$$

we shall have therefore, since $P = A$,

$$y = C'e^{mx} + e^{mx}\int e^{-(A+2m)x} dx \left(\int e^{(A+m)x} R dx + C\right)$$

$$= C'e^{mx} - \frac{Ce^{-(A+m)x}}{A+2m} + e^{mx}\int e^{-(A+2m)x} dx \int e^{(A+m)x} R dx.$$

Integrating by separation into parts, we shall find

$$\int e^{-(A+2m)x} dx \int e^{(A+m)x} R dx$$

$$= \frac{-e^{-(A+2m)x}\int e^{(A+m)x} R dx + \int e^{-mx} R dx}{A+2m} *;$$

if we substitute this value in the expression for y, and if after the proper reductions we put n in the place of $-(A+m)$, there will result

$$y = C'e^{mx} - \frac{Ce^{nx}}{m-n}$$

$$+ \frac{e^{mx}\int e^{-mx} R dx - e^{nx}\int e^{-nx} R dx}{m-n},$$

or changing the form of the arbitrary constants,

$$y = ce^{mx} + c'e^{nx} + \frac{e^{mx}\int e^{-mx} R dx - e^{nx}\int e^{-nx} R dx}{m-n}.$$

---

* The formula to integrate here is of this kind,

$$\int dU\int V dx = U\int V dx - \int U V dx \text{ (221)}.$$

We readily see that the quantity n is the second root of the equation $m^2+Am+B=0$, since, by the hypothesis $m+n=-A$.

When these roots are imaginary, we transform the expression for y by means of sines and cosines, as in No. 281; or else by supposng in the general expression for y,

$$z=e^{\alpha x}\cos\beta x, \text{ or } z=e^{\alpha x}\sin\beta x,$$

particular values which result from the second complete expression for z, in the No. just cited, when we make $c=0$, or $c'=0$, and by integrating by parts, we find

$$y=e^{\alpha x}[p\cos\beta x+q\sin\beta x]$$

$$\frac{e^{\alpha x}[\sin\beta x\int e^{-\alpha x}Rdx\cos\beta x-\cos\beta x\int e^{-\alpha x}Rdx\sin\beta x]}{\beta}.$$

When $m=n$, the expression for y preceding this last becomes incomplete, as in the No. just cited; and the second part of the expression presents itself under the form $\frac{0}{0}$; but we elude this difficulty by observing that A becomes equal to $-2m$, and that on this hypothesis the equation

$$y=C'e^{mx}+e^{mx}\int e^{-(A+2m)x}dx\left(\int e^{(A+m)x}Rdx+C\right)$$

is reduced to

$$y=C'e^{mx}+e^{mx}\int dx\left(\int e^{-mx}Rdx+C\right):$$

by integrating we find

$$y=C'e^{mx}+e^{mx}\left(Cx+x\int e^{-mx}Rdx-\int{}^{-mx}Rxdx\right),$$

or

$$y=e^{mx}(Cx+C')+e^{mx}\left(x\int e^{-mx}Rdx-\int e^{-mx}Rxdx\right).$$

In the applications of Analysis to Physical Astronomy, we frequently meet with the equation

$$\frac{d^2y}{dx^2}+a^2y=R,$$

for which we have

$$m=a\sqrt{-1}, \text{ or } \alpha=0, \text{ and } \beta=a;$$

Its integral will therefore be

$$y = p\cos a x + q \sin a x$$
$$+ \frac{\sin a x \int R d x \cos a x - \cos a x \int R d x \sin a x}{a}.$$

The function R has commonly the form

$$A + B\cos \beta x + C \cos \gamma x + \&c.$$

A, B, C, &c. being constant coefficients, and α, β, &c. representing whole numbers; and the integrations indicated are effected by the methods mentioned in No. 196.

283. The integration of the equation

$$d^2 z + P d z d x + Q z d x^2 = 0,$$

can sometimes, though rarely, be effected when the coefficients P and Q are variable quantities: we succeed in it, for example, when

$$P = \frac{A}{a+b x}, \qquad Q = \frac{B}{(a+b x)^2}.$$

We have (280)

$$dt + \left(t^2 + \frac{A t}{a+b x} + \frac{B}{(a+b x)^2}\right) d x = 0;$$

making

$$(a+b x) t = m,$$

there results

$$(a+b x) d m + \left(m^2 + (A-b) m + B\right) d x = 0.$$

We satisfy this equation by assuming

$$d m = 0, \text{ and } m^2 + (A-b) m + B = 0,$$

from whence we deduce two values of t, namely,

$$t = \frac{m'}{a+b x}, \quad t = \frac{m''}{a+b x};$$

but since

$$z = e^{\int t d x} = e^{\int \frac{m d x}{a+b x}} = (a+b x)^{\frac{m}{b}},$$

we shall have

$$z=C\,(a+bx)^{\frac{m'}{b}}+C'\,(a+bx)^{\frac{m''}{b}}.$$

284. The general equation

$$d^n y+Pd^{n-1}y\,dx+Qd^{n-2}y\,dx^2\ldots\ldots+Uy\,dx^n=V\,dx^n,$$

has properties analogous to those which we have shewn to belong to that of the first degree and second order.

1st. When the term $V\,dx^n$ is wanting, or the equation is of the form

$$d^n z+Pd^{n-1}z\,dx+Qd^{n-2}z\,dx^2\ldots\ldots+Uz\,dx^n=0,$$

it is only necessary to know n particular values of z, to obtain at once the general expression for this function, and if we denote these particular values by z_1, z_2, z_3, ... z_n, we shall have

$$z=C_1 z_1+C_2 z_2+C_3 z_3\ldots\ldots+C_n z_n,$$

C_1, C_2,......C_n, being arbitrary constants.

This proposition is easy of demonstration; for it is evident that each of the equations

$$C_1\,(d^n z_1+Pd^{n-1}z_1\,dx+Qd^{n-2}z_1\,dx^2\ldots+Uz_1\,dx^n)=0$$

$$C_2\,(d^n z_2+Pd^{n-1}z_2\,dx+Qd^{n-2}z_2\,dx^2\ldots+Uz_2\,dx^n)=0$$

....

$$C_n\,(d^n z_n+Pd^{n-1}z_n\,dx+Qd^{n-2}z_n\,dx^2\ldots+Uz_n\,dx^n)=0,$$

being, by hypothesis, identical, their sum will give an identical equation, which will be precisely the same as would have been obtained by substituting for z and its differentials in the proposed equation, the values which result from the above general expression of z.

2d. The integration of the equation

$$d^n y+Pd^{n-1}y\,dx+Qd^{n-2}y\,dx^2\ldots+Uy\,dx^n=V\,dx^n$$

may be made dependent on that of the equation

$$d^n z+Pd^{n-1}z\,dx+Qd^{n-2}z\,dx^2\ldots+Uz\,dx^n=0.$$

This is easily proved, by supposing the value of y to be of the same form with that of z; but that the quantities C_1, C_2, C_3, ... instead of being constant as above, are functions of x.

To fix our ideas, we will suppose the proposed equation to be of the third order only: and we shall have $y=C_1z_1+C_2z_2+C_3z_3$, where C_1, C_2, C_3, are to be determined so as to satisfy

$$d^3y+Pd^2y\,dx+Q\,dy\,dx^2+Uy\,dx^3=V\,dx^3.$$

If we form successively the values of dy, d^2y, and d^3y, considering C_1, C_2, C_3, as variable; we shall find, first

$$dy=C_1dz_1+C_2dz_2+C_3dz_3+z_1dC_1+z_2dC_2+z_3dC_3;$$

but since there are three quantities to be determined, and the question proposed affords only one condition, we are at liberty to fix on two others at pleasure, and consequently we may suppose

$$z_1dC_1+z_2dC_2+z_3dC_3=0,$$

which will give

$$dy=C_1dz_1+C_2dz_2+C_3dz_3.$$

This value, when differentiated, will become

$$d^2y=C_1d^2z_1+C_2d^2z_2+C_3d^2z_3+dz_1dC_1+dz_2dC_2+dz_3dC_3;$$

and again supposing

$$dz_1dC_1+dz_2dC_2+dz_3dC_3=0,$$

there will remain

$$d^2y=C_1d^2z_1+C_2d^2z_2+C_3d^2z_3,$$

whence we deduce

$$d^3y=C_1d^3z_1+C_2d^3z_2+C_3d^3z_3$$
$$+d^2z_1dC_1+d^2z_2dC_2+d^2z_3dC_3.$$

By the substitution of these values of y, dy, d^2y, d^3y, the proposed equation will become

$$\left.\begin{array}{l} C_1(d^3z_1+Pd^2z_1dx+Qdz_1dx^2+Uz_1dx^3) \\ +C_2(d^3z_2+Pd^2z_2dx+Qdz_2dx^2+Uz_2dx^3) \\ +C_3(d^3z_3+Pd^2z_3dx+Qdz_3dx^2+Uz_3dx^3) \\ +d^2z_1dC_1+d^2z_2dC_2+d^2z_3dC_3 \end{array}\right\} = Vdx^3,$$

which will reduce itself to

$$d^2z_1dC_1+d^2z_2dC_2+d^2z_3dC_3=Vdx^3,$$

because the functions z_1, z_2, z_3, satisfy the equation

$$d^3z + Pd^2zdx + Qdzdx^2 + Uzdx^3=0:$$

consequently there will exist among the differentials dC_1, dC_2, dC_3, the three equations

$$\left.\begin{array}{l} z_1dC_1+\ z_2dC_2+\ z_3dC_3=0 \\ dz_1dC_1+dz_2dC_2+dz_3dC_3=0 \\ d^2z_1dC_1+d^2z_2dC_2+d^2z_3dC_3=Vdx^3 \end{array}\right\}$$

whence we may derive the values of each of the differentials expressed in terms of x and dx, when those of z_1, z_2, z_3, are known. Hence we may obtain, by elimination, results of the form

$$dC_1=X_1dx, \qquad dC_2=X_2dx, \qquad dC_3=X_3dx,$$

whence

$$C_1=\int X_1dx+c_1, \quad C_2=\int X_2dx+c_2, \quad C_3=\int X_3dx+c_3;$$

and consequently

$$y=z_1(\int X_1dx+c_1)+z_2(\int X_2dx+c_2)+z_3(\int X_3dx+c_3)$$

will be the complete integral of the proposed equation.

Supposing we were acquainted with only two particular values of z, the proposed equation could not be integrated but by means of an equation of the second order. In fact, we should have in that case

$$y=C_1z_1+C_2z_2, \qquad dy=C_1dz_1+C_2dz_2,$$

by making

$$z_1dC_1+z_2dC_2=0;$$

but since we are not at liberty to dispose of more than one of the quantities C_1, C_2, we are obliged to use the complete developement of d^2y, which is

$$d^2y=C_1d^2z_1+C_2d^2z_2+dz_1dC_1+dz_2dC_2,$$

whence we get

$$d^3y=C_1d^3z_1+C_2d^3z_2+2\,d^2z_1dC_1+2\,d^2z_2dC_2 \\ +dz_1d^2C_1+dz_2d^2C_2.$$

Substituting these values in the equation proposed, and reducing, in the same manner as above, we should have

$$\left.\begin{array}{l} dz_1d^2C_1+dz_2d^2C_2+2\,d^2z_1dC_1 \quad +2\,d^2z_2dC_2 \\ \qquad\qquad +Pdz_1dC_1dx+Pdz_2dC_2dx \end{array}\right\}=Vdx^3,$$

an equation from which we may eliminate dC_2 and d^2C_2, by obtaining their values from the equation $z_1dC_1+z_2dC_2=0$, and its differential; the result which contains only d^2C_1, and dC_1, with functions of x, is reducible to the first order (277).

Lastly, when we have only one value of z, we fall upon an auxiliary equation of the third order, reducible to the second, as we may easily convince ourselves by putting, in the proposed,

$$C_1z_1,\quad C_1dz_1+z_1dC_1,\quad C_1d^2z_1+2\,dz_1dC_1+z_1d^2C_1,$$
$$C_1d^3z_1+3\,d^2z_1dC_1+3\,dz_1d^2C_1+z_1d^3C_1,$$

instead of

$$y,\ dy,\ d^2y,\ \text{and } d^3y:$$

the equation produced by these substitutions is reducible to

$$\left.\begin{array}{l} z_1d^3C_1+3\,dz_1d^2C_1 \quad +3\,d^2z_1dC_1 \\ \qquad +Pz_1d^2C_1dx+2Pdz_1dC_1dx \\ \qquad\qquad\qquad +Qz_1dC_1dx^2 \end{array}\right\}=Vdx^3.$$

If we suppose $V=0$, the equation in y becomes the same as that from which z is to be found, and thus the foregoing calculations shew in what manner we may obtain the general expression for this function by means of two, or only one particular value.

Since the method applied above to the equation of the first degree and the third order applies to equations of the same degree in every order we conclude, that *if we have* n *particular values of* z, *we can deduce the general expression for this function; and that we may derive the same expression from the integration of an equation of the first degree and order, when only* n − 1 *particular values are known.* This proposition, as well as the demonstration we have given of it, is due to Lagrange.

285. The equation

$$d^n z + P d^{n-1} z dx + Q d^{n-2} z dx^2 \ldots + U z dx^n = 0$$

is not integrable in all cases, but it may be satisfied when the coefficients P, Q, &c. are constant, by making $z = e^{mx}$, because upon this supposition it becomes divisible by $e^{mx} dx^n$, after the substitution of the values of z, dz, $d^2 z$, ... $d^n z$, which are

$$e^{mx}, \quad e^{mx} m dx, \quad e^{mx} m^2 dx^2 \ldots\ldots e^{mx} m^n dx^n:$$

we then find

$$m^n + P m^{n-1} + Q m^{n-2} \ldots\ldots + U = 0;$$

and if we call the n roots of this equation m_1, m_2, ... m_n, we shall have

$$z_1 = e^{m_1 x}, \qquad z_2 = e^{m_2 x}, \ldots . \, z_n = e^{m_n x};$$

and consequently

$$z = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} \ldots \ldots \ldots + C_n e^{m_n x}.$$

Such is the general expression for z, when the roots m_1, m_2, ... m_n, are all real and unequal.

Should imaginary roots occur, since they always appear in pairs of the form

$$\alpha + \beta \sqrt{-1}, \qquad \alpha - \beta \sqrt{-1},$$

we may get rid of the $\sqrt{-1}$, by changing the exponentials

into sines and cosines, by means of the formulæ in No. 164; and if m_1 and m_2 be two imaginary roots of the same pair, the terms $C_1e^{m_1x}+C_2e^{m_2x}$, which they introduce in the complete value of z, will become (281)

$$C_1e^{(\alpha+\sqrt{-1})x}+C_2e^{(\alpha-\beta\sqrt{-1})x}$$
$$=e^{\alpha x}[(C_1+C_2)\cos\beta x+(C_1-C_2)\sqrt{-1}\sin\beta x]$$
$$=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)=pe^{\alpha x}\sin(\beta x+q).$$

If any of the roots m_1, m_2, m_3, &c. become equal to each other, the complete value of z given above, loses its generality, since in that case several of the constants C_1, C_2, &c. will reduce themselves into one, as we have already seen for the second order (281). Suppose first $m_1=m_2$: in this case the two terms $C_1e^{m_1x}+C_2e^{m_2x}$ will give but one, viz. $(C_1+C_2)e^{m_2x}$; and the expression for z will contain no more than $n-1$ arbitrary constants; but if we suppose $m_2=m_1+k$, we get

$$C_1e^{m_1x}+C_2e^{m_2x}=e^{m_1x}(C_1+C_2e^{kx})$$
$$=e^{m_1x}\left[C_1+C_2\left(1+\frac{kx}{x}+\frac{k^2x^2}{1.2}+\&c.\right)\right]$$

which by the change of C_1+C_2 into c_1, and C_2k into c_2, becomes

$$e^{m_1x}\left(c_1+c_2x+c_2\frac{kx^2}{2}+\&c.\right),$$

and making $k=0$, we get $e^{m_1x}(c_1+c_2x)$, which being substituted for $C_1e^{m_1x}+C_2e^{m_2x}$, the value of z will re-attain the generality requisite to make it the complete integral of the proposed equation; so that we have

$$z=e^{m_1x}(c_1+c_2x)+C_3e^{m_3x}+\&c.$$

To proceed to the case where $m_1=m_2=m_3$, we must suppose in the foregoing result $m_3=m_1+k'$, and we shall have

$$z=e^{m_1x}(c_1+c_2x+C_3e^{k'x})+\&c.$$

Now by developing $e^{k'x}$, we shall find

$$z=e^{m_1x}\left[c_1+C_3+(c_2+C_3k')x+C_3\frac{k'^2x^2}{2}+C_3\frac{k'^3x^3}{2\,.\,3}+\&c.\right]+\&c.$$

whence, by changing the constants into

c_1+C_3, c_2+C_3k' and $C_3\frac{k'^2}{2}$, into c_1, c_2, and c_3,

there will result

$$z=e^{m_1x}\left(c_1+c_2x+c_3x^2+c_3\frac{k'x^3}{3}+\&c.\right)+\&c.$$

and when $k=0$,

$$z=e^{m_1x}(c_1+c_2x+c_3x^2)+\&c.$$

In the same way we find that if

$$m_1=m_2=m_3=m_4,$$

the general expression for z will be

$$z=e^{m_1x}(c_1+c_2x+c_3x^2+c_4x^3)+\&c.$$

and so on.

here

286. If we have any number m of differential equations of the first degree, containing $m+1$ variables, one only of these will be independent, and the m others will be functions of that one. When these, as well as their differential coefficients, rise no higher than to the first power in the proposed equations, which will then of course be of the first degree, we can always, by the method pointed out in No. 119, arrive at a differential equation of the first degree, between one of the functions to be determined, and the variable which we regard as independent; but we may sometimes avoid the process of elimination by integrating the proposed equations conjointly.

D'Alembert is the first who undertook the immediate

integration of a system of two or more differential equations, and the method which he devised upon this occasion is too ingenious to be passed over in silence.

Suppose we have, first the equations

$$du+(Au+Bx)dt=Tdt,$$

$$dx+(A'u+B'x)dt=T'dt,$$

where the variables u and x are regarded as functions of the independent variable t; if we multiply the second by a factor θ, a function of t, and then add it to the first, we get

$$du+\theta dx+[(A+A'\theta)u+(B+B'\theta)x]dt=(T+T'\theta)dt,$$

a result which comes under the form of an equation of the first degree and order, relative to two variables only, provided we have

$$du+\theta dx=d\left(u+\frac{B+B'\theta}{A+A'\theta}x\right),$$

since then, making

$$u+\frac{B+B'\theta}{A+A'\theta}x=z,$$

we have

$$dz+(A+A'\theta)zdt=(T+T'\theta)dt.$$

To satisfy the requisite condition we must suppose in general

$$\theta=\frac{B+B'\theta}{A+A'\theta},\quad d.\ \frac{B+B'\theta}{A+A'\theta}=0;$$

and eliminating θ from the second equation, we thence deduce a relation between the coefficients A, B, A', B', and $d\theta$, whence θ is given in functions of t.

When these coefficients are constant, it will be satisfied immediately by supposing θ constant; and this factor will be determined by the first equation, which rises only to the second degree. If we denote by θ_1 and θ_2 the values

of θ, which will also be those of $\frac{B+B'\theta}{A+A'\theta}$; by a_1 and a_2 those of $A+A'\theta$; and lastly by T_1 and T_2, those of $T+T'\theta$, we shall find (257.) the two primitive equations

$$u+\theta_1 x=e^{-a_1 t}(\int e^{a_1 t}T_1 dt+\gamma_1),$$

$$u+\theta_2 x=e^{-a_2 t}(\int e^{a_2 t}T_2 dt+\gamma_2),$$

γ_1 and γ_2 being the arbitrary constants, from which the general expressions for u and for x may be deduced.

The proposed equations appear at first not to be so general as they might be, since all the differentials are not found at once in each of them; but if we had the two following

$$M\,du+N dx+(Pu+Q\,x)\,dt=R\,dt,$$

$$M'du+N'dx+(P'u+Q'x)\,dt=R'dt,$$

we might easily reduce them to the first form, by reciprocally eliminating dx and du. The process itself is also immediately applicable; but the operation is simpler in the first form, which besides is of more frequent occurrence. We shall observe lastly, that we might have obtained one indeterminate quantity more, by multiplying the first of the proposed equations by a factor, as well as the second; this however would be useless in the case where the coefficients of the first member are constant, the only one we shall consider in this place.

2d. Suppose we have the equations

$$du+(A\ u+B\ x+C\ y)\,dt=Tdt,$$

$$dx+(A'\,u+B'\,x+C'\,y)\,dt=T'dt,$$

$$dy+(A''u+B''x+C''y)\,dt=T''dt,$$

in which the coefficients of the first members are constant; if we multiply the second by θ, and the third by θ'; and

add the results to the first, we shall obtain an equation which may be thrown into the form

$$du+\theta dx+\theta' dy+$$
$$(A+A'\theta+A''\theta')\left\{u+\frac{B+B'\theta+B''\theta'}{A+A'\theta+A''\theta'}x+\frac{C+C'\theta+C''\theta'}{A+A'\theta+A''\theta'}y\right\}dt$$
$$=(T+T'\theta+T''\theta')dt.$$

In order that this may become an equation of the first order between two variables, we must have

$$\theta=\frac{B+B'\theta+B''\theta'}{A+A'\theta+A''\theta'},\quad \theta'=\frac{C+C'\theta+C''\theta'}{A+A'\theta+A''\theta'}.$$

These latter equations, which determine θ and θ', by elimination lead to a final equation in θ, or θ', where the unknown quantity after the proper reductions rises but to the third degree.

If we consider, in particular, each of the three roots of this equation, we obtain three primitive equations of the form

$$u+\theta_1 x+\theta'_1 y=e^{-a_1 t}\left(\int e^{a_1 t}T_1 dt+\gamma_1\right)$$
$$u+\theta_2 x+\theta'_2 y=e^{-a_2 t}\left(\int e^{a_2 t}T_2 dt+\gamma_2\right)$$
$$u+\theta_3 x+\theta'_3 y=e^{-a_3 t}\left(\int e^{a_3 t}T_3 dt+\gamma_3\right).$$

287. D'Alembert applies this process to equations of the first degree and *any* order. It may be extended without difficulty to any number whatever of equations of the first degree and *first* order, and he therefore reduces those of the former kind to others of the latter. If, for instance, there be proposed two equations of the form

$$d^2u+(A\,du+B\,dx)\,dt+(Cu+Dx)\,dt^2=T\,dt^2,$$
$$d^2x+(A'du+B'dx)\,dt+(C'u+D'x)\,dt^2=T'dt^2,$$

he supposes $du=pdt$, $dx=qdt$; and he consequently obtains, between the five variables p, q, t, u, x, the four equations of the first order

$$dp+(Ap+Bq+Cu+Dx)dt=Tdt,$$
$$dq+(A'p+B'q+C'u+D'x)dt=T'dt,$$
$$du-pdt=0,$$
$$dx-qdt=0,$$

which are then treated as in the preceding No.

This artifice is alike applicable to equations of all orders, of the first degree, whatever be their number *.

Methods of resolving Differential Equations of the first and second Order of approximation.

288. After having exhausted the known methods of integrating a differential equation, we must seek to resolve it by approximation, that is to say, to derive from it the value of y in terms of x, by means of a series. The first idea which presents itself for the accomplishment of this purpose, is to assume a series for y, with indeterminate coefficients, and arranged according to the powers of x; but most commonly artifices of a particular kind are necessary for determining the exponents, which do not always follow the progression of the integral numbers. When the form of this series is known, we then find its coefficients, by substituting it as well as its differential, instead of y and dy respectively in the proposed equation.

If we had, for example, the equation

$$dy+ydx=mx^n dx,$$

we should suppose

$$y=Ax^\alpha+Bx^{\alpha+1}+Cx^{\alpha+2}+\&c.$$

substituting this value, and that of dy which results from it in the equation $dy+ydx=mx^n dx$, (taking care to arrange the terms so that a sufficient number of equations

---

* See Note (N).

shall be produced for determining the exponents and coefficients, without falling into contradictions) we should have

$$\left.\begin{array}{l}\alpha A x^{\alpha-1}+(\alpha+1) B x^{\alpha}+(\alpha+2) C x^{\alpha+1}+(\alpha+3) D x^{\alpha+2}+\&c.\\ -m x^{n}+ \qquad A x^{\alpha}+ \qquad B x^{\alpha+1}+ \qquad C x^{\alpha+2}+\&c.\end{array}\right\}=0;$$

an equation which becomes identical, provided we make $n=\alpha-1$, or $\alpha=n+1$,

$$\text{and } A=\frac{m}{\alpha},\ B=\frac{-m}{\alpha(\alpha+1)},\ C=\frac{m}{\alpha(\alpha+1)(\alpha+2)},$$

$$D=\frac{-m}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}, \&c.$$

and accordingly we should find

$$y=m\left\{\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{(n+1)(n+2)}\right.$$
$$\left.+\frac{x^{n+3}}{(n+1)(n+2)(n+3)}-\&c.\right\}$$

This value of y is incomplete, since it contains no arbitrary constant; and the same thing will happen in every case where the constant cannot be detached from the variable x, in the developement of the integral. We may, however, obtain one which shall have all the requisite generality in the following manner:

289. Let $f(x, y, c)=0$ be the integral of any differential equation; to determine the constant c, we should know the value of y which corresponds to a certain value of x; suppose, for instance, that $y=b$, when $x=a$: by means of this condition we obtain $f(a, b, c)=0$, from which the value of c might be found in terms of a and b. It is evident that the same end would be accomplished by preparing the expression for y, derived from the differential equation in such a manner that when x is made $=a$, y shall become $=b$; now this may be effected

by putting $x=a+t$, $y=b+u$, and taking such a series to represent u, that all its terms shall vanish when $t=0$.

The equation $dy+y\,dx=m\,x^n\,dx$ becomes, by this transformation, $du+(b+u)\,dt=m\,(a+t)^n\,dt$, and making

$$u=At^{\alpha}+Bt^{\alpha+1}+Ct^{\alpha+2}+\&c.$$

we shall obtain

$$\left.\begin{array}{l} \alpha A t^{\alpha-1}+(\alpha+1)\,Bt^{\alpha}+(\alpha+2)\,Ct^{\alpha+1}+\&c. \\ +\,b \quad + \quad At^{\alpha}+ \quad Bt^{\alpha+1}+\&c. \\ -\,ma^n-m\frac{n}{1}a^{n-1}t-m\frac{n\,(n-1)}{1\,.\,2}a^{n-2}t^2-\&c. \end{array}\right\}=0$$

in this equation we must suppose $\alpha-1=0$, or $\alpha=1$, and we shall then find

$$A=m\,a^n-b,\ B=\frac{m\,n\,a^{n-1}-m\,a^n+b}{1.2},$$

$$C=\frac{m\,n\,(n-1)\,a^{n-2}-mna^{n-1}+ma^n-b}{1\,.\,2\,.\,3},\ \&c.$$

290. Taylor's Series is immediately applicable to the same inquiry. Considering b as a function of a, that quantity will become

$$b+\frac{db}{da}\cdot\frac{t}{1}+\frac{d^2b}{da^2}\cdot\frac{t^2}{1\,.\,2}+\frac{d^3b}{da^3}\cdot\frac{t^3}{1\,.\,2\,.\,3}+\&c.$$

when a is changed to $a+t$; and we shall consequently have

$$y=b+u=b+\frac{db}{d}\cdot\frac{t}{1}+\frac{d^2b}{da^2}\cdot\frac{t^2}{1\,.\,2}+\frac{d^3b}{da^3}\cdot\frac{t^3}{1\,.\,2\,.\,3}+\&c.$$

But because a and b are two corresponding values of x and y, the same relation must exist between them and the coefficient $\frac{db}{da}$, as between x, y, and $\frac{dy}{dx}$: the value therefore of $\frac{db}{da}$ will be had by substituting a and b for x and y, in the proposed equation; and the successive differentials

of this resulting value, will give those of $\frac{d^2 b}{d a^2}$, $\frac{d^3 b}{d a^3}$, &c. by a process exactly similar to that of No. 42.

This series will generally converge, when t is very small; and to extend our numerical calculations to values of x more considerable than $a+t$, we must make $a+t=a_1$, $b+u=b_1$, and substitute these quantities in the proposed differential equation, in order from thence to deduce the values of the coefficients $\frac{d b_1}{d a_1}$, $\frac{d^2 b_1}{d a_1^2}$, &c. by whose assistance we may form a new value of y, corresponding to a_1+t, and expressed by a series similar to the foregoing. This must be employed so long as it continues convergent, and then another must be formed from it, as before explained.

This process ceases to be applicable when any one of the differential coefficients becomes infinite; but this will only take place either when the function y is really infinite when $x=a$, or when the series expressing this function contains fractional powers of x. The first case happens, for instance, in the equation $dy=\frac{dx}{x-a}$, whose integral is $y=c+\log.(x-a)$; we must here take the first value of x which differs from a. In the second case, the exponents of the series representing y must first be determined; knowing which, we may still employ Taylor's Series. If, for example, the series proceeded according to the powers of $x^{\frac{m}{n}}$, take $x^{\frac{1}{n}}=z$; and having thus transformed the proposed equation, the developement will then be by powers of z; we have only to remark that the first method (289) is free from this inconvenience.

291. The same processes which we have described are applicable to equations of higher orders. The most

general is that in which we assume for y a series whose exponents, as well as its coefficients, are undetermined. The following example will give an idea of the manner of obtaining both the one and the other.

Let the eqnation be

$$d^2y + ax^n y dx^2 = 0;$$

if we suppose

$$y = Ax^\alpha + Bx^{\alpha+\delta} + Cx^{\alpha+2\delta} + \&c.$$

and that the series of exponents is an increasing one, or that δ is positive, we may, when x is supposed very small, conceive y to reduce itself to its first term, since the others are too small to be compared with this first. On this supposition, we may confine ourselves to assuming

$$y = Ax^\alpha, \quad d^2y = \alpha(\alpha-1)Ax^{\alpha-2}dx^2,$$

and the proposed equation will become

$$\alpha(\alpha-1)Ax^{\alpha-2} + aAx^{\alpha+n} = 0.$$

It is not possible to determine α so as to make the two exponents $\alpha - 2$ and $\alpha + n$ equal, except in the particular case where $n = -2$; but the exponent of x being greater in the second term than the first, we may neglect one of these terms in comparison with the other, and the equation may then be verified in two ways (by approximation) viz. by taking $\alpha = 0$ and $\alpha = 1$, since either of these causes the term $\alpha(\alpha-1)Ax^{\alpha-2}$ (the greatest in the equation) to vanish: A therefore remains indeterminate, and we have two series, one beginning with A, and the other with Ax.

If we take successively

$$y = A + Bx^\delta + Cx^{2\delta} + \&c.$$

$$y = Ax + Bx^{1+\delta} + Cx^{1+2\delta} + \&c.$$

and substitute for y these values, and for d^2y its corresponding values, we shall find, by properly arranging the

terms, that δ must be equal to 2; and in either case, determining the values of the coefficients A, B, C, &c. we arrive at these two series:

$$A - \frac{aAx^{n+2}}{(n+1)(n+2)} + \frac{a^2Ax^{2n+4}}{(n+1)(n+2)(2n+3)(2n+4)}$$
$$- \frac{a^3Ax^{3n+6}}{(n+1)(n+2)(2n+3)(2n+4)(3n+5)(3n+6)} + \&c.$$
$$Ax - \frac{aAx^{n+3}}{(n+2)(n+3)} + \frac{a^2Ax^{2n+5}}{(n+2)(n+3)(2n+4)(2n+5)}$$
$$- \frac{a^3Ax^{3n+7}}{(n+2)(n+3)(2n+4)(2n+5)(3n+6)(3n+7)} + \&c.$$

The developements given above are only particular values, since they contain each but one arbitrary constant A; but (on account of the particular form of the proposed example) (279) we shall obtain a general expression for y by writing in the latter of them, A_1 for A, and taking their sum.

All that has been said in Art. 290. on equations of the first order may be applied to those of the second, with this difference alone, that the term of the series given in that article must be regarded as arbitrary, since the equation proposed determines only the coefficient $\frac{d^2b}{da^2}$, and those which follow it. To determine then the expression for y completely, we must know what this function and its first differential coefficient become when x has a particular value a assigned to it, or else to have two particular values of y corresponding to two given values of x; but this last method of proceeding is applicable only in general to the second integral, expressed by a finite number of terms.

292. The processes of approximation afforded by Taylor's Series, and which apply to all orders, shew that differential equations with two variables are always possible, that is to say, that we can always assign values, either

rigorously or approximatively exact: and the same thing appears from geometrical considerations. In fact if we consider an equation of the first order, we may deduce from it the value of the coefficient $\frac{dy}{dx}$ which expresses the trigonometrical tangent of the angle which is contained between the line of the abscissæ and the tangent of the curve represented by this equation; taking then the point
Fig. 50. M, fig. 50, corresponding to the co-ordinates $AP = a$, $PM = b$, draw MT making with MQ parallel to AB, the angle $M'MQ$ equal to that whose tangent is $\frac{db}{da}$; this line will touch the curve required at M. Considering the curve and its tangent as coincident in the immediate neighbourhood of the point of contact, the line TM will determine the length of the ordinate $P'M'$ corresponding to a point P infinitely near P', from which we calculate, by means of the proposed differential equation, the tangent of the angle $M''M'Q'$ formed at the point M' by the tangent $T'M'$ consecutive to TM. The continuation of this process will give a polygon which, in proportion as the number of its sides is increased, will differ less and less from the curve to which the proposed equation belongs. It follows also from this construction, that a differential equation of the first order represents an infinity of curves, since any point that we please may be assumed the point M.

In equations of the second order, which determine only $\frac{d^2y}{dx^2}$, we must substitute the osculating parabolas for tangents. Having assumed at pleasure a first point whose abscissa and ordinate are $x = a$, $y = b$, we form the equation

$$y - b = A(x - a) + B(x - a)^2$$

which represents a parabola passing through this point. Differentiating twice successively we derive from it,

$$\frac{dy}{dx} = A, \qquad \frac{d^2y}{dx^2} = 2B;$$

x being supposed $= a$. The coefficient A remains arbitrary; but B is determined by putting in the proposed equation a, b, and A instead of x, y, $\frac{dy}{dx}$: we then construct in the first place a parabola MN, fig. 53, passing through M, and whose tangent at that point makes with the abscissa an angle having a given trigonometric tangent. We next calculate the value of the ordinate $P'M'$ of this curve and also that of $\frac{dy}{dx}$, corresponding to a point P' taken very near to the point P upon the axis of abscissæ; then writing these values in the proposed differential equation, we thence deduce a new value of $\frac{d^2y}{dx^2}$. Let this be represented by $2B_1$ and making b_1 and A_1 the values of $P'M'$ and of $\frac{dy}{dx}$, we form the equation FIG. 53.

$$y - b_1 = A_1(x - a_1) + B(x - a_1)^2$$

of the second osculating parabola, from which we determine a third, and so on.

It is easy to modify this process so as to substitute the osculating circle for the osculating parabola, or to extend it to equations of all orders.

Of the particular Solutions of Differential Equations of the first Order.

293. In Art. 270, a *Particular Solution* of a differential equation presented itself which could not be derived from

the complete integral, and in Art. 288, we arrived at a value of y which contained no arbitrary constant; these two circumstances give rise to the following questions: *whence originate particular solutions?* and, *how are we to distinguish whether a primitive equation, which satisfies a proposed differential one, is or is not contained in its integral?* these are the subjects I proceed to examine.

The relation between a differential equation and its integral is such that the latter is equivalent to an infinite number of primitive equations of the former, which we should obtain by giving successively to the arbitrary constant all possible values, and each of which so obtained would satisfy the differential equation (43). These different primitive equations may be designated by the name of *Particular Integrals* because they are particular cases of the complete integral. *Particular Solutions* (whose number is always limited) are primitive equations essentially differing in their form from particular integrals. These solutions are of two sorts; those of the first are merely factors of the proposed differential equation, into which dx and dy do not enter, and which consequently being made $= 0$ give primitive equations, and establish relations between x and y which render the proposed equation identical. Solutions of this kind (if such there be) of the equation

$$M dx + N dy = 0,$$

may be found by seeking the common divisors of the functions M and N.

The second species of particular solutions, of which the equation $y\,dx - x\,dy = n\sqrt{dx^2 + dy^2} + dy^2$ (270.) has afforded us an instance, is intimately connected with the differential equation from which it originates, although it cannot be referred to any of the cases of the complete integral

whatever value we assign to the arbitrary constant, as it is easy to perceive by comparing the equations $y = cx + n\sqrt{1+c^2}$ and $x^2 + y^2 = n^2$.

The following theory of these latter solutions which before that time were regarded as forming a paradox in the Integral Calculus, has been given by Lagrange* in 1774.

294. The particular solutions, without being explicitly comprised in the complete integral, may nevertheless be deduced from it, if we cease to look upon the arbitrary constant as invariable. In fact, let $U = 0$, be a primitive equation containing the variables x, y, and a constant c; the corresponding differential equation, which we will designate by $V = 0$, will be the result of the elimination of this constant between the equations $U = 0$, and $\frac{dU}{dx}dx + \frac{dU}{dy}dy = 0$ (43); but if we suppose c to be any function whatever of x and y, we shall give the equation $U = 0$ generality sufficient to represent any given equation between the two variables, and consequently all the particular solutions of $V = 0$. This being premised, throwing the equation $\frac{dU}{dx}dx + \frac{dU}{dy}dy = 0$ into the form $dy = p\,dx$, we

---

* He calls them *Particular Integrals*, and gives the name of Particular Solutions to the different cases of the complete integral. Laplace, who successfully treated the same subject before Lagrange, employed these appellations in an inverted sense, and I have followed him. It appears to me improper that primitive equations which satisfy differential ones without being contained in their complete integrals and which cannot be obtained by the ordinary processes of integration, should bear a name which perpetually reminds us of these processes.

shall observe that since the equation $V=0$ results from the elimination of c between $U=0$ and $dy=p\,dx$, it ought to remain unchanged whatever value we assign to c, and consequently that we are at liberty to suppose c variable, provided that the law of its variation be such as to allow the equation $dy=p\,dx$ to continue true; now, although when c is regarded as variable as well as x, we have in general $dy=p\,dx+q\,dc$, p and q being functions of x and of c, still, provided $q=0$, we have, $dy=p\,dx$: determining then c 'in functions of x and y' by this equation, and substituting in $U=0$, the value which results, an equation will be obtained which will still satisfy the differential equation $V=0$.

In what has been said, y was regarded as a function of x and c: considering in its turn x, as a function of y and c, the equation $\frac{dU}{dx}dx+\frac{dU}{dy}dy=0$, may be thrown into the form $dx=m\,dy$; and reasoning as before, we shall find that if the value of dx, taken on the supposition that c varies, be $dx=m\,dy+n\,dc$, the equation resulting from the elimination of c between $n=0$ and $U=0$, will also satisfy the differential equation $V=0$.

The primitive equations afforded by both the foregoing processes are necessarily either particular solutions of $V=0$, if it be susceptible of any, or particular cases of its complete integral.

These two processes may be comprised in one, by getting rid of all fractional quantities in the equation $\frac{dU}{dx}dx+\frac{dU}{dy}dy+\frac{dU}{dc}dc=0$, the differential of $U=0$ taken with respect to x, y, and c. It will then have the form

$$M\,dx+N\,dy+P\,dc=0;$$

whence we derive

$$dy = -\frac{M}{N}dx - \frac{P}{N}dc, \quad dx = -\frac{N}{M}dy - \frac{P}{M}dc,$$

and if the integral functions M, N, are algebraic, or even transcendental, provided they be not susceptible of becoming infinite by some value of c, the coefficient of dc will not disappear but on the supposition that $P = 0$, which thus will give all the particular solutions of $V=0$.

When the equation $P=0$ contains only c and constant quantities, it gives a constant value for c, and of course conducts us only to a particular integral. When c rises only to the first degree in the expression of U, it will not enter at all into P, which therefore will be composed only of the variables x and y; but in this case the equation $P=0$, itself satisfies $V=0$; for, $U=0$ being of the form $Q+cP=0$, the equation $V=0$ becomes $PdQ - QdP = 0$. To decide now whether $P=0$ is a particular solution, or only a particular integral, we must eliminate one of the variables x, or y, between $U=0$ and $P=0$; the result will give c variable in the former case, and constant in the latter. If we were to find $c = \frac{0}{0}$, we must conclude that the equation $P=0$ is a factor of $U=0$ independently of the constant c, and consequently is extraneous to the differential equation $V = 0$.

295. I shall now apply this theory to the equation

$$y\,dx - x\,dy = n\sqrt{dx^2 + dy^2},$$

whose complete integral is $y - cx = n\sqrt{1+c^2}$ (270); if we make c vary at the same time with x and y, and reduce all the terms to the same denominator, we shall obtain

$$c\,dx\sqrt{1+c^2} - dy\sqrt{1+c^2} + (x\sqrt{1+c^2} + nc)\,dc = 0;$$

and putting the coefficient of dc equal to zero, it becomes

$$x\sqrt{1+c^2} + nc = 0,$$

whence we derive $c=\frac{x}{\sqrt{n^2-x^2}}$. This value of c changes the equation $y-cx=n\sqrt{1+c^2}$ into $x^2+y^2=n^2$, and gives the particular solution obtained in the article referred to.

All equations of the form $y=px+P$ (270), in which the foregoing is comprised, have an analogous particular solution. Their complete integral (represented by $y=cx+C$, C being composed of c, in the same manner that P is of p, or being the same function of it) gives

$$cdx-dy+\left(x+\frac{dC}{dc}\right)dc=0;$$

and making $x+\frac{dC}{dc}=0$, we thence obtain the value of c, on which the particular solution depends. This particular solution made its appearance in integrating the equation $y=px+P$; for in differentiating it this equation was obtained, composed of the two factors

$$x+\frac{dP}{dp}=0,\text{ and } dp=0,$$

and the result of the elimination of p between

$$y=px+P,\text{ and } x+\frac{dP}{dp}=0,$$

would be the same as that of the elimination of c between

$$y=cx+C,\text{ and } x+\frac{dC}{xc}=0.$$

The equations $y=px+P$ were remarked long ago by Clairaut, not only on account of the facility with which they are integrable after a second differentiation; but on account of the particular solution which this operation at once discloses.

Again, take the equation

$$xdx+ydy=dy\sqrt{x^2+y^2-a^2},$$

whose integral is

$$\sqrt{x^2+y^2-a^2}=y+c, \text{ or } x^2-2cy-c^2-a^2=0,$$

when the radical is taken away. We find

$$xdx-cdy-(y+c)dc=0,$$

whence

$$y+c=0,$$

and consequently

$$\sqrt{x^2+y^2-a^2}=0;$$

the particular solution is therefore in this example

$$x^2+y^2-a^2=0.$$

296. One property of particular solutions which presents itself easily in this latter example, and which is universal, is that the *differential equation may be so prepared that the particular solution shall become a factor of it.* In fact if we put

$$\sqrt{x^2+y^2-a^2}=u,$$

we shall have

$$xdx+ydy=udu,$$

and the proposed equation becomes

$$udu - udy = 0.$$

If we had taken $u=x^2+y^2-a^2$, the radical would have manifested itself in the transformed equation, which would have become

$$du-2dy\sqrt{u}=0;$$

and differentiating, we should have obtained

$$d^2u-2d^2y\sqrt{u}=\frac{dydu}{\sqrt{u}}-0,$$

and getting rid of the divisor, the result would be

$$d^2u\sqrt{u}-2ud^2y-dydu=0,$$

an equation which would still be verified by the supposition of $u=0$. As these transformations may be continued as far as we please, it follows that there are methods of pre-

paring all the successive differentials of the proposed equation, so that the particular solution shall satisfy them likewise, which could not be unless this proposition were true; for although when we make the constant c variable, and put $\frac{dy}{dc}=0$, we have $dy=pdx$, as well for the solution as for the complete integral, yet the value of d^2y, given by the particular solution, becomes

$$\frac{dp}{dx}dx^2+\frac{dp}{dc}\,\frac{dc}{dx}dx^2,$$

while it is simply $\frac{dp}{dx}dx^2$, for the complete integral; it is not even the same factor which these two values, generally speaking, satisfy: the equation

$$d^2u\sqrt{u}-2\,d^2yu-dydu=0,$$

is, as we see verified by the particular solution, independently of the differentials of the second order.

The developement and the demonstration of the circumstances I have pointed out, would lead me too far; they are to be met with in a memoir of M. Poisson, where he has elucidated with success many difficulties which still obscured the theory of the particular solutions of various differential equations. (See the *Journal de l'Ecole Polytechnique cah.* 13).

297. To decide, from what we have said, whether a primitive equation which contains no arbitrary constant, and which satisfies a given differential equation, is a particular integral, or only a particular solution, we require to have the complete integral; this circumstance, which is not always in our power, leads us naturally to the following enquiry:

Having given a value y=X, *which satisfies a differential equation, to determine whether it is comprised or not in the complete integral, and if possible to deduce this latter from it.*

If we suppose $y=V$ to be the value of y, derived from the complete integral, the function V must necessarily be so composed of the variable x and the arbitrary constant C, that it shall reduce itself to X, by assigning a proper value to C. Let C' denote this value of C, and since the supposition of $C=C'$ gives $V=X$, or the difference $V-X$ vanishes when $C-C'=0$, we may conclude that the expression $V-X$, or at least its developement, is susceptible of the form

$$V-X=V'(C-C')^{\mu}+V''(C-C')^{\nu}+\&c.$$

the exponents μ, ν, &c. being all positive, and the quantities V', V'', &c. being independent of $C-C'$. We may assume now $(C-C')^{\mu}=h$; and h will remain arbitrary as well as the quantity C; changing therefore $\frac{\nu}{\mu}$ to μ, it becomes

$$V-X=V'h+V''h^{\mu}+\&c.,$$

whence,

$$V=X+V'h+V''h^{\mu}+\&c.$$

an expression which we may look upon as the developement of the complete value of y.

This being premised, if we represent by $dy=pdx$ the proposed differential equation resolved with respect to dy, this new equation (which, by hypothesis is satisfied by $y=X$) ought to be verified independently of h, by the complete value of y. If we call this $X+k$, it will be necessary, in order to substitute it in $dy=pdx$, to enquire what p becomes when y is changed to $X+k$. Let then

$$P+P'k^{m}+P''k^{n}+\&c.$$

be the developement of this value of p, the exponents m, n, &c. which are supposed to be arranged according to their magnitude, will necessarily be positive; for p does not become infinite when $k=0$, since the equation $y=X$ (which

does not give dy infinite) renders the equations $dy = pdx$ identical, and consequently $dX = Pdx$.

When y is made $= X + k$, the result is

$$dX + dk = (P + P'k^m + P''k^n + \&c.)\,dx,$$

which the equation $dX = Pdx$ reduces to

$$dk = (P'k^m + P''k^n + \&c.)\,dx.$$

and substituting for k the developement

$$V'h + V''h^{\mu} + \&c.$$

it becomes

$$hdV' + h^{\mu}dV'' + \&c. = \left\{\begin{array}{l} P'h^m dx\,(V' + V''h^{\mu-1} + \&c.)^m \\ + P''h^n dx\,(V' + V''h^{\mu-1} + \&c.)^n \\ + \&c. \end{array}\right\}(A$$

an equation from which V', V'', &c. are to be determined independently of h. If we take only the term where this quantity has the least exponent, we form the equation

$$hdV' = P'V'^m h^m dx,$$

which cannot hold good, whatever be the value of h, unless $m = 1$; in this case h disappears, and it becomes

$$dV' = P'V'dx, \quad V' = e^{\int P'dx}.$$

When $m > 1$, we can no longer compare the first term $P'V'^m h^m dx$ of the second member with the term hdV' of the first; but we cause this latter to disappear by putting $dV' = 0$, which gives $V' = \text{const.}$ or more simply $V' = 1$; we then suppose $\mu = m$, and we get $dV'' = P'dx$, whence we obtain $V'' = \int P'dx$; and proceeding in this manner we find the other terms of the series.

When $m < 1$, it is no longer possible to satisfy the equation (A) in any way, since we cannot compare the term $P'h^m dx$ either to the term hdV', or to any of those which follow it, all whose exponents exceed unity; th

equation $y = X$ not admitting an arbitrary constant is therefore not a particular integral, but a particular solution.

298. This furnishes a process for discovering immediately the particular solutions of differential equations of the first order, without knowing their complete integral. In fact, the developement of p when y is changed to $y+k$, would be, in general, by Taylor's theorem,

$$p + \frac{dp}{dy}k + \frac{d^2p}{dy^2}k^2 + \&c.$$

and when it takes the form

$$P + P'k^n + \&c.$$

n being < 1, the coefficient $\frac{dp}{dy}$ becomes infinite (55); consequently the differentiation by which we derive this coefficient from p must introduce a divisor which vanishes. It follows from hence, that if we represent $\frac{dp}{dy}$ by $\frac{K}{L}$ every particular solution will make $L=0$, and will of course be a factor of L; and *vice versâ*; every factor of L which does not vanish at the same time with K, and which being made $=0$, verifies the proposed differential equation, will be a particular solution of it.

We may elude the necessity of resolving the proposed differential equation with respect to dy, by observing that if $Z=0$ represent this equation, Z being a function of x, y, and p, when we write pdx instead of y, we have

$$\frac{dZ}{dx}dx + \frac{dZ}{dy}dy + \frac{dZ}{dp}dp = 0,$$

whence

$$\frac{dp}{dy} = -\frac{\frac{dZ}{dy}}{\frac{dZ}{dp}};$$

and if the equation Z has been prepared so as to contain neither fractions nor radicals, it will suffice for rendering $\frac{dp}{dy}$ infinite, to make a factor of $\frac{dZ}{dp}$ vanish.

We should thus obtain only the particular solutions into which y as well as x enter; but we might obtain those which contain only x, and which are of the form $x = \text{const.}$ by considering x as a function of y in the proposed equation.

299. I proceed to investigate, by this method, first, the particular solutions of the equation

$$xdx + ydy = dy\sqrt{x^2 + y^2 - a^2}$$

of Art. (295). This equation becomes, after exterminating the radical,

$$x^2dx^2 + 2\,xydxdy + (a^2 - x^2)\,dy^2 = 0,$$

or

$$x^2 + 2\,xyp + (a^2 - x^2)\,p^2 = 0,$$

and by differentiating

$$\frac{dZ}{dp} = 2\,xy + 2p\,(a^2 - x^2):$$

the particular solution sought ought therefore to be such that by the assistance of the value of p, which its differential affords, it shall verify at once the two equations

$$x^2 + 2\,xyp + (a^2 - x^2)p^2 = 0,$$

$$xy + (a^2 - x^2)p = 0.$$

It follows from thence, that *without* the aid of its differential, it must verify the equation which results from eliminating p between the two foregoing ones. This being premised, the equation

$$xy + (a^2 - x^2)p = 0$$

multiplied by p, and subtracted from the proposed, leads to

$$x^2+xyp=0, \text{ or } p=-\frac{x}{y};$$

and substituting this value of p in the first, we find

$$x^2+y^2-a^2=0,$$

which we know is a particular solution of the proposed equation.

The more general equation $y=px+P$ being treated in the same manner, leads to $\frac{dZ}{dp}=x+\frac{dP}{dp}$; the equation therefore which the particular solutions ought to satisfy, is

$$x+\frac{dP}{dp}=0,$$

and they will result from the elimination of p between this and the proposed differential equation.

Lastly, to give an example of particular solutions of the form $y=$const. we will take the equation

$$\frac{dy}{dx}=b\,(y-a)^m,$$

whence we deduce immediately

$$\frac{dp}{dy}=mb\,(y-a)^{m-1}.$$

As this expression cannot become infinite, except when the exponent $m-1$ is negative, and at the same time $y=a$, a value which does not satisfy the proposed equation, except when m is positive, it follows that the exponent m must be a positive fraction. In this case $y=a$ is a particular solution, while the complete integral is

$$\frac{(y-a)^{1-m}}{1-m}-bx=\text{const.}$$

300. In general, among algebraic functions, radicals alone acquire a denominator by differentiation, and conse-

quently they alone can give $\frac{dp}{dy}=\frac{1}{0}$, when p has a finite value; it is therefore among radicals that we must look for particular solutions, by equating to zero the function which they affect. By following this process the equation

$$xdx+ydy=dy\sqrt{x^2+y^2-a^2}$$

gives immediately $x^2+y^2-a^2=0$; and the equation

$$ydx-xdy=n\sqrt{dx^2+dy^2},$$

from which we obtain

$$\frac{dy}{dx}=\frac{-xy}{n^2-x^2}\pm\frac{\sqrt{x^2+y^2-n^2}}{n^2-x^2}$$

leads to $x^2+y^2-n^2=0$, as we have already found in a variety of ways.

Resolution of some Geometrical Problems depending on Differential Equations.

301. As the method of throwing into an equation Geometrical Problems which depend on differential equations, rests solely on the properties of tangents, normals, radii of curvature, &c. and therefore presents no difficulties, but what are common to all other translations into the language of analysis, where the expressions of the lines to be considered are known; I shall only give a few examples of it.

It is first to be observed, that the integration of differential equations of the first order is also called the Method of Tangents, because every differential equation of this order, as it gives the value of $\frac{dy}{dx}$ in x and y, expresses the relation between the co-ordinates and the sub-tangent,

the tangent, or the normal, &c. in the curve which it represents. In fact, if we deduce from the proposed equation the value of $\frac{dy}{dx}=p$, the sub-tangent will be expressed by $\frac{y}{p}$, the tangent by $\frac{y\sqrt{1+p^2}}{p}$, &c. (65). The Differential Calculus was originally invented for the purpose of drawing tangents to curves; that is to say, for resolving the *direct problem of Tangents*. The Integral Calculus next employed the ingenuity of Mathematicians, for the purpose of arriving at the primitive equations of curves, by the properties of their tangents; but the progress and numerous applications of this calculus have caused the appellation of *the Inverse Method of Tangents*, which is adapted to only one of these applications, to be disused.

Originally, Mathematicians endeavoured to determine, by means of the areas, or even the arcs of certain known curves, the ordinate of the curve required: such constructions however have since been laid aside; because, elegant as they may have been considered in theory, they are always less convenient, and besides less exact in practice than the formulæ of approximation which have taken place of them.

A differential equation, generally speaking, cannot be constructed till the variables are separated, for the expression of one of them is then made to depend only on the quadrature of a curve, whose primitive equation is known.

302. I take, as an example, the construction of a curve, in which the sub-tangent is a given function of the abscissa x. The differential equation of this curve will be $\frac{ydx}{dy}=X$, X expressing the given function. The variables are here separable immediately, the equation having but

two terms, when it becomes $\frac{dy}{y} = \frac{dx}{X}$. Multiplying the two members by a constant quantity m, we get $\frac{mdy}{y} = \frac{mdx}{X}$, and denoting by Ly the logarithm of y taken in the system whose modulus is m, integration gives us

$$Ly = \int \frac{m\,dx}{X} = \frac{1}{m}\int \frac{m^2dx}{X}.$$

FIG. 51. Constructing now the curve DN, fig. 51, such that the ordinate corresponding to the abscissa AP, shall be $PN = \frac{m^2}{X}$, the area $ADNP$ will give the value of $\int \frac{m^2dx}{X}$. This area is reduced to a rectangle FQ, one of whose sides is m, and the other, AQ, will express $\frac{1}{m}\int \frac{m^2dx}{X}$: describing then the logarithmic curve ER, whose ordinates are perpendicular to the axis AC, and erecting at the point Q the perpendicular RQ, we shall have $L.\,RQ = AQ$ (101), or $L.\,RQ = \frac{1}{m}\int \frac{m^2dx}{X}$: RQ will therefore be equal to the ordinate of the curve required.*

We ought to remark, that this construction does not require that we should know the analytical expression of the function X: we may take, in its place, the ordinate of

* We shall not arrest our progress to detail the different methods of describing the logarithmic curve, which are all merely approximative, since it is more simple to construct it by points, by means of a table of logarithms. The areas included between the hyperbola and its asymptotes may also be employed (225).

any curve referred to the axis AB, and perform the graphical operations which the foregoing formulæ indicate upon this ordinate, and on the arbitrary line m. We see also that this line m has been introduced merely in order to render the formula homogeneous, and may be supposed equal to unity.

303. We proceed to state the solution of a problem celebrated from the earliest infancy of the Integral Calculus —*the problem of Trajectories*. Its object is to *determine a curve which shall intersect all curves of a given species at a given angle*. By curves of a given species are here meant the different particular curves obtained, by assigning to one of the constants of any primitive equation all possible values. If, for instance, the parameter of a parabola be made to vary, a series of parabolas will be obtained, referred to the same axis, having a common vertex, and whose concavities are turned the same way.*

Let $D_{,}N_{,}$, DN, $D'N'$, &c. fig. 52, be the curves so in- FIG. 52.
tersected, and MZ the curve which cuts them, or the trajectory required; if, through M, any one of its points we draw a tangent Mt, and draw also that of the curve cut by it, which passes through this point, the angle TMt, by the condition of the question must equal a given angle. Let us denote by x, y, the co-ordinates of the cutting curve, and by x', y', those of the curve cut; also, let a denote the trigonometrical tangent of the constant angle TMt, which is equal to the difference of the angles MTP, MtP, whose respective tangents are expressed by $\frac{dy'}{dx'}$, and $\frac{dy}{dx}$ (64). This relation

* In Mechanics, the name of trajectory is also given to the curve described by a body acted on by any forces; but such trajectories do not of course come under consideration in a work devoted solely to Analysis and Geometry.

$$\tan TMt = \tan (MtP - MTP),$$

leads to the following

$$a = \frac{\frac{dy}{dx} - \frac{dy'}{dx'}}{1 + \frac{dy}{dx}\frac{dy'}{dx'}} \text{ (Trig. 26).}$$

We here suppose that the primitive equation of the intersected curves is known; and we thence find, by differentiation $dy' = pdx'$, and the foregoing equation will thus become

$$a\left(1 + p\frac{dy}{dx}\right) + p - \frac{dy}{dx} = 0 \text{.........}(A).$$

It will be necessary now to write throughout x and y, instead of x' and y'; since at the point M the intersecting and intersected curves have the same co-ordinates. This being done, if we eliminate between the equation (A) and the primitive equation of the intersected curves the constant whose different values particularise each of those curves, we shall have a result which will include all their successive intersections with the trajectory, and will of course be the equation to this latter.

Take for example a series of parabolas, having the same axis, and the same vertex; and whose equation is $y'^n = ax'^m$; which gives $p = \frac{max'^{m-1}}{ny'^{n-1}}$. We may eliminate immediately from this expression, by means of the proposed equation, the parameter a, which particularises each parabola of the same degree; and substituting the result in the equation (A), after changing x' and y' into x and y, and dividing by $x^{m-1}y^{n-1}$, we shall find

$$a(nxdx + mydy) + mydx - nxdy = 0.$$

This equation being homogeneous may be treated by

the method in Art. 255. When $m=n=1$, it becomes integrable, by dividing by x^2+y^2, since

$$\frac{x\,dx+y\,dy}{x^2+y^2}=d\,.\,1\sqrt{x^2+y^2},$$

and $\dfrac{y\,dx-x\,dy}{x^2+y^2}=d.\,\text{arc}\left(\tan=\dfrac{x}{y}\right)$ (262); we have then

$$a\,1\sqrt{x^2+y^2}+\text{arc}\left(\tan=\frac{x}{y}\right)=C,$$

or
$$a\,1\frac{\sqrt{x^2+y^2}}{c}=\text{arc}\left(\tan=\frac{y}{x}\right),$$

by changing the arbitrary constant. If we then make

$$\sqrt{x^2+y^2}=u,\ \text{arc}\left(\tan\frac{y}{x}\right)=t,$$

we find, that the equation is that of the logarithmic spiral, whose property it is to cut the radius vector in a constant angle (114); and in fact, in the curve under consideration, the intersected curves are nothing else than a system of right lines drawn through the origin of the co-ordinates, and whose equation is $y'=\alpha x'$.

If we would suppose the angle TMt a right angle, a must be supposed infinite, and of course only the terms multiplied by it must be retained: thus the above equation reduces itself to $nx\,dx+my\,dy=0$, whose integral is $nx^2+my^2=c$, and indicates, that the curve which cuts all the proposed parabolas at right angles, is an ellipse described upon the same axis as the curves, and having the same vertex with them. The trajectories in which the angle TMt is a right angle, are called *orthogonal trajectories*; their general equation is $1+p\dfrac{dy}{dx}=0$, and is obtained by making a infinite in the equation (A).

304. The following problem will shew in what man-

ner Geometrical considerations lead to the theory of particular solutions, explained in (294). *To find a curve such, that all the perpendiculars dropped from a given point upon the tangents of this curve, shall be equal.* To obtain the differential equation of the problem, we must remember that, calling x and y the co-ordinates of a curve, and x' and y' those of its tangent, the equation of the latter is $y'-y=\frac{dy}{dx}(x'-x)$ (67); assuming therefore the given point from which the perpendiculars are to be dropped, for the origin of the co-ordinates, each of these perpendiculars will have for its equation $y'=-\frac{dx}{dy}x'$ (Trig. 86), and its length will be expressed by $\sqrt{x'^2+y'^2}$. If we put for x' and for y' the co-ordinates of the point where it meets the corresponding tangent, and whose values are found by the two equations above (Trig. 87), we shall have from these equations

$$x'=\frac{(xdy-ydx)\,dy}{dx^2+dy^2},\quad y'=-\frac{(xdy-ydx)\,dx}{dx^2+dy^2},$$

$$\text{and}\qquad \sqrt{x'^2+y'^2}=\frac{xdy-ydx}{\sqrt{dx^2+dy^2}}=n;$$

and the differential equation of the curve required is therefore

$$xdy-ydx=n\sqrt{dx^2+dy^2}.$$

This being known, it is easy to see, that the circle whose radius is n, and whose center is the origin of the co-ordinates, satisfies the conditions of the question. This circle having $x^2+y^2=n^2$, for its equation, is precisely the solution found in (270); but every right line situated so that its least distance from the origin of the co-ordinates, shall be equal to n, equally resolves the problem; and as an infinity of right lines may be drawn so as to satisfy this condi-

tion, it follows that it is in the equation comprehending all these lines that we are to look for the complete integral of the above differential equation, which in fact is

$$y-cx=n\sqrt{1+c^2} \text{ (270).}$$

A circumstance worthy of remark, and which is immediately perceived, is, that all the right lines of which we have spoken, are necessarily touched by the circle which represents the particular solution, since its radius is the perpendicular let fall upon each of them.

The same relation exists between the different curves which the complete integral of any differential equation of the first order represents, and that which is defined by a particular solution of this equation; the latter will touch all the former. In fact, the differential equation determines only the direction of the tangent; and every curve which at any assigned point has a common tangent with some one of the curves derived from the complete integral, will necessarily satisfy it. Now this is what actually takes place in the curve which touches all the others.

It follows from this, that the evolute of any curve is nothing more than a particular solution of the differential equation which represents all the normals of the curve, whose evolute it is; and in general, that curves given by particular solutions result from the successive intersections of the curves which correspond to the different values which the constant in the complete integral may assume.

The connexion established in Art. 294, between the complete integral and the particular solutions of differential equations, is thus deducible from Geometrical considerations; for every point in the circle of the preceding example may be regarded as the intersection of two consecutive tangents; that is to say, as the intersection of two right lines derived from two consecutive values of the constant c: the abscissa and ordinate of this intersection

depend upon the value of c, which is reciprocally a function of these quantities, or of x and y. It is evident, that to form the equation of a line consecutive to that which is represented by the equation

$$y-cx=n\sqrt{1+c^2},$$

this must be differentiated on the supposition that c alone varies; and as the intersection only of these two lines, or the point where the co-ordinates are common to both, is required, x and y must be regarded as constant. This intersection will therefore be given by the two equations

$$y-cx=n\sqrt{1+c^2}$$

$$-x=\frac{nc}{\sqrt{1+c^2}},$$

if a particular value be assigned to c; but if we eliminate c, the result, since it now corresponds to no one intersection in particular, must embrace the whole series of points produced by the crossings of the lines depending on all the values of c, combined two and two consecutively; that is to say, the circle which is the particular solution, and which is here derived from the variation of the arbitrary constant. The same remarks may be made on evolutes, when we consider them as produced by the intersections of the consecutive normals of their original curves.

Of the Integration of Functions of two or more Variables.

Investigation of a Function of more than one Variable, when all its Differential Coefficients of the same Order are given, explicitly or implicitly.

305. Functions which depend upon two or more variables, differ from those which contain only one, inas-

much as they have more than one differential coefficient of every order. If z, for instance, be a function of two variables, it will have two differential coefficients of the first order, namely $\frac{dz}{dx}$, $\frac{dz}{dy}$, the one being taken on the supposition, that x alone varies, the other that y alone is variable. For the second order the number of differential coefficients rises to three, and thus goes on increasing from one order to another (124). In re-ascending from the differential coefficients of a function of two or more variables to the function itself, several cases present themselves; 1st, We may have all the differential coefficients of the same order actually expressed in terms of the independent variables. 2d, The function itself may enter, together with the independent variables, in the expressions of the differential coefficients. 3d, We may have no more given than a certain relation between these coefficients, the function from which they are derived, and the independent variables. We shall first consider the simplest cases.

306. When the differential coefficient of the first order, of any function of two variables are known, we may thence deduce the first differential, and *vice versâ*: if $\frac{dz}{dx}=p, \frac{dz}{dy}=q$, we have $dz=p\,dx+q\,dy$. To find z we must integrate the differential $p\,dx+q\,dy$ by the process applied in Art. 261, to the differential $M\,dx+N\,dy$, which would be impossible, unless p and q satisfy the equation of condition $\frac{dp}{dy}=\frac{dq}{dx}$. When this is not the case, we conclude that the expression $p\,dx+q\,dy$ is not the differential of any function of two variables and signifies nothing whatever, so long as we regard at the same time the two variables x and y as independent.

307. We will now consider the functions of three variables.

Let $dz = n\,du + p\,dx + q\,dy$, that is to say, let

$$\frac{dz}{du} = n, \quad \frac{dz}{dx} = p, \quad \frac{dz}{dy} = q,$$

n, p, and q, being functions of u, x, y. There exist between these functions relations analogous to those which we have noticed in the differential $dz = p\,dx + q\,dy$. In fact, if we suppose successively dy, dx, and du, to vanish, that is, if we regard in succession, y, x, and u, as constant, the proposed differential ought to afford us three complete differentials relative to two variables; that is to say,

$$dz = n\,du + p\,dx, \quad dz = n\,du + q\,dy, \quad dz = p\,dx + q\,dy,$$

from which, as a necessary result

$$\frac{dn}{dx} = \frac{dp}{du}, \quad \frac{dn}{dy} = \frac{dq}{du}, \quad \frac{dp}{dy} = \frac{dq}{dx}.$$

The integration may then be performed considering at first only one of the variables, and proceeding as if the two others were constant. Let $\int n\,du = U + V$, V denoting a function into which u does not enter. If n at the same time contain u, x, y, we shall get by differentiation

$$dz = \frac{dU}{du}du + \frac{dU}{dx}dx + \frac{dU}{dy}dy + \frac{dV}{dx}dx + \frac{dV}{dy}dy,$$

or

$$dz = n\,du + \left(\frac{dU}{dx} + \frac{dV}{dx}\right)dx + \left(\frac{dU}{dy} + \frac{dV}{dy}\right)dy,$$

since $\frac{dU}{du}du = n\,du$; and in order that this value should be identical with the proposed, we must have, separately,

$$p = \frac{dU}{dx} + \frac{dV}{dx},$$

$$q = \frac{dU}{dy} + \frac{dV}{dy},$$

from which we deduce

$$\frac{dV}{dx} = p - \frac{dU}{dx},$$

$$\frac{dV}{dy} = q - \frac{dU}{dy}.$$

Now $\frac{dU}{dx}$ and $\frac{dU}{dy}$ are known functions: and we therefore obtain V by integrating, relatively to the two variables x and y, the differential

$$\left(p - \frac{dU}{dx}\right) dx + \left(q - \frac{dU}{dy}\right) dy,$$

by the process of Art. 261. Now this process supposes that the equation of condition

$$\frac{dp}{dy} - \frac{d^2 U}{dx\,dy} = \frac{dq}{dx} - \frac{d^2 U}{dy\,dx}.$$

is satisfied, but this equation, since $\frac{d^2U}{dx\,dy} = \frac{d^2U}{dy\,dx}$ (122) reduces itself to $\frac{dp}{dy} = \frac{dq}{dx}$. Also, since V cannot contain u, we must have

$$\frac{dV}{dx\,du} = 0, \qquad \frac{dV}{dy\,du} = 0,$$

which gives

$$\frac{dp}{du} = \frac{d^2 U}{dx\,du}, \quad \frac{dq}{du} = \frac{d^2 U}{dy\,du};$$

and consequently

$$\frac{dp}{du} = \frac{dn}{dx}, \quad \frac{dq}{du} = \frac{dn}{dy},$$

since

$$\frac{d^2 U}{dx\,du} = \frac{d.\frac{dU}{du}}{dx}, \qquad \frac{d^2 U}{dy\,du} = \frac{d.\frac{dU}{du}}{dy},$$

and $\frac{dU}{du} = n$.

The process of integration thus leading us back to the equations

$$\frac{dn}{dx} = \frac{dp}{du}, \quad \frac{dn}{dy} = \frac{dq}{du}, \quad \frac{dp}{dy} = \frac{dq}{dx},$$

it appears that three equations contain the only conditions which must of necessity be satisfied in order that the expression $dz = n\,du + p\,dx + q\,dy$ should be the differential of a function of three variables u, x, and y.

We shall pursue this subject no farther; what has already been said is sufficient to shew the manner of operating on a differential of the first order, containing any number of independent variables, and it is easy from thence to derive the method which must be employed for differentials of superior orders, in which d^2u, d^2x, d^2y, &c. must be regarded as new variables.

308. We proceed now to the case where the function required enters into the expression of its differential coefficients, which are thus all given *implicitly*. Let us first suppose that the function z depends only on two variables x and y, and we shall have $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$, p and q containing at once x, y, and z; and we thence deduce the differential equation $dz = p\,dx + q\,dy$, to which any equation whatever

$$P\,dx + Q\,dy + R\,dz = 0,$$

may be referred, by making $-\frac{P}{R} = p$, $-\frac{Q}{R} = q$.

In order that p and q may be the differential coefficients of a function of two variables, it is requisite that $\frac{d(p)}{dy}$ should be identical with $\frac{d(q)}{dx}$, or $\frac{d(p)}{dy} = \frac{d(q)}{dx}$, employing here the notation of Art. 126, to denote that not only x and y, but also the function z which implicitly contains those variables, must be made to vary. Thus, putting p and q in the place of $\frac{dz}{dx}$ and $\frac{dz}{dy}$, we shall find

$$\frac{dp}{dy}+q\cdot\frac{dp}{dz}=\frac{dq}{dx}+p\cdot\frac{dq}{dz},$$

or

$$\frac{dp}{dy}-\frac{dq}{dx}+q\cdot\frac{dp}{dz}-p\cdot\frac{dq}{dz}=0 \ldots\ldots\ldots (A).$$

If we substitute $-\frac{P}{R}, -\frac{Q}{R}$, in the place of p and q, we shall find, after all reductions,

$$P.\frac{dR}{dy}-R.\frac{dP}{dy}+R.\frac{dQ}{dx}-Q.\frac{dR}{dx}+Q.\frac{dP}{dz}-P.\frac{dQ}{dz}=0 \ldots (B),$$

an equation which comprehends the relation which must exist between P, Q, and R, in order that in the equation

$$P\,dx + Q\,dy + R\,dz = 0,$$

z may be regarded as a function of the two independent variables x and y, and that the integral of this equation may be consequently expressible by a single primitive equation between the three variables x, y, and z. It follows from this, that a differential equation taken at random cannot always be verified by a function of two independent variables.

For a long time those equations which did not satisfy the condition (B) were called *Absurd Equations*, and were regarded as having no meaning; but Monge has shewn

that every differential equation with three variables has a real signification; and that, while those whose integral is expressed by a single equation among the three variables belong to curve surfaces, every other represents an infinity of curves of double curvature possessed of a property common to all of them. We shall at present confine our attention to the first kind; and return afterwards to the consideration of the others.

309. Although the equation (B) should become identical by the substitution of the values of P, Q, R, it does not follow, as a necessary consequence, that the equation

$$P\,dx + Q\,dy + R\,dz = 0,$$

is an exact differential; but at least it may be rendered such by multiplication by some factor. In fact, let μ be this factor, and therefore

$$\mu P\,dx + \mu Q\,dy + \mu R\,dz$$

an exact differential, we shall have, then, by (307)

$$\frac{d.\mu R}{dy} = \frac{d.\mu Q}{dz}, \quad \frac{d.\mu R}{dx} = \frac{d.\mu P}{dz}, \quad \frac{d.\mu Q}{dx} = \frac{d.\mu Q}{dy},$$

equations, which when developed, become

$$\left.\begin{aligned} \mu\left(\frac{dR}{dy} - \frac{dQ}{dz}\right) + R.\frac{d\mu}{dy} - Q.\frac{d\mu}{dz} = 0 \\ \mu\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R.\frac{d\mu}{dx} - P.\frac{d\mu}{dz} = 0 \\ \mu\left(\frac{dQ}{dx} - \frac{dP}{dy}\right) + Q.\frac{d\mu}{dx} - P.\frac{d\mu}{dy} = 0 \end{aligned}\right\} \quad (C);$$

if we eliminate μ by multiplying the first by P, the second by $-Q$, the third by R, and adding the products together, their sum will be divisible by μ, and will give

$$P.\frac{dR}{dy} - P\frac{dQ}{dz} - Q\frac{dR}{dx} + Q\frac{dP}{dz} + R\frac{dQ}{dx} - R\frac{dP}{dy} = 0,$$

an equation identical with (B); and when this is satisfied,

the determination of μ will depend solely on any two of the three equations (C), taken together.

310. When the differentials dx, dy, and dz, rise above the first degree in the proposed equation, it cannot be integrated by the means above laid down unless it satisfies a new condition which we will next investigate. Take for instance the equation

$$Pdx^2+Qdy^2+Rdz^2+2Sdxdy+2Tdxdz+2Vdydz=0;$$

it cannot result from the differentiation of a primitive equation between the variables x, y, and z, unless it can be reduced to the form $P'dx+Q'dy+R'dz=0$: for, whatever be its integral, we may always derive from it by differentiation $dz=pdx+qdy$, p and q denoting any functions whatever of x, y, and z: consequently, if the proposed equation be resolved in dz (or if dz be found in functions of x, y, z, &c.) the differentials dx and dy must appear in the result entirely disengaged from the radical; which will not always take place; for we have

$$dz=\frac{1}{R}\{-Tdx-Vdy$$

$$\pm\sqrt{(T^2-PR)dx^2+2(TV-RS)dxdy+(V^2-QR)dy^2}\};$$

and if the quantity which is under the radical be not a perfect square, or at least if we have not

$$(TV-RS)^2=(T^2-PR)(V^2-QR),$$

the differentials dx, dy, will remain undisengaged from the radical. In general, whatever be the degree of the proposed equation with respect to dz, dx, dy, it is necessary that when arranged according to the powers of dz, it may be decomposed into factors of the form

$$dz-pdx-qdy=0.$$

Integration of partial Differential Equations of the first Order.

311. We now proceed to the third case in the investigation of functions of two or a greater number of variables. In this case we have only given, for the determination of the unknown function, some one or more of its differential coefficients of a certain order, or one equation alone between them. In this consists what is called the *Calculus of Partial Differences*, and which ought to be denominated, according to the remarks in Art. 124, *the Integral Calculus of Partial Differentials*, for the differential coefficients separately considered lead us to a knowledge of the partial differentials alone and not the differences, which are the object of a separate calculus which will be found in the Treatise on Series, which terminates this work. The coefficient $\frac{d^{m+n} z}{d x^m \, d y^n}$, multiplied by $d x^m d y^n$ becomes $\frac{d^{m+n} z}{d x^m \, d y^n} d x^m \, d y^n$, and then expresses the mth differential (taken relative to x) of the nth differential of z relative to y, and *vice versâ*.

312. The simplest partial differential equation is that which contains only one of the coefficients of the first order, together with the independent variables. Suppose, for example, $\frac{d z}{d x} = R$, R not containing z; multiplying by $d x$, we obtain $\frac{d z}{d x} d x = R \, d x$, or $d z = R \, d x$, provided that in the differentiation of z, x is regarded as the only variable, and therefore, integrating relative to x alone, it will become

$$z = \int R \, d x + C.$$

In this result C denotes, not a simple arbitrary constant, but a function of all the variables, except x, which z may contain, of a form absolutely indeterminate. If, for instance, z depended at the same time on x and y, we should have $z = \int R\,dx + \phi(y)$, $\phi(y)$ denoting an arbitrary function, composed of y and constants, combined in any way we please. When z depends on three variables, x, y, u, we shall have $z = \int R\,dx + \phi(u, y)$, and $\phi(u, y)$ will represent an arbitrary function, in which the variables u, y, may be combined in any manner whatever, either with each other or with constants. In general, for any number of independent variables s, t, u, x, y, &c. the integral of $\frac{dz}{dx} = R$ will be $z = \int R\,dx + \phi(s, t, u, y, \&c.)$, for it is evident that the function $\phi(s, t, u, y, \&c.)$, whatever be its form, will not change by the variation of x alone, and therefore in all cases $\frac{dz}{dx}$ will be equal to $\frac{d\int R\,dx}{dx} = R$.

We have supposed that z does not enter into R. If however this were the case, the equation must be treated as if x and z were the only variables, and integrated as an ordinary differential equation, by some of the methods delivered before. The integral of the equation

$$\frac{dz}{dx}\,dx - R\,dx = 0, \text{ or } dz - R\,dx = 0;$$

being denoted then by $V = \text{const.}$ we shall have

$$V = \phi(s, t, u, y, \&c.)$$

for the primitive equation on which the function z depends. In fact, if this equation be differentiated, relatively to x and z, the result will be of the form

$$P\,dz + Q\,dx = 0,$$

and it will be such that $-\frac{Q}{P} = R$, which gives $\frac{dz}{dx} = R$.

Suppose for instance that $\frac{dz}{dx} = \frac{x^2-y^2}{x^2+y^2} \times \frac{z}{x} = R.$

If we regard y as invariable, and integrate the equation $\frac{dz}{z} = \frac{dx}{x} \cdot \frac{x^2-y^2}{x^2+y^2}$, as a common differential equation between z and x, we have

$$\frac{dz}{z} = \frac{dx}{x} \cdot \frac{2x^2-(x^2+y^2)}{x^2+y^2} = \frac{2x\,dx}{x^2+y^2} - \frac{dx}{x};$$

whence $l\,\frac{z}{c} = l(x^2+y^2) - l\,x$; and $z = c \cdot \frac{x^2+y^2}{x}$:

in which, as y was supposed invariable, the constant c may be any function of y whatever; and we get, for the integral of the equation $\frac{dz}{dx} = R$,

$$z = \frac{x^2+y^2}{x} \cdot \phi(y),$$

$\phi(y)$ denoting an arbitrary function of y.

313. The equation $Pp+Qq=R$, in which P, Q, and R contain at once x, y, and z, is the most general which it is possible to have between the coefficients p and q $\left(\frac{dz}{dx} \text{ and } \frac{dz}{dy}\right)$, when they do not rise above the first degree. If we take the value of p in this equation, and substitute it in

$$dz = p\,dx + q\,dy,$$

we shall find

$$P\,dz - R\,dx = q\,(P\,dy - Q\,dx),$$

the coefficient q remaining undetermined. Two cases here present themselves: 1st, the composition of P, Q, R may be such, that the function $Pdz-Rdx$ contains only the variables z and x, whose differentials it involves, at the same time that $Pdy-Qdx$ contains only x and y. 2d, one or the other, or both of these functions may involve the three variables x, y, z.

In the first case there exists a factor μ, which renders $Pdy-Qdx$ a complete differential, and a factor μ', which performs the same office to $Pdz-Rdx$; and if we denote these complete differentials by $d\,M$ and $d\,N$, we have

$$Pdy-Qdx=\frac{1}{\mu}\,d\,M, \qquad Pd\,z-Rd\,x=\frac{1}{\mu'}\,d\,N,$$

and the foregoing equation will become $d\,N=\frac{q\,\mu'}{\mu}\,d\,M$.

It cannot then be integrable, unless $q\,.\,\frac{\mu'}{\mu}$ be a function of M; but this being the case, the form of that function is perfectly arbitrary; putting then $q.\frac{\mu'}{\mu}=\phi'(M)$, we have $dN=\phi'(M)\,d\,M$, and integrating, it becomes $N=\phi\,(M)$, ϕ denoting an arbitrary function.

As an example of this case, we shall take the equation $px+qy=n\,z$, or $x\frac{d\,z}{d\,x}+y\frac{d\,z}{dy}=n\,z$; from which we find

$$P=x,\ Q=y,\ R=n\,z,$$
$$Pdy-Q\,d\,x=x\,d\,y-y\,d\,x,$$
$$Pd\,z-Rd\,x=x\,d\,z-n\,z\,d\,x;$$

and by the integration of the equations

$$x\,d\,y-y\,d\,x=0,\ \text{and}\ x\,d\,z-n\,z\,d\,x=0,$$

we find, that the factors μ and μ' are respectively $\frac{1}{x^2}$ and $\frac{1}{x^{n+1}}$, and consequently, that $M=\frac{y}{x}$, $N=\frac{z}{x^n}$; whence it follows, that $\frac{z}{x^n}=\phi\left(\frac{y}{x}\right)$, or $z=x^n\,.\,\phi\left(\frac{y}{x}\right)$, that is to say, z is an homogeneous function of x and y of the nth degree. In fact, the equation $px+qy=n\,z$ is no other than the expression of the theorem relating to homoge-

neous functions given in No 266, and of which the foregoing process, in the case of two variables, furnishes a demonstration.

314. When the variables x, y, and z are intermixed without distinction in the functions $Pdy-Qdx$, $Pdz-Rdx$, it is no longer possible to render them separately integrable, by means of factors, and this, because the equations

$$Pdy-Qdx=0, \text{ and } Pdz-Rdx=0,$$

are not integrable separately and independently; for we must observe, that z cannot be regarded as constant in the former, nor x in the latter. Lagrange was the first who observed that if, nevertheless, these equations were integrated *conjointly*, and that from them, two primitive equations were derived, each containing an arbitrary constant, so that M and N being given functions of x, y, z, we should have $M=a$, $N=b$; then the integral of the proposed equation $Pp+Qq$ would be $N=\varphi\,(M)$, φ denoting always an arbitrary function. This important proposition seems to be demonstrated in a manner sufficiently simple, as follows.

Since the equations $M=a$, $N=b$ are supposed to be deduced from the equations $Pdx - Qdy=0$, and $Pdz-Rdx=0$, their differentials must hold good at the same time with these latter; that is, if in the equations

$$\frac{dM}{dx}dx+\frac{dM}{dy}dy+\frac{dM}{dz}dz=0,$$

$$\frac{dN}{dx}dx+\frac{dN}{dy}dy+\frac{dN}{dz}dz=0,$$

the values of dy and dz deduced from the equations $Pdy-Qdx=0$, and $Pdz-Rdx=0$, be substituted, we shall arrive at results which are identically equal to zero. These results are

$$\frac{dM}{dx}P+\frac{dM}{dy}Q+\frac{dM}{dz}R=0,$$

$$\frac{dN}{dx}P+\frac{dN}{dy}Q+\frac{dN}{dz}R=0;$$

whence we deduce

$$\frac{dM}{dx}=-\frac{dM}{dy}\cdot\frac{Q}{P}-\frac{dM}{dz}\cdot\frac{R}{P},$$

$$\frac{dN}{dx}=-\frac{dN}{dy}\cdot\frac{Q}{P}-\frac{dN}{dz}\cdot\frac{R}{P};$$

Now the equation $N=\phi\,(M)$ gives $d\,N=\phi\,(M)\,d\,M$, or

$$dx+\frac{dN}{dy}dy+\frac{dN}{dz}dz=\phi'(M)\left\{\frac{dM}{dx}dx+\frac{dM}{dy}dy+\frac{dM}{dz}dz\right\}$$

and substituting in this equation the above values of $\frac{dM}{dx}$, $\frac{dN}{dx}$, we shall find

$$\frac{dN}{dy}(Pdy-Qdx)+\frac{dN}{dz}(Pdz-Rdx)=$$

$$=\phi'(M)\,.\left\{\frac{dM}{dy}(Pdy-Qdx)+\frac{dM}{dz}(Pdz-Rdx)\right\}$$

whence we obtain

$$Pdz-Rdx=-\frac{\frac{dN}{dy}-\phi'(M)\,.\,\frac{dM}{dy}}{\frac{dN}{dz}-\phi'(M)\,.\,\frac{dM}{dz}}(Pdy-Q(dx).$$

If then, for the sake of brevity, we represent by ω the quantity which multiplies $(Pdy-Qdx)$, and which is indeterminate (containing $\phi'(M)$), the above equation will become

$$Pdz-Rdx=-\omega\,(Pdy-Qdx);$$

whence (308) we get

$$p=-\frac{dz}{dx}=\frac{R+\omega Q}{P},\ q=\frac{dz}{dy}=-\omega,$$

values which satisfy the proposed equation, independently of ω, and consequently of $\phi\,(M)$.

When we make $\phi\,(M) = a$, the integral $N = \phi\,(M)$ reduces itself to $N=b$, which shews that $N=b$ is a particular integral of the proposed equation. The same may be said of $M=a$; for, from the equation $N=\phi\,(M)$ we derive $M = \phi_1(N)$, ϕ_1 indicating the inverse operation of ϕ, and consequently $\phi_1\,(N)$ denoting an arbitrary function of N, and which we may therefore suppose $=a$. *

315. In a great number of cases the integration of partial differential equations of the first order with three variables, is greatly facilitated by separating them into two others by the introduction of an indeterminate quantity, as may be seen in the following example:

Let the equation be $f\,(p,\ x)=F(q,y)$: if we make $f\,(p,\ x)=\omega$, we shall have at the same time $F(q,\ y)=\omega$, and from these two equations we deduce

$$p=f_{,}\,(\omega,\ x), \qquad q=F_{,}\,(\omega,\ y),$$

$f_{,}$ and $F_{,}$ denoting the inverse functions of those which f and F represent. The equation $dz = p\,dx + q\,dy$ will then become

$$dz=dx\ .\ f_{,}\,(\omega,\ x)+dy\ .\ F_{,}\,(\omega,y)\,;$$

but if we represent the integrals $\int dx\ .\ f_{,}(\omega,\ x)$, and $\int dy\ .\ F_{,}\,(\omega,\ y)$ taken only with respect to the variables x and y, by P and Q; these last being also functions of ω, we shall have

$$dx\ .\ f_{,}(\omega,\ x)=\frac{dP}{dx}dx=dP-\frac{dP}{d\omega}d\omega$$

$$dy\ .\ F_{,}(\omega,y)=\frac{dQ}{dy}dy=dQ-\frac{dQ}{d\omega}d\omega,$$

and consequently

* See Note (O).

$$dz = dP + dQ - \left(\frac{dP}{d\omega} + \frac{dQ}{d\omega}\right) d\omega.$$

As this last equation cannot become a complete differential, except upon the supposition that $\frac{dP}{d\omega} + \frac{dQ}{d\omega} = \phi'(\omega)$, from which it would follow, that

$$\int \left(\frac{dP}{d\omega} + \frac{dQ}{d\omega}\right) d\omega = \phi(\omega):$$

we must therefore have

$$z + \phi(\omega) = P + Q, \quad \phi'(\omega) = \frac{dP}{d\omega} + \frac{dQ}{d\omega},$$

two equations, between which ω must be eliminated, when the arbitrary function $\phi(\omega)$ is determined.

It is often sufficient to substitute in the equation $dz = p\,dx + q\,dy$, the value of p or q, derived immediately from the proposed equation, and then to integrate the results by parts. For instance, if we have $p = f(q)$, it becomes

$$dz = dx\,f(q) + q\,dy:$$

we therefore find

$$z = xf(q) + qy - \int (xf'(q) + y)\,dq;$$

and since the integration indicated cannot take place, unless $xf'(q) + y = \phi'(q)$, it follows that

$$z + \phi(q) = x \cdot f(q) + qy; \quad \phi'(q) = x \cdot f'(q) + y,$$

and assigning any form to q, the elimination of q from these equations, gives an equation between z, x, and y.

Of the Integration of partial Differential Equations of an Order superior to the first.

316. When we proceed to the second order, the differential coefficients of this order are three in number, for a function of two variables, and a partial differential equation of the same order may express in general a relation between the independent variables, the function sought, and its differential coefficients, as well of the first order as the second. We see by analogy, that the general equation of any order, and with any number of variables, must contain the independent variables, the function sought, and its differential coefficients from the first order to that of the equation inclusively.

Before we consider the general case, we shall mention some which are reducible to inferior orders.

1st. Every equation among three variables of the form

$$f\left\{x, y, \frac{d^n z}{dy^n}, \frac{d^{n+1} z}{dx\,dy^n}, \ldots\ldots\ldots \frac{d^{n+m} z}{dx^m dy^n}\right\} = 0,$$

although of the order $m+n$, may be reduced to the order m, by making $\frac{d^n z}{dy^n} = v$, as it will then become

$$f\left\{x, y, v, \frac{dv}{dx}, \frac{d^2 v}{dx^2}, \ldots\ldots\ldots \frac{d^m v}{dx^m}\right\} = 0.$$

In the solution of this equation we must suppose first y constant, since all the differential coefficients of v, which are found in it, are taken relative to x; and it may therefore be treated as a differential equation between two variables x and v: but it is evident, that to give the expression of v all the generality of which it is susceptible, it will be necessary to replace the m arbitrary constants, which it ought to contain, by so many arbitrary functions of the vari-

able y, which was originally supposed constant; v being thus obtained, we find the expression of z by the equation $\frac{d^n z}{dy^n}=v$; in which we must now regard x as constant, and which, by this means, becoming an equation of the nth order, between two variables, may be treated as an equation of this kind, observing only to change the n arbitrary constants introduced by the new integration into as many arbitrary functions of x.

2d. Equations of the form

$$f\left(x, y, z, \frac{dz}{dx}, \frac{d^2z}{dx^2}, \ldots\ldots \frac{d^nz}{dx^n}\right)=0,$$

$$f\left(x, y, z, \frac{dz}{dy}, \frac{d^2z}{dy^2}, \ldots\ldots \frac{d^nz}{dy^n}\right)=0,$$

may be always treated immediately, as if there entered into them no more than two variables, viz. x and z in the former, y and z in the latter; and after this integration functions of y must be substituted for the constants in the one, and of x in the other.

The equations of the second order,

$$\frac{d^2z}{dxdy}+P\frac{dz}{dx}=Q, \qquad \frac{d^2z}{dxdy}+P\frac{dz}{dy}=Q,$$

where P and Q contain only x and y, belong to the first of these forms. If we make $\frac{dz}{dx}=v$, the first of them becomes $\frac{dv}{dy}+Pv=Q$, an equation of the first order and degree, with respect to v and y, and whose integral is therefore

$$v=e^{-\int Pdy}\left(\int e^{\int Pdy}Qdy+C\right) \text{ (257).}$$

If we put for v its value $\frac{dz}{dx}$, and change C to $\phi(x)$, we shall have

$$\frac{dz}{dx}=e^{-\int P dy}[\int e^{\int P dy} Q dy+\phi(x)];$$

and integrating now, with respect to y and x only, we get

$$z=\int dx\, e^{-\int P dy}[\int e^{\int P dy} Q dy+\phi(x)]+\psi(y);$$

and treating the second equation in the same manner, we find

$$z=\int dy\, e^{-\int P dx}[\int e^{\int P dx} Q dx+\phi(y)]+\psi(x).$$

when $P=0$, the above results reduce themselves to

$$z=\int dx\int Q dy+\int dx\,\phi(x)+\psi(y),$$

in the one case, and in the other to

$$z=\int dy\int Q dx+\int dy\,\phi(y)+\psi(x);$$

but since the function ϕ is arbitrary, these may be written

$$z=\int dx\int Q dy+\phi(x)+\psi(y),$$

$$z=\int dy\int Q dx+\phi(x)+\psi(x).$$

We may also observe, that the latter cases depend on the integration of functions of one variable only, and have been considered in this point of view, in No. 247.

We have instances of the second general form in the equation

$$\frac{d^2z}{dx^2}+P\frac{dz}{dx}=Q,\quad \frac{d^2z}{dy^2}+P\frac{dz}{dy}=Q.$$

where P and Q are supposed to contain x, y, and z. The former must be treated as an equation of the second order, between x and z; and the arbitrary constants arising from its integration are functions of y. The latter may be operated on in the same manner relative to the variables z and y, and the arbitrary constants must be changed to functions of x. To take only the simplest cases, let us reduce the proposed equations to

$$\frac{d^2z}{dx^2}=Q,\qquad \frac{d^2z}{dy^2}=Q.$$

where we will suppose Q to contain only x and y; and the formulæ of No. 220. will immediately give

$$z=\int dx\int Q\,dx+Cx+C'; \qquad z=\int dy\int Q\,dy+Cy+C',$$

whence we conclude that

$$z=\int dx\int Qdx+x\phi(y)+\psi(y); \quad z=\int dy\int Qdy+y\phi(x)+\psi(x).$$

317. In considering equations of the second order with three variables, and which contain all the differential coefficients of this order, but of the first degree only, we shall use, for simplicity, the following substitutions:

$$dz=p\,dx+q\,dy$$
$$dp=r\,dx+s\,dy \qquad dq=s\,dx+t\,dy\ (*)$$
$$d^2z=dp\,dx+dq\,dy \ = r\,dx^2+2\,s\,dx\,dy+t\,dy^2.$$

The partial differential equation of this order with three variables, generally considered, can give only one of the coefficients r, s, and t, in functions of the others and of the quantities p, q, x, y, z; which is not sufficient for the determination of the differentials dp and dq. We may also, by the aid of three differentials eliminate from the proposed equation two of the three coefficients r, s, t, and the result will express the relation which the proposed equation denotes as existing between dp and dq. It is this process which Monge has followed.

We shall apply it to the equation

$$Rr+Ss+Tt=V,$$

where R, S, T, V, are supposed to involve, in any way whatever the quantities x, y, z, p, q. If we substitute the values of r and of t, derived from the equations

* The coefficient of dx in dq is the same as that of dy in dp on account of the condition $\frac{dp}{dy}=\frac{dq}{dx}$, which the differential dz ought to satisfy.

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$

which values are

$$r = \frac{dp - s\,dy}{dx}, \qquad t = \frac{dq - s\,dx}{dy};$$

we find,

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = s\,(R\,dy^2 - S\,dx\,dy + T\,dx^2),$$

an equation in which it would seem that it was necessary to integrate separately the two members, on account of the indeterminate differential coefficient s, which multiplies the second; but here, as in No. 314, it is sufficient if we can arrive at two primitive equations $M=a$, and $N=b$, which satisfy at the same time the two equations

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0$$
$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0.$$

The integral of the equation proposed being then $N=\varphi\,(M)$.

To demonstrate this, we shall first of all transform the preceding equations into others where the differentials do not rise above the first degree, and we therefore suppose $dy = m\,dx$. The second of the above equations becoming by this substitution

$$R\,m^2 - S\,m + T = 0 \ldots\ldots\ldots (A).$$

determines the quantity m; and putting for dy its value in $R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0$, we shall have, for each of the values of which m is susceptible a system of equations of the form

$$\left.\begin{aligned} dy - m\,dx &= 0 \\ R\,m\,dp + T\,dq - V\,m\,dx &= 0 \end{aligned}\right\} (1).$$

with which must be joined the equation

$$dz = p\,dx + q\,dy,$$

which expresses the relation between the function z and the coefficients q, p.

This being premised, if the equations $M=a$, and $N=b$, satisfy the equations (1), and if in the differentials

$$0=\frac{dM}{dx}dx+\frac{dM}{dy}dy+\frac{dM}{dz}dz+\frac{dM}{dp}dp+\frac{dM}{dq}dq$$

$$0=\frac{dN}{dz}dx+\frac{dN}{dy}dy+\frac{dN}{dz}dz+\frac{dN}{dp}dp+\frac{dN}{dq}dq,$$

we substitute the value of dz obtained from the equation $dz=p\,dx+q\,dy$, and instead of dy and dq, the values given by the equations (1), the resulting equations

$$\left(\frac{dM}{dx}+m\frac{dM}{dy}+(p+qm)\frac{dM}{dz}+\frac{Vm}{T}\frac{dM}{dq}\right)dx$$
$$+\left(\frac{dM}{dp}-\frac{Rm}{T}\frac{dM}{dq}\right)dp=0,$$

$$\left(\frac{dN}{dx}+m\frac{dN}{dy}+(p+qm)\frac{dN}{dz}+\frac{Vm}{T}\frac{dN}{dq}\right)dx$$
$$+\left(\frac{dN}{dp}-\frac{Rm}{T}\frac{dN}{dq}\right)dp=0,$$

must be identical, and must therefore be equivalent to the following, separately:

$$\frac{dM}{dx}+m\frac{dM}{dy}+(p+qm)\frac{dM}{dz}+\frac{Vm}{T}\frac{dM}{dq}=0,$$

$$\frac{dM}{dp}-\frac{Rm}{T}\frac{dM}{dq}=0,$$

$$\frac{dN}{dx}+m\frac{dN}{dy}+(p+qm)\frac{dN}{dz}+\frac{Vm}{T}\frac{dN}{dq}=0,$$

$$\frac{dN}{dp}-\frac{Rm}{T}\frac{dN}{dq}=0.$$

The equation $N=\varphi(M)$ being differentiated, gives

$$dN=\varphi'(M)\,dM,$$

or

$$\frac{dN}{dx}dx + \frac{dN}{dy}dy + \frac{dN}{dz}dz + \frac{dN}{dp}dp + \frac{dN}{dq}dq =$$

$$\varphi'(M)\left\{\frac{dM}{dx}dx + \frac{dM}{dy}dy + \frac{dM}{dz}dz + \frac{dM}{dp}dp + \frac{dM}{dq}dq\right\};$$

and if we substitute in this last, the values of

$$\frac{dM}{dx}, \quad \frac{dM}{dp}, \quad \frac{dN}{dx}, \quad \frac{dN}{dp},$$

taken from the four preceding equations, and at the same time change dz into $p\,dx + q\,dy$, we shall obtain

$$\left(\frac{dN}{dy} + q\frac{dN}{dz}\right)(dy - m\,dx)$$

$$+ \frac{1}{T}\frac{dN}{dq}(Rm\,dp + T\,dq - Vm\,dx) =$$

$$\varphi'(M)\left\{\left(\frac{dM}{dy} + q\frac{dM}{dx}\right)(dy - m\,dx)\right.$$

$$\left.+ \frac{1}{T}\frac{dM}{dq}(Rm\,dp + T\,dq - Vm\,dx)\right\},$$

which comes to the same as

$$Rm\,dp + T\,dq - Vm\,dx = \omega\,(dy - m\,dx),$$

if we make

$$\omega = -\frac{\dfrac{dN}{dy} + q\dfrac{dN}{dz} - \varphi'(M)\left(\dfrac{dM}{dy} + q\dfrac{dM}{dz}\right)}{\dfrac{1}{T}\left(\dfrac{dN}{dq} - \varphi'(M)\dfrac{dM}{dq}\right)}.$$

If we now put $r\,dx + s\,dy$ and $s\,dx + t\,dy$ for dp and dq, and put the coefficients of the independent differentials dx and dy each equal to zero, we should obtain

$$Rmr + Ts - Vm = -\omega m, \quad Rms + Tt = \omega;$$

and from these equations deducing the values of the differential coefficients r and t, in order to substitute them in the proposed, it will become, after all reductions,

$$s(Rm^2 - Sm + T) = 0;$$

and therefore (by reason of the equation A) it will be satisfied independently of the quantities ω and s.

318. The theorem above demonstrated, as well as those analogous to it in the higher orders, is not equally general with that of No. 314; for it must be remarked that the equations (1) may contain at once the five variables x, y, z, p, and q, and if we join to these equations, the equation $dz = pdx + qdy$ it will be impossible by elimination to arrive at a resulting equation containing less than three of them, and which consequently cannot be derived from one single primitive equation, except under certain conditions (308). We shall not however be warranted in concluding from thence, that if these conditions be not fulfilled the proposed partial differential equation itself cannot be deduced from any single primitive equation.

319. Suppose, for instance, we have given the equation

$$Ar + Bs + Ct = V,$$

in which A, B, C, &c. are constant, and V is a function of x and y. The equation (A) becomes, in this case,

$$Am^2 - Bm + C = 0;$$

and its roots, which we will designate by m' and m'', are constant, and will thus furnish two systems of equations (1) whose integration gives

$$\left.\begin{aligned} y - m'x &= a \\ Am'p + Cq - m'\textstyle\int Vdx &= b \end{aligned}\right\}$$

$$\left.\begin{aligned} y - m''x &= a' \\ Am''p + Cq - m''\textstyle\int Vdx &= b' \end{aligned}\right\}$$

and in which the integral $\int Vdx$ depends upon one variable only, because y may be exterminated from it by substituting its value derived from the first equation of each system.

We have therefore at once the following two first integrals of the proposed equation

$$Am'p + Cq - m'\int V dx = \phi\,(y - m'x),$$

$$Am''p + Cq - m''\int V dx = \psi\,(y - m''x);$$

and by integrating any one of these equations, we shall arrive at the second integral. If we take, for instance, the first, it gives

$$p = -\frac{C}{Am'}q + \frac{1}{A}\int V dx + \phi\,(y - m'x);$$

in which, for greater simplicity, we may write m'' instead of $\frac{C}{Am'}$, because by the equation (A) we have $m'm'' = \frac{C}{A}$, and by substituting this in the equation $dz = p\,dx + q\,dy$, we find

$$dz - \frac{dx}{A}\int V dx - dx\,.\,\phi\,(y - m'x) = q\,.\,(dy - m''dx):$$

consequently the equations to be integrated (314) will be

$$dy - m''dx = 0, \quad dz - \frac{dx}{A}\int V dx - dx\,\phi\,(y - m'x) = 0,$$

the former gives $y - m''x = a'$, by which the latter is changed into

$$dz - \frac{dx}{A}\int V dx - dx\,.\,\phi\,[a' + (m'' - m')\,x] = 0,$$

whose integral is

$$z - \frac{1}{A}\int dx \int V dx - \frac{1}{m'' - m'}\,\phi\,\{a' + (m'' - m')\,x\} = b',$$

and which reduces itself to

$$z + \frac{1}{A}\int dx \int V dx - \phi\,(y - m'x) = b',$$

when for a' we substitute its value, observing that ϕ is an arbitrary function whose differential coefficients are consequently arbitrary too, and which may also be supposed to

comprehend any constant quantity we please. We must also remark here, that in order to obtain $\int dx \int V dx$, the first integration must be performed with respect to x, substituting for y its value deduced from the equation $y - m'x = a$, as was before said; but having obtained this result, we must replace a by its value, $y - m'x$, and previous to the second integration, change y into $a' + m''x$, its value, deduced from the equation $y - m''x = a'$. In general, when we have several of these successive integrations to perform, we can never employ, in order to simplify them, more equations than subsist cotemporaneously. With attention to these circumstances the second integral of the proposed equation, $Ar + Bs + Ct = V$, will be

$$z - \frac{1}{A} \cdot \int dx \int V dx = \phi\,(y - m'x) + \psi\,(y - m''x).$$

If we had $A = 1$, $B = 0$, $C = -c^2$, and $V = 0$, the proposed equation would be

$$r - c^2 t = 0, \quad \text{or } \frac{d^2 z}{dx^2} = c^2 \frac{d^2 z}{dy^2}, *$$

and the integral would become

$$z = \phi\,(y - cx) + \psi\,(y + cx).$$

320. The arbitrary functions which enter into the integrals of partial differential equations, are determined by supposing that the function z takes particular forms when certain relations between x and y are assigned. We shall give two examples of the method of performing this:

1st. If we have $1 = M\phi\,(V)$, M and V denoting functions whose form is given in x, y, z; and if we would determine the function represented by the characteristic ϕ, so that when $F(x, y, z) = 0$, we shall have at the same time $f(x, y, z) = 0$, the characteristics F and f, denoting

* This is the equation of vibrating chords.

functions of a given form; we make $V=t$, and combine the three equations

$$V=t, \quad F(x, y, z)=0, \quad f(x, y, z)=0,$$

in order to deduce from thence the values of x, y, and z, in terms of t. If now these values are substituted in M, it will become a function of t, which we will designate by T, and we shall then have

$$1=T\phi(t), \quad \text{or } \phi(t)=\frac{1}{T},$$

and the form of the function ϕ is therefore determined, and the value of the expression $\phi(V)$ is known, if in this equation we write instead of t and T their values in terms of x, y, and z.

2d. Suppose $\quad 1=M\phi(V)+N\psi(V);$

since there are two functions to be determined, there must also be two conditions: suppose then that

$F(x, y, z)=0$, gives $f(x, y, z)=0$, and that

$F'(x, y, z)=0$, gives $f'(x, y, z)=0$, for the two conditions.

If we make $V=t$, and from the three equations

$$V=t, \quad F(x, y, z)=0, \quad f(x, y, z)=0,$$

deduce the values of x, y, z, in t, the functions M, N, will be changed into functions of t. Let T and θ be these functions, and we shall have

$$1=T\phi(t)+\theta.\psi(t)\ldots\ldots\ldots(1);$$

again, combining the three equations

$$V=t, \quad F'(x, y, z)=0, \quad f'(x, y, z)=0,$$

to obtain the values of x, y, z, in t; we shall, by means of these values, change the quantities M, N, into functions of t, which we will denote by T' and θ', and it will become

$$1=T'.\phi(t)+t'.\psi(t)\dots\dots\dots(2).$$

By means of the equations (1) and (2) we determine the functions $\phi(t)$ and $\psi(t)$ in t, and then we have only to substitute V for t, to get the values of $\phi(V)$ and $\psi(V)$ in x, y, z. *

On Equations of total Differentials, which do not satisfy the Conditions of Integrability.

321. We have shewn in No. 308, that a differential equation of the first order, between three variables of the form $Pdx+Qdy+Rdz=0$ cannot be satisfied by any function of two variables, except the equation

$$P.\frac{dR}{dy}-R\frac{dP}{dy}+R\frac{dQ}{dx}-Q\frac{dR}{dx}+Q\frac{dP}{dz}-P\frac{dQ}{dz}=0$$

be identical of itself; but when once a mutual dependance of whatever kind is established between x, y, and z, the proposed is changed into another containing no more than two of these variables, and will therefore suffice for the determination of one of them in functions of the other.

If we had, for example, the equation

$$\frac{dz}{z-c}=\frac{x\,dx+y\,dy}{x(x-a)+y(y-b)},$$

which, provided a and b are not equal to zero, cannot satisfy the condition above-mentioned, we suppose $y=\phi(x)$,

* The determination of the arbitrary functions comes to the same thing as making the curve surface represented by the proposed equation pass through given curves; and these curves may be continuous or discontinuous, as well as the functions themselves.

* See Note (P).

$\phi(x)$ denoting any function of x whatever, and it will become

$$\frac{dz}{z-c}=\frac{[x+\phi(x)\,\phi'(x)]\,dx}{x(x-a)+\phi(x)\,[\phi(x)-b]},$$

which gives as many different relations between z and x, as we please to assign particular forms to the function ϕ. If, for instance, we make $\phi(x)=x$, we shall have

$$\frac{dz}{z-c}=\frac{2\,x\,dx}{x(x-a)+x(-b)}=\frac{2\,dx}{2\,x-a-b},$$

whence we find

$$z-c=C\,(2\,x-a-b),$$

and the proposed will be satisfied by the system of equations

$$\left.\begin{aligned} y&=x \\ z-c&=C\,(2\,x-a-b) \end{aligned}\right\}$$

Newton, in his Treatise on Fluxions*, had already pointed out this method of resolving differential equations containing more than two variables; but it has the inconvenience of requiring an integration for every result we would obtain, and Monge has remarked, that by the proper introduction of an arbitrary function we may arrive at a general system of equations, which shall give an infinite number of particular ones, all satisfying the proposed.

322. The process to be followed for integrating the equation

$$P dx+Q\,dy+R d z=0,$$

by one primitive equation, when this is possible, leads us also to the most general solution it admits of in the contrary case. In fact, if we first of all perform the integration regarding one of the variables, z for instance, as constant; and if we represent by $U=c$ the primitive equation which

---

* Newtoni Opuscula, tom. I, page 83, edit. of 1744.

corresponds to $Pdx+Qdy=0$, if then we differentiate this primitive equation, making at the same time x, y, z, and C to vary. and compare the result with the proposed equation, we shall obtain

$$\frac{dC}{dz}=\frac{dU}{dz}=\mu R,$$

μ being the factor which renders $Pdx+Qdy$ an exact differential. The second member of this equation does not indeed reduce itself in this case to a function of z only, as it does when the condition of integrability is satisfied, and therefore is incapable of giving C in functions of z, as this condition requires; but it is evident that supposing always C to be a function of z, the proposed equation will be satisfied by the primitive equation $U=C$, provided we have at the same time

$$\frac{dC}{dz}=\frac{dU}{dz}-\mu R.$$

If then we make $C=\phi(z)$, the system of equations

$$\left.\begin{array}{r} U=\phi(z) \\ \dfrac{dU}{dz}-\mu R=\phi'(z) \end{array}\right\}$$

will satisfy the proposed, whatever be the form of the function ϕ; and an infinite number of particular cases may be obtained by assuming ϕ at pleasure.

If we apply what has been said to the equation

$$\frac{dz}{z-c}=\frac{xdx+ydy}{x(x-a)+y(y-b)},$$

which we took as an example in the preceding No. we have

$$Pdx+Qdy=\frac{xdx+ydy}{x(x-a)+y(y-b)},\quad R=-\frac{1}{z-c};$$

and if we make

$$\mu=x(x-a)+y(y-b),$$

we find $U=x^2+y^2$, and thus we obtain the equations

$$\left.\begin{array}{r} x^2+y^2=\phi(z) \\ \dfrac{x(x-a)+y(y-b)}{z-c}=\phi'(z) \end{array}\right\}$$

Of the Method of Variations.

Investigation of the Variation of any Function whatever.

323. All the applications of the Differential Calculus, hitherto presented to the reader's view, suppose that the relation between the variables remains unaltered during the whole course of the investigation; but there is a variety of problems, in which it is necessary to conceive, that this relation changes. For example, if V denote a function containing x, y, and the differential coefficients of y, the integral $\int V\,dx$, taken between the same values of x, is susceptible of an infinite number of values, which depend on the relation established between x and y; so that it may be asked, among all the possible relations between x and y, which is that which gives the integral $\int V\,dx$, between given limits, the greatest or the least possible value. The integral $\int V\,dx$, so long as we assign no particular relation between x and y, expressing the measure of a certain property common to all curves, we enquire then for what particular curve this property is a *maximum* or a *minimum*. It is evident, that if CE, fig. 54, represent this curve, then for every other, as $\gamma\epsilon$, the integral $\int V dx$ must have a less value in the former case, and a greater in the latter. In order to satisfy this condition, the first thing requisite is to investigate the difference which any change in the relation between y and x, or in the nature of the curve which this relation represents, produces in the integral $\int V\,dx$. This change may be expressed, by making y vary independently of x; for when we consider two curves,

CE and $\gamma\epsilon$, the same abscissa x corresponds to the two ordinates, PM and $P\mu$, and their difference $M\mu$ ought to be distinguished from the differences $M'R$ and $\mu'\rho$, which exist between two consecutive ordinates, taken in the same curve.

Lagrange, whose first researches produced the Calculus of Variations, has also made an application of it to Mechanics, of the highest importance, the force of which will be easily seen, if we observe that we may regard the co-ordinates of the different points of a body in motion, either for the purpose of comparing at the same instant two different points of the body, or two consecutive positions of the same point. In the one case there exists no relation between the co-ordinates, but that which defines the surfaces which terminate the body; in the other the co-ordinates vary according to the conditions of the motion once commenced, and as functions of a new variable, which is the measure of the time. Here are again two modes of varying the same quantity, and which it is highly convenient to denote by two distinct signs. That which succeeds the other, and is as it were superimposed on it, constitutes the Calculus of Variations, whose various applications can scarcely be understood, unless we regard it as *having for its object the differentiation, under a new point of view, of quantities which have already been differentiated in some other manner*. In this second system of differentiation an hypothesis must be made agreeable to the nature of the questions we propose to resolve. (See the *Mecanique Analytique*, pages 51 and 195).

324. Lagrange has designated this new differentiation by the characteristic δ, and this practice has been adopted. In order not to overstep the limits of our subject, we shall confine ourselves to the application of the principles of the Calculus of Variations to Geometrical questions.

In these questions the characteristic d is used to denote

the transition from one point to another in the same curve, and the characteristic δ is applied to the change of the curve. Thus MR being represented by dy, $M\mu$ will be δy; whence it follows that

$$P'M'=y+dy, \quad P\mu=y+\delta y.$$

If we pass to the point μ' from M', we find, by what we have before said,

$$P'\mu'=y+dy+\delta(y+dy)$$
$$=y+dy+\delta y+\delta dy;$$

but the point μ' being consecutive to μ in the curve $\gamma\iota$, we have also

$$P'\mu'=y+\delta y+d(y+\delta y)$$
$$=y+\delta y+dy+d\delta y;$$

and the comparison of these two expressions for the same line, gives this remarkable consequence:

$$\delta dy=d\delta y.$$

The same thing may also be proved without the consideration of curves, if we denote by $\phi(x)$ the primitive form of y, and by another function $\psi(x)$, the result of the variation*. Consequently $\delta y=\psi(x)-\phi(x)$ will be

* In order to assign a common origin to the functions $\phi(x)$ and $\psi(x)$, Euler, who hastened to adopt, and to illustrate the Calculus of Variations, regarded the primitive value of y, or $\phi(x)$ as being deduced from another function, containing at the same time with x a new variable, t, and becoming $\phi(x)$ when $t=0$ (*Novi. Comm. Acad. Petrop. tom.* XVI. page 35). By this mean $y+\delta y$ becomes $y+\frac{dy}{dt}dt$, and $\frac{dy}{dt}$ being taken on the hypothesis that $t=0$ represents, as long as we assign no particular relation between y and t, an arbitrary function of x. The general value of y would be expressed by the series

$$y+\frac{dy}{dt}\cdot\frac{t}{1}+\frac{d^2y}{dt^2}\cdot\frac{t^2}{1.2}+\&c.$$

th

a certain function of x, and of course a function of y, by reason of the primitive relation between these variables. Denoting then by π this last function, we have

$$\delta y = \pi\,(y).$$

According to this law, if we denote $y + dy$ by y', we shall have

$$\delta y' = \pi\,(y');$$

whence we conclude that

$$\delta y' - \delta y = \pi\,(y') - \pi\,(y) = d\,.\,\pi\,(y) = d\,\delta y;$$

but since

$$dy = y' - y,$$

if we take the variations, we have

$$\delta\,dy = \delta y' - \delta y,$$

whence we obtain, as before,

$$\delta\,dy = d\,\delta y.$$

It follows from this, that $\delta\,d^2y = d\,\delta\,dy = d^2\,\delta\,y$; and continuing in this way, we arrive at this fundamental theorem:

$$\delta\,d^n y = d^n\,\delta y,$$

in consequence of which *we are at liberty to transpose the characteristics* d *and* δ.

In order to give greater symmetry to our operations, as well as to embrace circumstances relating to the limits of the integrals, of which we shall give some examples far-

---

the variable t being supposed equal to zero in y and its differential coefficients; and if we take the differential coefficients of this series, with regard to x, we shall form all the quantities which it is necessary to substitute, in order to obtain the varied state of the integral $\int V d\,x$, arranged in powers of t. It is under this form that Lagrange, in the new edition of his *Leçons sur le Calcul des Fonctions*, presents the Calculus of Variations, respecting which he enters into a variety of highly-interesting details.

ther on, we make x vary as well as y; but the above theorem does not on this account cease to remain true; because the law of variation, although arbitrary, being constant, δx is a function of x, from which $\delta x'$ is deduced by changing x into x': we have therefore $\delta d x = d \delta x$, and similarly $\delta d V = d \delta V$, for any function V, dependent on x.

325. There is an analogous theorem relative to the sign $\int$. In fact if we represent $\int U$ by U_1, we shall have

$$d U_1 = U, \quad \text{whence } \delta d U_1 = \delta U;$$

if now we transpose the characteristics δ and d, and pass to the integrals, we find successively

$$d \delta U_1 = \delta U, \quad \text{and } \delta U_1 = \int \delta U;$$

and replacing U_1 by its equal $\int U$, we find

$$\delta \int U = \int \delta U.$$

326. This being premised, we see that in order to obtain the variation of any function U, containing x, y, and their differentials, of any orders, we must suppose that x and y respectively become $x + \delta x$, and $y + \delta y$, and regard δx and δy, as arbitrary functions, the one of x, and the other of y. The operation breaking off at the terms which contain higher powers of δx and δy than the first, the process is evidently the same as that of differentiating in the ordinary manner the function U, as well relative to x and y, as to their differentials, regarded as distinct variables, only in this last differentiation, making use of the characteristic δ instead of d. It is evident, in fact, that on this hypothesis the differentials of

$$x, \quad y, \quad dx, \quad dy, \quad \&c.$$

are

$$\delta x, \quad \delta y, \quad \delta dx, \quad \delta dy, \quad \&c.$$

If therefore the ordinary differential of U is

$$d U = M dx + N d^2x + P d^3x + Q d^4x + \&c.$$
$$+ m dy + n d^2y + p d^3y + q d^4y + \&c.$$

it will suffice to change the last d into δ, and we shall have

$$\delta U = M\delta x + N\delta dx + P\delta d^2x + Q\delta d^3x + \&c.$$
$$+ m\delta y + n\delta dy + p\delta d^2y + q\delta d^3y + \&c.$$

If the function U be of the form Vdx, V containing only

$$x,\ y,\ \frac{dy}{dx} = p,\ \frac{dp}{dx} = q,\ \&c.$$

we shall have

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \&c.$$

and the variation will be

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \&c.$$

observing that the quantities p, q, r, &c. are to be considered as involving two independent variables, x and y (preceding No.), and that therefore their variation may be taken on two different hypotheses, viz. making either one of them vary alone, or both together. We shall proceed upon this latter supposition, because as we have already remarked, it is more general, and because it is easy from thence to deduce the results of the other, by suppressing the terms relative to the variable which we would consider as constant. Now if we differentiate by the characteristic δ the functions

$$\left.\begin{aligned} p &= \frac{dy}{dx} \\ q &= \frac{dp}{dx} \\ r &= \frac{dq}{dx} \end{aligned}\right\} \text{ we find } \left\{\begin{aligned} \delta p &= \frac{dx\delta dy - dy\delta dx}{dx^2} = \frac{d\delta y - pd\delta x}{dx} \\ \delta q &= \frac{dx\delta dp - dp\delta dx}{dx^2} = \frac{d\delta p - qd\delta x}{dx} \\ \delta r &= \frac{dx.\delta dq - dq\delta dx}{dx^2} = \frac{d\delta q - rd\delta x}{dx} \end{aligned}\right.$$

&c. &c.

and by the assistance of these expressions we may find the variation of any expression whatever, containing x, y, and their differentials of any order.

327. When we consider an integral formula as $\int U$, in which U is, as above, a function of x, y, and their differentials, we have $\delta \int U = \int \delta U$ (325); and by the preceding No.

$$\int \delta U = \int(M\,\delta x + N\,\delta dx + P\,\delta d^2x + Q\,\delta d^3x + \&c.)$$
$$+ \int(m\,\delta y + n\,\delta dy + p\,\delta d^2y + q\,\delta d^3y + \&c.)$$

This expression is not reduced to the most simple form it can be exhibited in: it must be so transformed, to this effect, that no term shall remain under the sign $\int$ containing at once the characteristics δ and d applied to each other; and this may be accomplished by transposing the δ after the d, and then integrating by parts, as follows:

$$\int M\,\delta x = \int M\,\delta x$$
$$\int N\,\delta dx = \int N\,d\delta x = N\,\delta x - \int dN\,\delta x$$
$$\int P\,\delta d^2x = \int P\,d^2\delta x = P\,d\delta x - \int dP\,d\delta x = P\,d\delta x - dP\,\delta x + \int d^2P\,\delta x$$
$$\int Q\,\delta d^3x = \int Q\,d^3\delta x = Q\,d^2\delta x - \int dQ\,d^2\delta x = Q\,d^2\delta x - dQ\,d\delta x + \int d^2Q\,d\delta x$$
$$= Q\,d^2\delta x - dQ\,d\delta x + d^2Q\,\delta x - \int d^3Q\,\delta x$$
$$\&c. = \&c.$$

Similarly, we have

$$\int m\,\delta y = \int m\,\delta y$$
$$\int n\,\delta dy = \int n\,d\delta y = n\,\delta y - \int dn\,\delta y$$
$$\int p\,\delta d^2y = \int p\,d^2\delta y = p\,d\delta y - dp\,\delta y + \int d^2p\,\delta y$$
$$\int q\,\delta d^3y = \int q\,d^3\delta y = q\,d^2\delta y - dq\,d\delta y + d^2q\,\delta y - \int d^3q\,\delta y$$
$$\&c. = \&c.$$

and, by substitution, we obtain

$$\int \delta U = (N - dP + d^2Q - \&c.)\,\delta x + (P - dQ + \&c.)\,d\delta x$$
$$+ (Q - \&c.)\,d^2\delta x + \&c.$$
$$+ (n - dp + d^2q - \&c.)\,\delta y + (p - dq + \&c.)\,d\delta y$$
$$+ (q - \&c.)\,d^2\delta y + \&c.$$
$$+ \int(M - dN + d^2P - d^3Q + \&c.)\,\delta x$$
$$+ \int(m - dn + d^2p - d^3q + \&c.)\,\delta y.$$

This result is composed of two similar parts; the one due to the variation of x, the other to that of y; and it is easy to

see that it might be extended to any number of variables, by adding to it, for each, a series of terms similar to those which the variable x, or the variable y has produced.

328. When the expression $\int U$ is put under the form $\int V\,dx$, that is to say, when only the variables x and y and differential coefficients of y enter into V, the investigation of the variation appears rather more complicated, but it has led to consequences sufficiently remarkable. We must first observe that

$$\delta\int V\,dx = \int\delta(V\,dx) = \int V\,d\delta x + \int dx\,\delta V,$$
$$\int V\,d\delta x = V\delta x - \int dV\,\delta x,$$

and consequently, that

$$\delta\int V\,dx = V\delta x + \int(dx\,\delta V - dV\,\delta x).$$

The quantity $dx\,\delta V - dV\,\delta x$ is formed by writing for dV and δV their values stated in (326); and it becomes

$$dx\,.\,\delta V - dV\,.\,\delta x = N(dx\,\delta y - dy\,\delta x) + P(dx\,\delta p - dp\,\delta x)$$
$$+ Q(dx\,\delta q - dq\,\delta x) + \&c.$$

Putting now $p\,dx$ for dy in the quantity which is multiplied by N, and for δp its value (326) in that which is multiplied by P, we find

$$dx\,\delta y - dy\,\delta x = dx\,(\delta y - p\,\delta x)$$
$$dx\,\delta p - dp\,\delta x = d\,\delta y - p\,d\delta x - dp\,\delta x = d(\delta y - p\,\delta x),$$

whence it follows, that

$$dx\,\delta p - dp\,\delta x = d\left(\frac{dx\,\delta y - dy\,\delta x}{dx}\right).$$

If we change y into p, and p into q, we have similarly

$$dx\,\delta q - dq\,\delta x = d\left(\frac{dx\,\delta p - dp\,\delta x}{dx}\right),$$

and so on: making then

$$\delta y - p\,\delta x = \omega,$$

ve

$$- dy\,\delta x = \omega\,dx, \quad dx\,\delta p - dp\,\delta x = d\omega,$$

$$- dq\,\delta x = d\frac{d\omega}{dx}, \text{ \&c.}$$

uently

$$dx\,\delta V - dV\,\delta x) = \int N\omega\,dx + \int P\,d\omega + \int Q\,d\frac{d\omega}{dx} + \text{\&c.}$$

e second member of this is affected by the charac-

$$= \quad - \int \frac{dP}{dx}\,.\,\omega\,dx$$

$$\int \frac{dQ}{dx}\,d\omega$$

$$= Q\frac{d\omega}{dx} - \frac{dQ}{dx}\omega + \int \frac{1}{dx}\,d\frac{dQ}{dx}\,.\,\omega\,dx,$$

&c.

By means of these expressions we obtain

$$\delta\int V\,dx = V\,\delta x + \left\{P - \frac{dQ}{dx} + \text{\&c.}\right\}\omega + \left\{Q - \text{\&c.}\right\}\frac{d\omega}{dx} + \text{\&c.} + \int \left\{N - \frac{dP}{dx} + \frac{1}{dx}\,d\frac{dQ}{dx} - \text{\&c.}\right\}\omega\,dx.$$

We may extend this result without any difficulty to the case of a greater number of variables dependent on x, by adding for each of them, a series of terms similar to those which have been obtained by considering y alone; but it is important to observe, that if we replace ω by its value

y − p δx, the part affected by the sign ∫ may then be ritten thus,

$$\int\left\{N-\frac{dP}{dx}+\frac{1}{dx}d\frac{dQ}{dx}-\&c.\right\}dx\,\delta y$$
$$-\int\left\{N-\frac{dP}{dx}+\frac{1}{dx}d\frac{dQ}{dx}-\&c.\right\}p\,dx\,\delta x,$$

id we now see that in this case the coefficient of δy and at of δx have a relation which could not be perceived in e preceding No. and in virtue of which, if one of these efficients be made equal to zero, the other also vanishes.

329. A remark not less worthy of attention is, that if, the developement of $\delta\int U$ (327), we had

$$M - dN + d^2P - d^3Q + \&c. = 0$$
$$m - dn + d^2p - d^3q + \&c. = 0,$$

e variation $\int\delta U$ would be entirely freed from the integral gn; but these equations are precisely those which ought hold good, in order that the function U should be im-ediately integrable of itself: this may be shewn *à priori*, applying the method of variations itself to determine ese conditions.

In fact, let U be the differential of a function U_1; or, t $U = dU_1$, and consequently

$$\delta U = \delta d U_1 = d \delta U_1,$$

follows then that if U be a complete differential, δU ught also to be one; and consequently when we have rought out from under the integral sign all those terms hich are susceptible of integration, those which remain ust, collectively, be equal to zero; and that, without the ecessity of supposing any relation between $x, y, \delta x, \delta y$.

The developement of $\delta\int V dx$, as it furnishes only he single condition

$$N-\frac{dP}{dx}+\frac{1}{dx}d\frac{dQ}{dx}-\&c. = 0,$$

shews, that that which relates to the variable x becomes of no use when the function U is reduced to the form $V dx$, V containing only x, y, and the differential coefficients of y.

330. These remarks are not limited to the expression for $\int U$; they equally extend to those of $\int\int U, \int^3 U$, &c. whatever be the number of the signs of integration; and if we seek the variations of these latter formulæ, as we have done with respect to $\int U$, we shall find the equations of condition, which must hold good in order that U shall be the complete differential of a function U_1, of an order immediately inferior to U_1, of a function U_2 whose order is inferior to that of U by two units, &c. and so on. Let

$$\left.\begin{array}{l}\delta U = M\delta x + Nd\delta x + Pd^2\delta x + Qd^3\delta x + \&c. \\ \quad + m\delta y + nd\delta y + pd^2\delta y + qd^3\delta y + \&c.\end{array}\right\};$$

we shall have then, by what we have already shewn,

$$\begin{array}{l}\delta\int U = (N - dP + d^2Q - \&c.)\,\delta x + (P - dQ + \&c.)\,d\delta x \\ \qquad\qquad + (Q - \&c.)\,d^2\delta x + \&c. \\ \quad + (n - dp + d^2q - \&c.)\,\delta y + (p - dq + \&c.)\,d\delta y \\ \qquad\qquad + (q - \&c.)\,d^2\delta y + \&c. \\ \quad + \int(M - dN + d^2P - d^3Q + \&c.)\,\delta x \\ \quad + \int(m - dn + d^2p - d^3q + \&c.)\,\delta y;\end{array}$$

but, $\delta\int U = \delta U_1$; and, since $U_1 = dU_2$, it becomes

$$\delta U_2 = \delta\int U_1 = \int\delta U_1 = \int\delta\int U:$$

δU_2 is therefore obtained by integrating $\delta\int U$ once more, and bringing from under the first sign of integration all which it is possible to integrate. Thus we find

$$\begin{array}{l}\delta U_2 = \int(N - dP + d^2Q - \&c.)\,\delta x + \int(P - dQ + \&c.)\,d\delta x \\ \qquad\qquad + \int(Q - \&c.)\,d^2\delta x + \&c. \\ \quad + \int(n - dp + d^2q - \&c.)\,\delta y + \int(p - dq + \&c.)\,d\delta y \\ \qquad\qquad + \int(q - \&c.)\,d^2\delta y + \&c. \\ \quad + \int\int(M - dN + d^2P - d^3Q + \&c.)\,\delta x \\ \quad + \int\int(m - dn + d^2p - d^3q + \&c.)\,\delta y;\end{array}$$

and, integrating by parts the terms which contain the differentials of δx and δy, we have

$$\begin{aligned}\delta U_2 = {} & (P - 2\,dQ + 3\,d^2R - \&c.)\delta x + (Q - 2\,dR + \&c.)\,d\delta x \\ & + (R - \&c.)\,d^2\delta x + \&c. \\ & + (p - 2\,dq + 3\,d^2r - \&c.)\delta y + (q - 2\,dr + \&c.)\,d\delta y \\ & + (r - \&c.)\,d^2\delta y + \&c. \\ & + \int (N - 2\,dP + 3\,d^2Q - 4\,d^3R + \&c.)\,\delta x \\ & + \int (n - 2\,dp + 3\,d^2q - 4\,d^3r + \&c.)\,\delta y \\ & + \int\int (M - dN + d^2P - d^3Q + d^4R - \&c.)\,\delta x \\ & + \int\int (m - dn + d^2p - d^3q + d^4r - \&c.)\,\delta y.\end{aligned}$$

Such is the expression for the required variation, which will not be entirely freed from the two signs $\int$ unless the equations

$$\begin{aligned} N - 2\,dP + 3\,d^2Q - 4\,d^3R + \&c. & = 0 \\ n - 2\,dp + 3\,d^2q - 4\,d^3r + \&c. & = 0 \\ M - dN + d^2P - d^3Q + d^4R - \&c. & = 0 \\ m - dn + d^2p - d^3q + d^4r - \&c. & = 0, \end{aligned}$$

be identical; in which case δU_2, being once integrated relative to the characteristic δ will at once give U_2, or the second integral of the proposed function U.

For instance, let $U = x\,d^2y + 2\,dx\,dy + y\,d^2x$;

and we have

$$\begin{aligned}\delta U = {} & d^2y\,\delta x + 2\,dy\,d\delta x + y\,d^2\delta x \\ & + d^2x\,\delta y + 2\,dx\,d\delta y + x\,d^2\delta y,\end{aligned}$$

$$M = d^2y, \quad N = 2\,dy, \quad P = y,$$
$$m = d^2x, \quad n = 2\,dx, \quad p = x;$$

and the equations of condition given above, become

$$\begin{aligned} 2\,dy - 2\,dy & = 0 \\ 2\,dx - 2\,dx & = 0 \\ d^2y - 2\,d^2y + d^2y & = 0 \\ d^2x - 2\,d^2x + d^2x & = 0: \end{aligned}$$

the proposed function is therefore immediately integrable. The part $y\delta x + x\delta y$, freed from the sign $\int$, gives, when integrated relative to the characteristic δ, $U_2 = xy$.

The foregoing processes are sufficient to shew that the first integration of a differential function of m variables, requires m conditions to be satisfied, when these variables are regarded as independent, and, that, for a number n of successive integrations there will be $m\,n$ equations of condition. There would be but $m-1$ for the first integration, and $n(m-1)$ for all together, were the proposed function of the form $\int^n V\,d\,x$, where V contains only differential coefficients.

On the Maxima and Minima of indeterminate integral Formulæ.

331. Integrals, such as $\int y\,d\,x$, $\int\sqrt{d\,x^2 + d\,y^2}$, may be called indeterminate, when no particular form is assigned to the function y; but, in order that they may be susceptible of a *maximum* or a *minimum*, these integrals must be *definite* (209), since it is only when taken between given limits that they have any fixed value, when y is determined in functions of x.

The principles laid down in No. 134, respecting functions whose form is given, will apply also, with the aid of the Calculus of Variations, to indeterminate integrals. In fact, pursuing the course traced in No. 121, the result of the substitution of

$$x+\delta x,\quad y+\delta y,\quad d\,x+\delta d\,x,\quad d\,y+\delta d\,y, \text{ \&c.}$$

in the place of the quantities x, y, $d\,x$, $d\,y$, &c. in any function u of these quantities may be arranged according to the powers of the variations δx, δy, $\delta d\,x$, $\delta d\,y$, &c.; and δu will contain all the terms of this developement in which the variations do not rise above the first degree. These terms, since they change their sign at the same time with the variations which affect them, ought, according to

heory above cited, to vanish in the case of a *maximum* or *num*, whatever be the variations δx and δy; and con-ently we must have $\delta u = 0$. When $u = \int U$, since $= \int \delta U$, (325); we must have, at the *maximum* and *num* of $\int U$, the equation $\int \delta U = 0$, noticing however it is only when taken between the limits assigned for hat $\int \delta U$ is to vanish.

t follows also from the same theory that the condition 0 does not necessarily indicate the existence of a *max-* or *minimum*, since this requires, besides, that the s in which the variations rise to the second degree always preserve the same sign; the discussion of these conditions is too complicated, and too delicate to be duced in this place.

32. The developement of $\int \delta U$ is composed of two entirely distinct from each other (327), since one of is freed from the sign $\int$ and the other remains af-d by it; the first may be represented by

$$\alpha \delta x + \beta \delta y + \alpha_1 d \delta x + \beta_1 d \delta y + \text{\&c.}$$

he latter by $$\int \{ \chi \delta x + \psi \delta y \}.$$

e two parts cannot be compared with each other, since latter is unintegrable, so long as δx and δy preserve independence which the nature of the problem re-s; and in this state we cannot cause the integral to h by any other means than by making separately

$$\chi = 0, \quad \psi = 0,$$

umber of which equations is universally equal to that e independent variations; but, when there are no than two variables, and U may be thrown into the $V dx$, the developement of the variation of $\int V dx$ in , shews that $\chi = -\psi p$, and consequently that

$\chi\, d x + \psi\, d y = 0$, a condition easily verified in every particular case. It follows from this, that the equations $\chi = 0$ and $\psi = 0$ are in fact one and the same, and that one relation only exists between x and y which we might equally well have obtained by making $\delta x = 0$, that is to say, by supposing x not to vary; but this hypothesis (as we shall soon see) would greatly restrict the properties of that part of the variation which is unaffected by the sign $\int$.

It appears then that the equations noticed in (329), as expressing the conditions which render the formulæ $\int U$ and $\int V\, d x$ integrable, and which in that case are identical, determine, in every other case, the relation between y and x by which the proposed integrals attain their *maximum* or *minimum* values. We easily see that these equations may rise to an order whose exponent is double that of the highest differential involved in either U or V.

333. The expression for $\int \delta U$ becomes, by the disappearance of the part affected by the integral sign,

$$\int \delta U = \alpha\, \delta x + \beta\, \delta y + \alpha_1 d\, \delta x + \beta_1 d\, \delta y + \&c.$$

and making, for the sake of brevity $\int \delta U = \phi$, the complete value of this integral is obtained, by taking the difference between those of ϕ at each of the limits (209); so that if ϕ' represent the value of ϕ at the first limit, and ϕ'' at the second, we shall have $\int \delta U = \phi'' - \phi'$, whence it follows, that in the case of a *maximum* or *minimum* of the integral $\int U$, the condition

$$\phi'' - \phi' = 0,$$

must also be satisfied; but we must be careful to notice, that this equation contains no quantities but such as are relative to the limits of the integral $\int U$; and that the variations δx, δy, δdx, $\delta\, d y$, &c. may there be either nothing, or simply connected with each other by given relations, ac-

cording as these limits are fixed or variable. The Géometrical application of these different circumstances will illustrate them sufficiently.

The former case takes place when the curve which renders the proposed integral a *maximum* or *minimum*, is to be determined among all curves subjected to the condition of passing through two points, whose co-ordinates are given, as well as every thing else relating to them; and when the integral is to commence at one of these points, and terminate at the other. If x' and y' denote the co-ordinates of the first, and x'' and y'' those of the second, these quantities, since they belong to all the curves which can come under consideration in the question, will not undergo any variation. When therefore we change x and y first into x' and y', and then into x'' and y'', we must make $\delta x'=0$, $\delta y'=0$, $\delta x''=0$, $\delta y''=0$. The terms affected with these variations will therefore disappear of themselves, from the equation $\phi''-\phi'=0$, which will in consequence be verified, if it contain only these terms; and the curve deduced from the equation $\chi=0$, will completely resolve the problem, provided we subject it to the condition of passing through the two given points; which may universally be effected by a proper determination of the arbitrary constants, comprised in the integral of the equation cited, which in that case will be of the second order.

If the equation $\phi''-\phi'=0$ contain besides, terms affected with $\delta dx'$, $\delta dy'$, $\delta dx''$, $\delta dy''$; and if, in addition to the preceding condition, the tangents of the curve sought be required to have a given inclination to the line of the abscissæ, at the limits of the integral, these terms will disappear of themselves; because, as the differentials dx and dy undergo no change at the limits, the variations $\delta dx'$, $\delta dy'$, $\delta dx''$, $\delta dy''$, will be equal to zero, and will therefore cause the products into which they enter to vanish: but in

order to make the curve required satisfy this condition, its equation ought to contain two arbitrary constants more than in the case last examined, and consequently the differential equation $\chi=0$ ought to be of the fourth order. This will suffice to point out in what manner the equation $\phi''-\phi'=0$ will be be satisfied, when the co-ordinates of the limits and their differential coefficients have fixed values: we now proceed to consider the cases where the limits themselves must be regarded as variable.

334. It may be required, that the curve which has the proposed *maximum* or *minimum* property, shall be taken, not among all curves which can be drawn through two given points, but among all which can be drawn between two
FIG. 55. given curves AA' and BB', fig. 55, without determining the points in which these latter are intersected by it. It is evident, that in passing from one curve AB to another $A'B'$, the extremities A and B will have changed their places, the abscissæ which correspond to the beginning and the end of the integral, have not therefore the same values in its varied, as in its primitive state; and the ordinates of those points will have varied according to the law which determines the nature of the curves AA' and BB'. Under these circumstances, the variations of the ordinates, and of their abscissæ, must have the same relations as the differentials relative to the curves AA' and BB', which relations are expressed by the equations of these curves, and are therefore given. It is requiste then to introduce these relations into the equation $\phi''-\phi'=0$, and afterwards, in order to verify it, we must make the coefficients of those variations which remain independent, equal to zero.

In proportion as the function $\int U$ contains differentials of higher orders, the number of terms of the equation $\phi''-\phi'=0$ will increase, and we are at liberty to subject the limits to new conditions; as for instance, if the curve AB were required to be taken among all which can touch

at once the two curves AA' and BB'. In virtue of this condition, not only the co-ordinates x and y must have, at the limits of the integral, relations expressed by the equations of these curves; but the same must also hold with respect to their differentials. Thus the variations $\delta dx'$, $\delta dy'$, $\delta dx''$, $\delta dy''$, are no longer independent; but must coincide with the second differentials, relative to the given curves. We may then, with the help of these relations, eliminate some of the variations $\delta dx'$, $\delta dy'$, $\delta dx''$, $\delta dy''$, from the equation $\phi'' - \phi' = 0$; and it must then be satisfied, by putting separately equal to zero, the coefficients of the variations remaining, which will be entirely arbitrary.

The equations obtained by this process, since they establish relations between the co-ordinates of the extreme points of the proposed curve, will necessarily have reference to the constants introduced by the integration of the equation $\chi = 0$, and will serve to determine their values.

335. A few particular applications will illustrate the preceding theory; but we must first remark, that since there are circumstances where it is necessary to consider the variations of the limits, if th co-ordinates x', y', x'', y'', of these limits enter into the expression for U, they must be made to vary in that expression, as well as x and y, and consequently to add to δU the terms

$$A'\delta x' + B'\delta y' + A''\delta x'' + B''\delta y''$$
$$+ A'_1 d\delta x' + B'_1 d\delta y' + A''_1 d\delta x'' + B''_1 d\delta y'' + \&c.$$

and since the variations $\delta x'$, dy', $\delta x''$, $\delta y''$, are independent of the indeterminate co-ordinates x and y, they may be brought from under the sign $\int$, while the functions A', A'', &c. A'_1, A''_1, &c. remain affected by it: and it will be requisite to introduce into the first part of the variation $\int\delta U$, the terms

$$\delta x' \int A' + \delta y' \int B' + \delta x'' \int A'' + \delta y'' \int B''$$
$$+ d\delta x' \int A'_1 + d\delta y' \int B'_1 + d\delta x'' \int A''_1 + d\delta y'' \int B''_1 + \&c.$$

these integrals being taken between the same limits as that proposed at first.

It is not at once evident what the preceding terms become, if one of the limits be at the same time the origin of the co-ordinates. This difficulty may be avoided, by making first of all

$$x = X - x', \quad y = Y - y',$$

and then supposing, that the origin of the co-ordinates X, Y, is fixed, while the quantities x' and y' are variable; thus it becomes

$$\delta x = \delta X - \delta x', \quad \delta y = \delta Y - \delta y'.$$

As to the differentials dx, dy, &c. since they do not depend on the quantities x' and y', they consequently have no variation, and the expression for δU becomes simply

$$M(\delta X - \delta x') + N \delta d X + \&c.$$
$$+ m(\delta Y - \delta y') + n \delta d Y + \&c.$$

It is then allowable to put x', y', equal to zero, provided we allow the variations $\delta x'$, $\delta y'$, which may be considered as the first degree of magnitude of these quantities, to remain. On this supposition X and Y again become x and y, and the change which takes place in $\int \delta U$ reduces itself to the terms $-\delta x' \int M - \delta y' \int m$, whose integrals must be taken between the limits originally proposed.

336. Suppose it were required to determine y in functions of x, in such a manner that $\int \sqrt{dx^2 + dy^2}$, between given limits, should be a *minimum*, which comes to the same thing as *to find the nature of the shortest line which can be drawn between two points upon a plane.* Now we have

$$\delta U=\delta\sqrt{dx^2+dy^2}=\frac{dx\,\delta dx+dy\,\delta dy}{\sqrt{dx^2+dy^2}},$$

and $$\int\delta U=\int\frac{dx}{ds}\,d\delta x+\int\frac{dy}{ds}\,d\delta y,$$

making $\sqrt{dx^2+dy^2}=ds$, and transposing the characteristics d and δ. If we now integrate by parts, we get

$$\int\delta U=\frac{dx}{ds}\,\delta x+\frac{dy}{ds}\,\delta y-\int\left(d\frac{dx}{ds}\,\delta x+d\,\frac{dy}{ds}\,\delta y\right);$$

and the part affected with the sign $\int$ gives (332),

$$d\frac{dx}{ds}=0,\text{ whence }\frac{dx}{ds}=C,\ \frac{dy}{dx}=C',\ y=C'x+C''.$$

This result, as might be expected, indicates a straight line; and the constants which it includes, will suffice to satisfy the conditions relative to the points between which it must be drawn.

The part freed from the sign $\int$, or ϕ (333), containing only the variations of the co-ordinates of the extreme points, vanishes when they are fixed; and the constants C' and C'' are in that case determined by subjecting the right line to pass through these points. When they are not fixed, but only subject to the condition of being situated in given curves, the quantities x' and y', x'' and y'', which are unknown, must then, as well as their variations, satisfy the equation $\phi''-\phi'=0$, which becomes

$$\frac{dx''}{ds''}\,\delta x''+\frac{dy''}{ds''}\,\delta y''-\frac{dx'}{ds'}\,\delta x'-\frac{dy'}{ds'}\,\delta y'=0,$$

and also the equations of the given curves, whose differentials we will denote by

$$dy=m\,dx,\qquad dy=n\,dx;$$

we have then (334),

$$\delta y' = m \delta x, \quad \delta y'' = n'' \delta x'',$$

$$\left(\frac{d x''}{d s''} + n'' \frac{d y''}{d s''}\right) \delta x'' - \left(\frac{d x'}{d s'} + m' \frac{d y'}{d s'}\right) \delta x' = 0.$$

and on account of the independence of the variations $\delta x''$ and $\delta x'$, this equation separates itself into the following:

$$d x'' + n'' d y'' = 0, \text{ or } \frac{d y''}{d x''} = -\frac{1}{n''},$$

$$d x' + m' d y' = 0, \text{ or } \frac{d y'}{d x'} = -\frac{1}{m'},$$

which denote that the proposed right line must meet each of the given curves at right angles.

In consequence of the equations $y = C'x + C''$, we have $d y = C' d x$, for every point in the right line, and the preceding equations become

$$1 + C' n'' = 0, \quad 1 + C' m' = 0;$$

but the constant C' depends on the co-ordinates of the extreme points, because the equation of the right line drawn through these points, is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'), \text{ which gives } C' = \frac{y' - y''}{x' - x''};$$

and substituting this value for C', there result the equations

$$x' - x'' + n''(y' - y'') = 0, \quad x' - x'' + m'(y' - y'') = 0,$$

the combination of which with those of the given curves determines the points through which the shortest distance between these curves passes, and completes the solution of the problem.

We should have arrived at the same equations, by first supposing the extreme points fixed, in which case we should have the equation

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'),$$

between y and x. In fact, by reason of this relation, the integral $\int\sqrt{dx^2+dy^2}$, when taken between the abscissæ x' and x'', becomes

$$(x'-x'')\sqrt{1+\left(\frac{y'-y''}{x'-x''}\right)^2}=\sqrt{(x'-x'')^2+(y-y'')^2};$$

and the mere application of the Differential Calculus suffices to determine the *minimum* of this expression, if we take into consideration the mutual dependence which the equations of the given curves establish between x' and y', x'' and y''.

It is thus that we may arrive at the solution of problems similar to the foregoing, without the aid of the equation $\phi''-\phi'=0$ (which is not deducible by the methods of Euler and Bernoulli), as often as we are able to obtain the integral proposed; but by considering that this integral is an implicit function of the quantities relative to its limits, M. Poisson has immediately investigated the relations which an absolute *maximum* of the proposed integral requires to be established between these quantities, making use of the method of differentiation under the sign $\int$ (note, page 334). Thus he has arrived at the same equation $\phi''-\phi'=0$, which results from the method of variations.

337. The problem of the preceding No., when considered as relative to space of three dimensions, leads to the determination of z and y, in functions of x, in the expression $\int\sqrt{dx^2+dy^2+dz^2}$, which if we put $=\int ds$, we have

$$\int\delta\sqrt{dx^2+dy^2+dz^2}=\int\frac{dx}{ds}d\delta x+\int\frac{dy}{ds}d\delta y+\int\frac{dz}{ds}d\delta z=$$

$$\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}dz-\int\left(d\frac{dx}{ds}\delta x+d\frac{dy}{ds}\delta y+d\frac{dz}{ds}\delta z\right).$$

The part affected by the sign $\int$, affords the three equations,

$$d\frac{dx}{ds}=0,\quad d\frac{dy}{ds}=0,\quad d\frac{dz}{ds}=0,$$

whose combinations by 2 and 2, agree in giving

$$\frac{dz}{dx}=\text{const.},\qquad \frac{dz}{dy}=\text{const.},$$

and denote that the line required is a straight one.

If this right line be drawn between a fixed point and a curve, whose differential equation is

$$dz=p\,dx+q\,dy,$$

we must have, at the second limit $\delta z''=p\,\delta x''+q\,\delta y''$, and the first being fixed, gives $\phi'=0$. The value of $\delta z''$ therefore changes $\phi''=0$ into

$$(dx''+p''dz'')\,\delta x''+(dy''+q''dz'')\,\delta y''=0;$$

which, putting the coefficients of the independent variations equal to zero, gives

$$dx''+p''dz''=0,\qquad dy''+q''dz''=0,$$

whence we see (by No. 143), that the line required is a normal to the given curve surface.

If the whole course of the shortest line to be determined, is subjected to the condition of lying upon a given curve surface, the variations δx, δy, δz, under the sign $\int$, must satisfy the differential equation of that surface, which we shall represent by $dz=p\,dx+q\,dy$; we therefore make

$$\delta z=p\,\delta x+q\,\delta y$$

in the expression for $\int dU$, which thus becomes

$$\left(\frac{dx}{ds}+p\frac{dz}{ds}\right)\delta x+\left(\frac{dy}{ds}+q\frac{dz}{ds}\right)\delta y,$$

$$-\int\left\{\left(d\frac{dx}{ds}+p\,d\frac{dz}{ds}\right)\delta x+\left(d\frac{dy}{ds}+q\,d\frac{dz}{ds}\right)\delta y\right\}$$

From the part affected by the sign $\int$, we derive the equations

$$d\frac{dx}{ds}+p\,d\frac{dz}{ds}=0,\quad d\frac{dy}{ds}+q\,d\frac{dz}{ds}=0,$$

one of which, together with that of the given surface, is sufficient to determine the nature of the shortest line which can be drawn upon this surface between two points in it.

If this line is to be drawn from a fixed point to a curve traced on the same surface, we shall first have $\phi'=0$, and if we denote by $dy=n\,dx$, the differential equation of the projection of the proposed curve on the plane of the x and y, we shall have $\delta y''=n''\delta x''$; and the equation $\phi''=0$, which will be changed by reason of this to

$$dx''+p''dz''+(dy''+q''dz'')n''=0,$$

will denote that the two curves under consideration must cut each other at right angles.

338. We proceed to investigate the relation between x and y, which shall render the expression $\int\frac{\sqrt{dx^2+dy^2}}{\sqrt{2(y-Y)}}$, a *minimum* where Y is considered as a function of the coordinates x' and y', x'' and y'', relative to the limits.*

To resolve this question in all its generality, we must make Y vary as well as y (335). Suppose

* This is the problem of the Brachystochrone, or the curve along which a body will descend in the least possible time from one point to another.

$$\sqrt{2(y-Y)}=u, \qquad \sqrt{dx^2+dy^2}=ds,$$

and we shall have

$$\delta u=\frac{\delta y-\delta Y}{u}, \qquad \delta s=\frac{dx}{ds}\,d\,\delta x+\frac{dy}{ds}\,d\,\delta y,$$

$$[illegible]\frac{s}{\ }=-\int\frac{ds}{u^3}(\delta y-\delta Y)+\int\frac{dx}{uds}\,d\,\delta x+\int\frac{dy}{uds}\,d\,\delta y$$

$$[illegible]\,Y\int\frac{ds}{u^3}+\frac{dx}{[illegible]}\,\delta x+\frac{dy}{uds}\,\delta y$$

$$[illegible]\frac{ds}{u^3}+d\frac{dy}{uds}\Big)\,\delta y\Big\}$$

Now [illegible] ms [illegible] with the sign $\int$, we derive

$$d\frac{[illegible]}{uds}=0, \qquad \frac{ds}{u^3}+d\frac{dy}{uds}=0;$$

the first of these, which is [illegible] most simple, gives

$$\frac{dx}{uds}=C, \text{ whence } \frac{dx}{\sqrt{dx^2+dy^2}}=C\sqrt{2(y-Y)}.$$

This result denotes a cycloid (102); for if we make $y-Y=z$, we shall obtain

$$dx=\frac{dz\sqrt{2C^2}.\sqrt{z}}{\sqrt{1-2C^2z}}=\frac{z\,dz}{\sqrt{\frac{1}{2C^2}z-z^2}}.$$

When $\delta Y=0$, the quantity ϕ gives, for the limits, the equation

$$dx''\,\delta x''+dy''\,\delta y''=0, \qquad dx'\,\delta x'+dy'\,\delta y'=0,$$

from which we conclude, as in No. 336, that if the curve required be drawn between two others, it must meet them at right angles.

When δY is not nothing, we must calculate the value

of $\int \frac{ds}{u^3}$, between the limits of the proposed integral.

Now the equation

$$\frac{ds}{u^3} + d\,\frac{dy}{u\,ds} = 0,$$

furnished by the coefficient of δy under the sign $\int$, gives

$$\int \frac{ds}{u^3} = -\,\frac{dy}{u\,ds} + \text{const.}$$

and if we take notice that δY, since it does not depend on the indeterminate variables x and y, must have the same value at both limits, the equation $\phi'' - \phi' = 0$ becomes

$$\left.\begin{array}{l} -\,\dfrac{dy''}{u''ds''}\,\delta Y + \dfrac{dx''}{u''ds''}\,\delta x'' + \dfrac{dy''}{u''ds''}\,\delta y'' \\ +\,\dfrac{dy'}{u'ds'}\,\delta Y - \dfrac{dx'}{u'\,ds'}\,d\,x' - \dfrac{dy'}{u'\,ds'}\,\delta y' \end{array}\right\} = 0.$$

If we take $Y = y'$, which gives $\delta Y = \delta y'$, we have, after reducing and separating the variations relative to each limit,

$$\frac{dx''}{u''ds''}\,\delta x'' + \frac{dy''}{u'ds''}\,\delta y'' = 0, \qquad \frac{dx'}{u'ds'}\,\delta x' + \frac{dy''}{u''ds''}\,\delta y' = 0;$$

and, if we then make, as in No. 336,

$$\delta y'' = n''\,\delta x'', \qquad \delta y' = m'\,\delta x',$$

and call to mind that $\frac{dx}{u\,ds} = C$, the equations above will take the form

$$C + \frac{dy''}{u''ds''}\,n'' = 0, \qquad C + \frac{dy''}{u''ds''}\,m' = 0,$$

from which it follows, that $n'' = m'$. This result shews, that at the points where the required curve cuts the given ones, these latter must have their tangents parallel. More-

over the equation relative to this last limit comes to the same as

$$d x'' \delta x'' + d y'' \delta y'' = 0,$$

and therefore proves, that the required curve must cut the second given curve at right angles.

339. The preceding problems relate to absolute *maxima* and *minima*; but the question *To find among all possible relations between* x and y, *which give the same value of the indeterminate integral* $\int U_1$, *taken from* $x = x'$ to $x = x''$, *that which renders the expression* $\int U$ *a* maximum *or* minimum *under the same circumstances*, belongs to the class of *relative maxima* and *minima*. It may be resolved by making the variation of the function $\int U + a \int U_1$ equal to zero, a being a constant but indeterminate coefficient. This is not the place to demonstrate this rule in detail; we may however easily perceive that if the above function be a *maximum* or *minimum*, and we suppose $\int U_1 = A$, the integral $\int U$ will always have the greatest or least value which it is susceptible of on this hypothesis. The indeterminate coefficient a serves to supply the condition $\int U_1 = A$.

If for instance the curve were required which, under a given perimeter, shall include the greatest or least space, we should have

$$\int U + a \int U_1 = \int \{ y\, d x + a \sqrt{d x^2 + d y^2} \} :$$

and if we put $\sqrt{d x^2 + d y^2} = d s$, the part of the variation affected by the sign $\int$ would be

$$- \int' \left\{ \left(d y + a d \frac{d x}{d s} \right) \delta x - \left(d x - a d \frac{d y}{d s} \right) \delta y \right\},$$

and would give, for the determination of the curve required

$$d x - a d \frac{d y}{d s} = 0,$$

whose integral

$$x - a\frac{dy}{ds} = C, \text{ or } dy = \frac{(x - C)\,dx}{\sqrt{a^2 - (x - C)^2}}$$

evidently denotes the circle whose radius is a.

This radius is to be determined from the value assigned to the perimeter $\int \sqrt{dx^2 + dy^2}$; and the constant C, and that which would be introduced by the integration we have left unperformed, will suffice to make the circle pass through the fixed limits. Its area is a *maximum* or *minimum*, according as it turns its concavity or its convexity towards the axis of the abscissæ. Such is the simplest case of the "*Isoperimetrical Problems*" so called, because at first only curves of the same length were considered *.

---

* See Note (Q).

APPENDIX.

On Differences and Series.

340. A SERIES being a regular progression of terms increasing or diminishing in magnitude according to a certain law, it follows, that when that law is given, and the place of any term in the series assigned, its magnitude may be determined; and thus the successive terms of the series may be produced in order. To assign the place of any term in a series, it is sufficient to indicate the number of terms by which it is removed from some one which we regard as fixed. This number is usually called the *index* of the term to which it corresponds. For example, in the series

$$0,\ 1,\ 4,\ 9,\ 16\ \ldots\ldots\ldots\ x^2,$$

if we assume the first term as our point of departure, we shall have for the corresponding series of indices

$$0,\ 1,\ 2,\ 3,\ 4\ \ldots\ldots\ldots\ x.$$

If the series be continued backwards, the indices must be considered as negative; thus in the series

$$\ldots -x^3, \ldots -1,\ 0,\ 1,\ 8,\ 27, \ldots\ldots x^3, \ldots$$

$$\ldots -x,\ \ldots -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\ldots\ x, \ldots$$

the numbers in the lower line represent the indices of the terms immediately above them.

341. Since a regular progression or uniform law is essentially included in our notion of a series, it follows that the magnitude of every term is determined solely by its index, and by the law which the series observes; in other words, that any term is a certain function of its corresponding index, the form of which does not change in passing from one term to another, but remains the same throughout the whole extent of the series. Thus, in the former of the above series, each term is the square, in the latter, the cube of its index. This function analytically expressed, is called the *general term*; and it is evident, that by substituting successively in its expression, instead of the symbol (x), which denotes the index the progression of natural numbers,

$$\ldots\ldots -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ldots\ldots$$

the terms of the series will be produced in this order: the general term may therefore be considered as characterizing the series, and as including all its properties, and our reasonings respecting series will, in consequence, be wholly confined to their general terms.

The general terms of an arithmetic and geometric progression are respectively $a+bx$ and a^x, for when each of the above progression of numbers is successively written for x, in these expressions we obtain the series

$$\ldots\ldots a-2b,\ a-b,\ a,\ a+b,\ a+2b,\ \ldots\ldots$$

$$\ldots\ldots \frac{1}{a^2},\ \frac{1}{a},\ 1,\ a,\ a^2,\ \ldots\ldots$$

In general, when we speak of a series in future, we shall consider it as continued indefinitely both ways, unless the contrary is expressly mentioned. This is essential to all reasonings in which only the general term is employed, and is precisely analogous to the expression of a curve by its equation, in which no limitation of its extent is ad-

mitted. The *sum* of a series is the aggregate of all its terms connected by the sign +, which the word *series*, alone is frequently used to signify.

342. The general term of any series is usually denoted by subscribing its index to a certain letter, (called a characteristic), towards the right-hand, thus, u_x, A_x; or affixing it in any other part, as ${}^xu$, according to the necessity of the case, just as in the foregoing pages, $f(x)$ and $\phi(x)$ have been used to denote functions of x. Convenience of writing where the same letters frequently occur, arising from the rejection of superfluous parentheses, is the only reason for the distinction.

u_x then being any function whatever of x, if for x we write successively the progression

$$\ldots -2,\ -1,\ 0,\ 1,\ 2, \ldots\ldots x,\ x+1, \ldots$$

we shall produce the series

$$\ldots u_{-2},\ u_{-1},\ u_{-0},\ u_1,\ u_2, \ldots\ldots u_x,\ u_{x+1}, \ldots$$

of which it is the general term.

The excess of the term u_{x+1} over that which immediately precedes it (u_x), or the function $u_{x+1}-u_x$, is called *the difference of the function* u_x, and is denoted by the sign Δ, thus

$$u_{x+1}-u_x=\Delta u_x.$$

Now it is obvious that $x_{x+1}-u_x$ is a certain function of x, the nature of which is entirely dependent on that of the original function u_x, from which it is derived: Δu_x is therefore a certain function of x, derived in this particular manner from u_x, and as the latter has u, the former must be considered as having Δu for its characteristic. The difference of the function Δu_x (or the *second difference* of u_x) is therefore $\Delta u_{x+1}-\Delta u_x$. This second difference, or $\Delta(\Delta u_x)$, by an abbreviation of the same kind as that used in the Differential Calculus, is written $\Delta^2 u_x$, and thus we have

$$\Delta^2 u_x = \Delta u_{x+1} - \Delta u_x.$$

In like manner

$$\Delta^3 u_x = \Delta^2 u_{x+1} - \Delta^2 u_x,$$

....

$$\Delta^n u_x = \Delta^{n-1} u_{x+1} - \Delta^{n-1} u_x.$$

343. When the particular form of the function u_x is assigned, nothing is easier than to deduce the values of the successive differences Δu_x, $\Delta^2 u_x$, &c. in terms of x. It is only *to write* x + 1 *for* x *in the expression of the function, and to subtract from the result the original expression: the remainder is the first difference of the proposed function, and the others may be obtained in like manner.*

If for example we had $u_x = a + bx$, we should obtain

$$\Delta u_x = \left\{a + b(x+1)\right\} - (a + bx) = b; \quad \Delta^2 u_x = b - b = 0.$$

More generally, if $u_x = a + b \cdot v_x$, where v_x is any function of x, we should find

$$\Delta u_x = (a + b \cdot v_{x+1}) - (a + b \cdot v_x) = b\,(v_{x+1} - v_x) = b \cdot \Delta v_x.$$

Suppose again $u_x = a^x$. In this case

$$\Delta u_x = a^{x+1} - a^x = a^x\,(a-1)$$

$$\Delta^2 u_x = a^{x+1}(a-1) - a^x\,(a-1) = a^x\,(a-1)^2$$

and so on to

$$\Delta^n u_x = a^x (a-1)^n.$$

The following results will be useful to us in our future enquiries, and will serve at present as exercises for the reader. Supposing then u_x and v_x to represent any functions whatever of x, we have

$$\Delta\,(u_x \cdot v_x) = u_{x+1} \cdot v_{x+1} - u_x \cdot v_x = (u_x + \Delta u_x)\,(v_x + \Delta v_x)$$

$$= u_x \cdot \Delta v_x + v_x \cdot \Delta u_x + \Delta u_x \cdot \Delta v_x$$

$$= u_x \cdot \Delta v_x + v_{x+1} \cdot \Delta u_x$$

$$\Delta\,\frac{u_x}{v_x} = \frac{u_{x+1}}{v_{x+1}} - \frac{u_x}{v_x} = \frac{v_x \cdot \Delta u_x - u_x \cdot \Delta v_x}{v_x \cdot v_{x+1}},$$

by reducing to a common denominator, and substituting

for u_{x+1} and v_{x+1}, in the numerator, their values, $u_x + \Delta u_x$ and $v_x + \Delta v_x$.

Again,

$$\Delta \left\{ u_x . u_{x+1} . u_{x+2} \ldots u_{x+n} \right\} = u_{x+1} \ldots u_{x+n+1} - u_x \ldots u_{x+n} =$$
$$= u_{x+1} \ldots \ldots u_{x+n} . (u_{x+n+1} - u_x).$$

If u_x be the general term of an arithmetical progression, or if $u_x = a + bx$, the factor $u_{x+n+1} - u_x$ becomes $(n+1)b$, which being constant, it appears that the difference of a factorial function of this kind is of the same form with the function itself, only having one factor less; in like manner

$$\Delta \frac{1}{u_x \ldots \ldots u_{x+n}} = - \frac{u_{x+n+1} - u_x}{u_x \ldots \ldots u_{x+n+1}},$$

to which result the same observation may be applied, only that the number of factors in the denominator of the difference is greater by an unit than in the function proposed. Again, we have

$$\Delta \frac{u_x \ldots u_{x+n}}{v_x \ldots v_{x+m}} a^x = \frac{u_{x+1} \ldots u_{x+n}}{v_x \ldots v_{x+m+1}} a^x \left\{ a v_x . u_{x+n+1} - u_x . v_{x+m+1} \right\}.$$

344. The successive differences of any rational integral function of x are easily obtained, and, as they lead to certain very remarkable results, we shall proceed to investigate them. To begin with a simple case, let us take $u_x = x^2$, whence we get $\Delta u_x = (x+1)^2 - x^2 = 2x + 1$, and $\Delta^2 u_x = 2(x+1) + 1 - (2x+1) = 2$, $\Delta^3 u_x = 0$, &c. If we suppose $u_x = x^3$, we have in like manner,

$$\Delta . x^3 = (x+1)^3 - x^3 = 3x^2 + 3x + 1$$
$$\Delta^2 . x^3 = 3 . \Delta (x^2) + 3 . \Delta x = 6x + 6,$$

and $\quad \Delta^3 . x^3 = 6, \quad \Delta^4 . x^3 = 0,$ &c. and so on.

The general expression of any rational integral function being

$$u_x = Ax^n + Bx^{n-1} + \ldots Ix^2 + Kx + L,$$

its first difference will be

$$\Delta u_x = A\{(x+1)^n - x^n\} + B\{(x+1)^{n-1} - x^{n-1}\} + \ldots + K$$
$$= n\,A\,.x^{n-1} + B'.x^{n-2} + \ldots + I'.x + K',$$

where $B', \ldots I', K'$, are certain constant coefficients.

In like manner, since *this is a rational integral function, one degree lower than the original function,* we shall have for the difference of this, or the second difference of u_x,

$$\Delta^2 u_x = n\,(n-1)\,A\,.\,x^{n-2} + B''x^{n-3} + \ldots I'',$$

and the nth will be

$$\Delta^n u_x = n\,(n-1)\ldots 2\,.\,1\,.\,A,$$

so that in general, *the* n*th difference of a rational integral function of the* n*th degree is constant, and of course all the higher orders of differences vanish.*

In order to see more clearly the meaning of this result, suppose $u_x = x^3 + 2\,x + 3$, and putting 0, 1, 2, 3, &c. successively for x, we form the series u_0, u_1, &c. Subtracting then each from that which follows it, we obtain the series of values Δu_0, Δu_1, &c. and from this in like manner, the series $\Delta^2 u_0$, $\Delta^2 u_1$, &c. is derived, as follows

$$u_0 = 3,\ u_1 = 6,\ u_2 = 15,\ u_3 = 36,\ u_4 = 75,\ \&c.$$
$$\Delta u_0 = 3,\ \Delta u_1 = 9,\ \Delta u_2 = 21,\ \Delta u_3 = 39,\ \&c.$$
$$\Delta^2 u_0 = 6,\ \Delta^2 u_1 = 12,\ \Delta^2 u_2 = 18,\ \&c.$$
$$\Delta^3 u_0 = 6,\ \Delta^3 u_1 = 6,\ \&c.$$
$$\Delta^4 u_0 = 0,\ \&c.$$

where we see that the terms of the series composing the third order of differences are all equal to each other, and to 1 . 2 . 3, the degree of the proposed function having 3 for its exponent.

345. If in the equation $\Delta^2 u_x = \Delta u_{x+1} - \Delta u_x$ (342), we substitute for Δu_x and Δu_{x+1}, their values, $u_{x+1} - u_x$, and $u_{x+2} - u_{x+1}$, we find

$$\Delta^2 u_x = u_{x+2} - 2\,u_{x+1} + u_x.$$

If in this equation we substitute Δu_x for x, we get

$$\Delta^3 u_x = \Delta u_{x+2} - 2\,\Delta u_{x+1} + \Delta u_x,$$

and writing for

Δu_{x+2}, Δu_{x+1}, Δu_x, their values, $u_{x+3} - u_{x+2}$, &c.

we obtain

$$\Delta^3 u_x = u_{x+3} - 3\,u_{x+2} + 3\,u_{x+1} - u_x.$$

A little attention to the steps of the above process will convince us, that the manner in which the coefficients in the expressions for these successive differences are produced from each other, is precisely the same as in the operation of raising the binomial $u-1$ to its successive powers by multiplication, and that we ought therefore to have

$$\Delta^n u_x = u_{x+n} - \frac{n}{1}\,u_{x+n-1} + \frac{n(n-1)}{1\,.\,2}\,u_{x+n-2} - \&c.$$

which enables us to express the nth difference of any function in terms of its successive values. Conversely, if we would express any term in the progression of these values in terms of the function, and its successive differences, we have

$$u_{x+1} = u_x + \Delta u_x.$$

In this equation for x write $x+1$, and we get

$$u_{x+2} = u_{x+1} + \Delta u_{x+1},$$

whence, by substituting for u_{x+1} its value, $u_x + \Delta u_x$, and for Δu_{x+1}, its equal $\Delta u_x + \Delta^2 u_x$ obtained from this, we find

$$u_{x+2} = u_x + 2\,\Delta u_x + \Delta^2 u_x:$$

again, writing $x+1$ for x, and proceeding as before

$$u_{x+3} = u_x + 3\,\Delta u_x + 3\,\Delta^2 u_x + \Delta^3 u_x,$$

and we conclude by the analogy which subsists between the production of the constant coefficients in these expressions, and in those of $(1+\Delta)^2$, $(1+\Delta)^3$, &c. when developed by

actual multiplication, in powers of the symbol Δ, that in general

$$u_{x+n} = u_x + \frac{n}{1} \Delta u_x + \frac{n(n-1)}{1 \, . \, 2} \Delta^2 u_x + \&c.$$

346. As the above demonstration of these two theorems, founded on something like an inductive process, may not appear satisfactory, we shall now present the reader with one which has the advantage of setting in the clearest light the analogy above-mentioned. It will also afford us an opportunity of introducing to the notice of the English student, the principles of the "Calculus of Generating Functions."

Suppose $\phi(t)$ to be a function of t, susceptible of developement, as follows:

$$\phi(t) = \ldots + u_{-1} t^{-1} + u_0 + u_1 t + \ldots u_{x-1} t^{x-1} + u_x t^x + u_{x+1} t^{x+1} + \ldots (a).$$

in which it is evident, that u_x may represent any function of x whatever, by considering this equation as the *definition* of $\phi(t)$. The developement of $\phi(t)$ therefore must produce, or *generate*, the coefficients $\ldots u_0, \ldots u_x$, annexed to their proper powers of t, which may be looked upon as a mere instrument for the purpose of keeping them distinct from each other, and presenting them to the eye in their order. This function is therefore called the generating function of u_x. For instance, $-\log(1-t)$ is the generating function of $\frac{1}{x}$, and $\frac{t}{(1-t)^2}$ of x, for the developements of these functions are respectively

$$\frac{t}{1} + \frac{t^2}{2} + \ldots \frac{t^x}{x} + \&c.$$

$$1 \, . \, t + 2 \, . \, t^2 + \ldots x \, . \, t^x + \&c.$$

If we multiply both sides of the equation (a) by t, we find

$$t\,\phi(t) = \ldots + u_{-1} + u_0 t + u_1 t^2 + \ldots u_{x-1} t^x + \&c.$$

where the coefficient of t^x being u_{x-1}, it appears that the generating function of u_{x-1} is $t.\phi(t)$, and in like manner that of u_{x-n} is $t^n.\phi(t)$. Again

$$\frac{1}{t}\phi(t) = \ldots u_{-1}t^{-2} + u_0 t^{-1} + u_1 + \ldots u_{x+1}t^x + \&c.$$

which shews that $\frac{1}{t}.\phi(t)$ is the generating function of u_{x+1}, and in the same manner we find $\left(\frac{1}{t}\right)^n.\phi(t)$ for that of u_{x+n}.

Hence it follows, that the generating function of Δu_x, or of $u_{x+1} - u_x$ will be $\left(\frac{1}{t} - 1\right).\phi(t)$ for this function, being developed, will produce the difference of two series, whose general terms are respectively $u_{x+1}.t^x$, and $u_x t^x$. To find then the generating function of Δu_x we have only to multiply that of u_x by $\frac{1}{t} - 1$. Now Δu_x itself may be considered as a new function of x, whose generating function is $\left(\frac{1}{t} - 1\right).\phi(t)$, and therefore that of *its* difference, or of $\Delta^2 u_x$, will be

$$\left(\frac{1}{t} - 1\right)^2.\phi(t),$$

and in like manner, that of $\Delta^n u_x$ will be

$$\left(\frac{1}{t} - 1\right)^n.\phi(t),$$

that is

$$\left(\frac{1}{t}\right)^n.\phi(t) - \frac{n}{1}\left(\frac{1}{t}\right)^{n-1}.\phi(t) + \frac{n(n-1)}{1.2}\left(\frac{1}{t}\right)^{n-2}.\phi(t) - \&c.$$

Now it is evident that the sum of the generating functions of any number of functions, such as u_{x+n}, u_{x+n-1}, &c. (or any others) connected by any constant coefficients, is

the same with the generating functions of their sum, when affected respectively with the same coefficients. But we have seen that $\left(\frac{1}{t}\right)^n . \phi(t)$ is the generating function of u_{x+n}, $\left(\frac{1}{t}\right)^{n-1} . \phi(t)$ of u_{x+n-1}, and so on. The above expression is therefore the generating function of

$$u_{x+n} - \frac{n}{1} u_{x+n-1} + \&c.$$

as well as of $\Delta^n u_x$, whence it follows that

$$\Delta^n u_x = u_{x+n} - \frac{n}{1} u_{x+n-1} + \&c.$$

for the generating functions of both being the same, the coefficients of t^x in their developements must be identical, however these developements may have been performed.

347. The generating function of u_{x+n} being $\left(\frac{1}{t}\right)^n . \phi(t)$, or $\left\{1 + \left(\frac{1}{t} - 1\right)\right\}^n . \phi(t)$, if we developе $\left\{1 + \left(\frac{1}{t} - 1\right)\right\}^n$ in powers of $\left(\frac{1}{t} - 1\right)$ it will take the form

$$\phi(t) + \frac{n}{1}\left(\frac{1}{t} - 1\right) . \phi(t) + \frac{n(n-1)}{1.2}\left(\frac{1}{t} - 1\right)^2 . \phi(t) + \&c.$$

which is also the generating function of

$$u_x + \frac{n}{1} \Delta u_x + \frac{n(n-1)}{1.2} \Delta^2 u_x + \&c.; \quad (346)$$

and hence it appears that this latter expression is equal to u_{x+n}.

348. It will easily be seen that the extent of this method is not confined to the cases here presented. It would surpass the limits of an Elementary Essay like the present to enter much farther into the subject. Our intention has been rather to excite the curiosity of the intelligent reader respecting one of the most refined inven-

tions of analytical genius, than to satisfy it by a detail suited to its importance. We can only accompany him one step farther, after which we must refer him to the "*Theorie Analytique des Probabilites*" of Laplace, a work indispensably necessary to all who would thoroughly understand the nature of differences and series, or form an elegant taste in analytical composition *.

Instead of considering the function $u_{x+1} - u_x$, we might, with greater generality have supposed Δu_x to represent any other combination of the successive values u_x, u_{x+1}, u_{x+2}, &c., of the first degree. For example, if we had originally defined Δu_x to mean $au_x + b u_{x+1} + cu_{x+2}$ where a, b, c, are any constant coefficients; we should have

$$\Delta^2 u_x = a \Delta u_x + b \Delta u_{x+1} + c \Delta u_{x+2}$$
$$\Delta^3 u_x = a \Delta^2 u_x + b \Delta^2 u_{x+1} + c \Delta^2 u_{x+2},$$
$$\&c. = \&c.$$

and the expression for $\Delta^n u_x$ in terms of u_x, u_{x+1}, &c. may be obtained as follows:

The generating function of Δu_x is evidently in this case (346),

$$\left(a + \frac{b}{t} + \frac{c}{t^2}\right) \cdot \phi(t),$$

and if we regard this as the generating function of a new function Δu_x that of $\Delta \Delta u_x$ or $\Delta^2 u_x$ will be found by multiplying it again by the factor $\left(a + \frac{b}{t} + \frac{c}{t^2}\right)$ which produces

* The reader is referred also to the *Mecanique Celeste*, tom. IV. to the *Journal de l'Ecole Polytechnique*, No. 15. Also to the *Memoires de l'Acad. des Sciences*, 1779. In our own language we are not aware that any mention of the subject is to be found, if we except a paper by the Author of this Essay, in the Transactions of the Royal Society, 1815, where the *English* reader will find some further developement of it.

$$\left(a + \frac{b}{t} + \frac{c}{t^2}\right)^2 \cdot \phi(t),$$

and, in like manner the generating function $\Delta^n u_x$ will be

$$\left(a + \frac{b}{t} + \frac{c}{t^2}\right)^n \cdot \phi(t).$$

If the factor $\left(a + \frac{b}{t} + \frac{c}{t^2}\right)^n$ be developed in powers of t, by the usual methods, it will produce an expression

$$A_0 + A_1 \cdot \left(\frac{1}{t}\right) + A_2 \cdot \left(\frac{1}{t}\right)^2 + \ldots\ldots A_{2n} \cdot \left(\frac{1}{t}\right)^{2n}$$

where A_0, A_1, ... A_{2n} are certain known functions of n, a, b, c, and we therefore have

$$A_0 \cdot \phi(t) + A_1 \cdot \left(\frac{1}{t}\right) \cdot \phi(t) + \ldots\ldots A_{2n} \cdot \left(\frac{1}{t}\right)^{2n} \cdot \phi(t)$$

for the generating function of $\Delta^n u_x$, whence we see that

$$\Delta^n u_x = A_0 u_x + A_1 u_{x+1} + \ldots\ldots A_{2n} u_{x+2n}$$

and the same reasoning may be applied when

$$\Delta u_x = a u_x + b u_{x+1} + \ldots\ldots k u_{x+m}.$$

349. Instead of developing $\left(a + \frac{b}{t} + \frac{c}{t^2}\right)^n$ in powers of $\frac{1}{t}$, we might have chosen any other function of t, or any other method of developement. Suppose for instance we had taken $z = \frac{1}{t} + 1$ for the function of t according to whose powers we would have the developement performed. This gives $\frac{1}{t} = z - 1$, and,

$$\left(a + \frac{b}{t} + \frac{c}{t^2}\right)^n = \left\{(a - b + c) + (b - 2c)z + cz^2\right\}^n$$

whose developement we may represent by

$$B_0 + B_1 z + \ldots\ldots B_{2n} z^{2n}.$$

Now, when we affix $\phi(t)$ to this, we must consider first that $z . \phi(t)$ or $\left(\frac{1}{t}+1\right) . \phi(t)$ is the generating function of $u_{x+1}+u_x$: as we have used Δu_x to denote $a u_x + b u_{x+1} + c u_{x+2}$, it will be necessary to denote $u_{x+1}+u_x$ by some other symbol, and as it is indifferent what we employ, we will suppose

$$u_{x+1}+u_x = \nabla u_x$$
$$\nabla u_{x+1} + \nabla u_x = \nabla^2 u_x, \text{ \&c.}$$

thus we have $z . \phi(t)$ for the generating function of ∇u_x, $z^2 . \phi(t)$ for that of $\nabla^2 u_x$, &c. (348), whence it is easy to conclude that

$$\Delta^n u_x = B_0 . u_x + B_1 \nabla u_x + B_2 \nabla^2 u_x + \ldots B_{2n} \nabla^{2n} u_x,$$

and in like manner an indefinite number of expressions for the same function may obtained, differing in form according to the different manner of developing $\left(a+\frac{b}{t}+\frac{c}{t^2}\right)^n$.

The reason of the analogy observed in (345) between the process of substitution and the elevation of a binomial to its powers is now sufficiently evident, and we will therefore dwell upon the subject no longer.

350. Returning to our original definition of Δu_x, in the equation

$$\Delta^n u_x = u_{x+n} - \frac{n}{1} u_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \text{\&c.}$$

if we assign particular values to u_x we shall obtain very readily the expressions for the nth differences of the functions so given. Let us take for instance $u_x = x^m$, and we get

$$\Delta^n . x^m = (x+n)^m - \frac{n}{1}(x+n-1)^m + \text{\&c.*}$$

* The full point after the Δ in $\Delta^n . x^m$, serves to distinguish $\Delta^n (x^m)$ from $(\Delta^n x)^m$ with which it might be confounded, and

We have already seen that when $n = m$, this becomes $1 . 2 . 3 ... m$, (344), hence we have

$$(n + x)^n - \frac{n}{1}(n + x - 1)^n + \&c. = 1 . 2 ... n$$

whatever be the value of x; or supposing $n + x = k$

$$k^n - \frac{n}{1}(k - 1)^n + \frac{n(n-1)}{1 . 2}(k - 2)^n - \&c. = 1 . 2 ... n.$$

If we suppose $x = 0$, and denote by $\Delta^n . o^m$ the particular value which $\Delta^n . x^m$ has in that case, we find

$$\Delta^n o^m = n^m - \frac{n}{1}(n-1)^m + \frac{n(n-1)}{1 . 2}(n-2)^m - \&c.$$

when n is greater than m this expression is therefore constantly equal to zero, and when $n = m$, we see that

$$n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1 . 2}(n-2)^n - \&c. = 1 . 2 ... n.$$

These are the singular results we alluded to in our investigation of the successive differences of any rational integral function, (344): the numbers comprehended under the form $\Delta^n o^m$ possess a vast variety of very curious properties, and are intimately connected with many of the most interesting enquiries in the pure mathematics *.

351. The remarkable form of the expressions for $\Delta^n u_x$ and u_{x+n} deduced in (345) affords us an opportunity

---

and which, as in the Differential Calculus, is occasionally (but inelegantly) written thus $\Delta^n x^m$, without the point. Thus $\Delta^n . u^m{}_x = \Delta^n (u^m{}_x)$.

* The reader who is curious upon this subject, will find much satisfaction in the perusal of a paper by Dr. Brinkley, in the Phil. Trans. 1807, where they are employed with great effect. In a paper also, by the Author of this Essay, in Phil. Trans. 1816, i. a great number of their properties (some of them of a very singular nature), is demonstrated.

to develope the principles of a method of notation which seems to unite in the most perfect manner the properties of conciseness, simplicity and elegance, and appears peculiarly well adapted to open new and enlarged views of the extent and meaning of analytical operations.

If the expression $(1 + \Delta)^n$, regarded as a function of a certain symbol Δ, be developed in powers of Δ, it will produce the series

$$1 + \frac{n}{1}\Delta + \frac{n(n-1)}{1.2}\Delta^2 + \&c.$$

It is of no consequence to our present purpose in what light we regard that symbol Δ, whether as a quantity, or merely as an instrument by means of which, and by a process purely mechanical, we are enabled to produce the numerical coefficients of the series affected with their proper powers of the same Δ. In this point of view, the expression $(1 + \Delta)^n$ must be considered as having no other meaning than as an abbreviated expression for its developement, and when prefixed to the function u_x, each term of this developement is understood to be applied, separately, to the same function, so that the following expressions

$$(1+\Delta)^n u_x$$

$$\left\{1 + \frac{n}{1}\Delta + \frac{n(n-1)}{1.2}\Delta^2 + \&c.\right\} u_x$$

$$u_x + \frac{n}{1}\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \&c.$$

may be indiscriminately used for one another, the two former being regarded as having no meaning, but as abbreviations of the latter. In general, if $f(\Delta)$ be a function of Δ developable in a series of powers of Δ, such as

$$A.\Delta^\alpha + B.\Delta^\beta + \&c.$$

then the expression $f(\Delta)u_x$ is used as an abbreviation of

$$A.\Delta^\alpha u_x + B.\Delta^\beta u_x + \&c.$$

and the same notation is applicable to other characteristic letters, such as d, f, δ, &c.

It follows from this, that the successive performance of two or more series of operations, represented by $f(\Delta)$, $f'(\Delta)$, &c. upon the same function u_x, is equivalent to the performance of that series of operations which is represented by their product. Suppose, for instance, $f(\Delta) = 1+\Delta$, and $f'(\Delta) = \Delta - \Delta^3$; then

$$f(\Delta)u_x = u_x + \Delta u_x$$

and $f'(\Delta)\cdot\left\{f(\Delta)u_x\right\} = \Delta\left\{u_x + \Delta u_x\right\} - \Delta^3\left\{u_x + \Delta u_x\right\}$

$$= \Delta u_x + \Delta^2 u_x - \Delta^3 u_x - \Delta^4 u_x,$$

which is also the expanded expression for

$$\left\{f(\Delta) \times f'(\Delta)\right\} u_x.$$

The reason of this is evident, since $f(\Delta) \times f'(\Delta) = (\Delta - \Delta^3)(1+\Delta) = \Delta(1+\Delta) - \Delta^3(1+\Delta)$, to which, if we affix u_x, the same terms are produced in the same order, and the same holds good in general. This observation is the ground-work of the whole system, and should be carefully attended to by the reader.

352. We have then

$$u_{x+n} = (1+\Delta)^n u_x,$$

and it will not be amiss to shew how the other equation, expressing the value of $\Delta^n u_x$ in terms of u_x, u_{x+1}, &c. may be deduced from this. For this purpose we have only to consider, that Δ^n and $\left\{(1+\Delta)-1\right\}^n$ are identically the same, and the latter of these is also identical with

$$(1+\Delta)^n - \frac{n}{1}(1+\Delta)^{n-1} + \frac{n(n-1)}{1\,.\,2}(1+\Delta)^{n-2} - \text{\&c.}$$

so that this expression, *if developed in powers of* Δ *by a mere mechanical process*, would produce simply Δ^n. Affixing now the function u_x, it appears that

$$\Delta^n u_x = (1+\Delta)^n u_x - \frac{n}{1}(1+\Delta)^{n-1} u_x + \&c.$$

but $(1+\Delta)^n u_x$ has no other meaning than the series

$$u_x + \frac{n}{1}\Delta u_x + \&c.$$

or u_{x+n}, and so on for the rest; consequently

$$\Delta^n u_x = u_{x+n} - \frac{n}{1} u_{x+n-1} + \&c.$$

353. This process depends on the principle, that it is the same thing whether we affix the u_x to the sum of any number of functions of Δ, or take the sum of the results produced by affixing it to each separate function, which is obviously the case. For instance, suppose

$$\left.\begin{array}{l} F(\Delta) = A + B\Delta + C\Delta^2 + \&c. \\ f(\Delta) = a + b\Delta + c\,\Delta^2 + \&c. \end{array}\right\}$$

then we shall have $\left\{ f(\Delta) + F(\Delta) \right\} u_x =$

$$= \left\{ (A+a) - (B+b)\Delta + \&c. \right\} u_x$$

$$= (A+a) u_x + (B+b) \,.\, \Delta u_x + \&c.$$

which is the sum of the two series

$$F(\Delta) u_x = A u_x + B \Delta u_x + \&c.$$

$$f(\Delta) u_x = a u_x + b \,\Delta u_x + \&c.$$

354. We proceed to the investigation of other relations. We have seen that the difference of the product $u_x \,.\, v_x$ of two functions u_x and v_x, is represented by

$$\Delta \,.\, u_x v_x = u_x \Delta v_x + v_x \Delta u_x + \Delta u_x \Delta v_x. \qquad (a)$$

In order to find the expression for the nth difference of the same product, we may proceed as follows:

$$\Delta^2 \,.\, u_x v_x = \Delta\,(u_x \Delta v_x) + \Delta\,(v_x \Delta u_x) + \Delta\,(\Delta u_x \Delta v_x),$$

each term of which being expanded by the help of the equation (a), as follows,

$$\Delta\,(u_x\Delta v_x)=u_x\Delta^2 v_x+\Delta v_x\Delta u_x+\Delta u_x\Delta^2 v_x,\ \&c.$$

we find

$$\Delta^2 .\, u_x v_x = u_x\Delta^2 v_x+v_x\Delta^2 u_x+2\,\Delta v_x\Delta u_x+\Delta^2 v_x\Delta^2 u_x+$$
$$2\,\Delta v_x\Delta^2 u_x+2\,\Delta u_x\Delta^2 v_x,$$

and so on; but the process soon becomes troublesome from the vast multitude of terms, the law of whose coefficients is not immediately visible. The system of notation above explained, *by separating the symbols of operation from those of quantity*, will here afford us a remarkable assistance.

We first observe, that if we suppose an accent applied to the letter Δ to indicate that it is to be referred solely to v_x, while the unaccented Δ refers to u_x only, we shall then have

$$\Delta^m\Delta'^n u_x v_x=\Delta^m u_x\times\Delta^n v_x,$$

the accent over the Δ being understood to make no alteration whatever in the nature of the operation denoted by that letter; but simply to indicate, that in the developement of the ultimate result, the powers of the accented Δ are to be placed immediately before the v_x, and those of the unaccented before u_x. Thus we have

$$\Delta(u_x v_x)=u_x\Delta v_x+v_x\Delta u_x+\Delta u_x\Delta v_x=(\Delta+\Delta'+\Delta\,\Delta')u_x v_x.$$

Now the difference of this is equal to the sum of those of its separate terms, that is

$$\begin{aligned}\Delta^2\,(u_x v_x)=&(\Delta+\Delta'+\Delta\,\Delta')\,v_x\,\Delta u_x\\ &+(\Delta+\Delta'+\Delta\,\Delta')\,u_x\Delta v_x\\ &+(\Delta+\Delta'+\Delta\,\Delta')\,\Delta u_x\Delta v_x.\end{aligned}$$

But according to the system of notation we have adopted, if any combination, such as $\Delta^m\ \Delta'^n$ be prefixed to $\Delta^p u_x\ \Delta^q v_x$, it will produce $\Delta^{m+p}u_x\ \Delta^{n+q}v_x$, which is also represented by $\Delta^{m+p}\ \Delta'^{n+q}u_x v_x$. The above expression therefore becomes

$$\Delta^2 . u_z v_z = (\Delta + \Delta' + \Delta\,\Delta')\,\Delta u_z v_z$$
$$+ (\Delta + \Delta' + \Delta\,\Delta')\,\Delta' u_z v_z$$
$$+ (\Delta + \Delta' + \Delta\,\Delta')\,\Delta\,\Delta' u_z v_z,$$

in which the symbols of operation may now, without confusion, be separated from those of quantity, when we find

$$\Delta^2 . u_z v_z = (\Delta + \Delta' + \Delta\,\Delta')\,(\Delta + \Delta' + \Delta\,\Delta')\,u_z v_z,$$

that is,

$$(\Delta + \Delta' + \Delta\,\Delta')^2 u_z v_z;$$

for the same combinations of Δ and Δ' must result from this, as from the other. Just in the same way we may shew from this, that

$$\Delta^3 . u_z v_z = (\Delta + \Delta' + \Delta\,\Delta')^3 u_z v_z,$$

and so on. Now $\Delta + \Delta'\,\Delta\,\Delta' = (1+\Delta)\,(1+\Delta') - 1$, and therefore

$$\Delta^n . u_z v_z = \{(1+\Delta)(1+\Delta') - 1\}^n u_z v_z.$$

This result might also have been deduced from what we proved in (351), for since the performance of any number of successive operations is denoted by prefixing the product of the functions representing them separately, if these operations be all the same, and n times repeated, the nth power of the function representing them must be prefixed.

355. What we have demonstrated in the case of two functions u_z and v_z, holds good, *mutatis mutandis*, for any number u_z, u'_z, u''_z, &c. If we suppose Δ to refer to u_z, Δ' to u'_z, Δ'' to u''_z, and so on, we shall have

$$\Delta^n . (u_z . u'_z . u''_z . \&c.) = \{(1+\Delta)(1+\Delta') . \&c. - 1\}^n u_z u'_z . u''_z$$

where, in the developement of the second member, each term, such as $\Delta^m \Delta'^n \Delta''^p \ldots$ being applied immediately before $u_z u'_z u''_z \ldots$ is supposed to have no other meaning than

$$\Delta^m u_z \times \Delta^n u'_z \times \Delta^p u''_z \times \&c.$$

the accents over the letters Δ being used only as a temporary

contrivance, to keep their powers distinct, and at the same time to point out their application to the proper functions, and in no way altering the meaning of the characteristics themselves.

The difference of the proposed function or $\Delta.(u_z u'_z u''_z \ldots)$ being $u_{z+1} u'_{z+1} u''_{z+1}$. &c. $- u_z u'_z u''_z$. &c., if for u_{z+1}, &c., we substitute their values $u_z + \Delta u_z = (1+\Delta) u_z$, $u'_z + \Delta u'_z = (1+\Delta') u'_z$, &c.

we get

$$\Delta.(u_z u'_z.\,\&c.) = (1+\Delta) u_z \times (1+\Delta') u'_z \times \&c. - u_z u'_z.\,\&c.$$
$$= \{(1+\Delta)(1+\Delta')(1+\Delta'').\,\&c. - 1\} u_z u'_z u''_z.\,\&c.$$

from which it may be shewn, just as in the case of two functions (354) that

$$\Delta^n.(u_z u'_z u''_z.\,\&c.) =$$
$$\{(1+\Delta)(1+\Delta')(1+\Delta'').\,\&c. - 1\}^n u_z u'_z u''_z.\,\&c.$$

the complete developement of which is easily obtained by the theorem for raising a multinomial to the nth power, the expression within the brackets being equal to

$$\Delta + \Delta' + \&c.\, \Delta\Delta' + \&c. + \ldots\ldots + \Delta\Delta'\Delta''.\,\&c.$$

containing all the possible combinations of Δ, Δ', &c. by ones, twos, threes, &c.

356. The same differences may be expressed in another and much more simple form, by uniting several of their terms together by means of the successive values of one of the functions. For if we set out from the equation (343),

$$\Delta.(u_z u'_z) = u_z \Delta u'_z + u'_{z+1} \Delta u_z,$$

we find

$$\Delta^2.(u_z u'_z) = \Delta.(u_z \Delta u'_z) + \Delta.(u'_{z+1} \Delta u_z)$$

but the foregoing equation gives by making the proper substitutions

$$\Delta.(u_z \Delta u'_z) = u_z \Delta^2 u'_z + \Delta u'_{z+1}.\Delta u_z$$
$$\Delta.(u'_{z+1} \Delta u_z) = \Delta u_z.\Delta u'_{z+1} + u'_{z+2}.\Delta^2 u_z,$$

and if these values be substituted in the expression for $\Delta^2.(u_z u'_z)$ it becomes

$$\Delta^2.(u_x u'_x) = u_x \Delta^2 u'_x + 2 \Delta u_x . \Delta u'_{x+1} + \Delta^2 u_x . u'_{x+2},$$

and, in like manner it may be shewn that

$\Delta^n.(u_x u'_x) =$

$$u_x \Delta^n u'_x + \frac{n}{1} \Delta u_x . \Delta^{n-1} u'_{x+1} + \frac{n(n-1)}{1.2} \Delta^2 u_x . \Delta^{n-2} u'_{x+2} + \&c.$$

The separation of symbols of operation from those of quantity affords a very elegant demonstration of this theorem, for if we throw the expression

$$\{(1 + \Delta)(1 + \Delta') - 1\}^n u_x u'_x,$$

into the form

$$\{\Delta' + \Delta(1 + \Delta')\}^n u_x u'_x$$

and then develope it by the binomial theorem, we find

$\Delta^n.(u_x u'_x) =$

$$\left\{\Delta'^n + \frac{n}{1} \Delta \Delta'^{n-1}(1 + \Delta') + \frac{n(n-1)}{1.2} \Delta^2 \Delta'^{n-2}(1 + \Delta')^2 + \&c.\right\} u_x u'_x.$$

Now, any term of this, such as

$$\Delta^m \Delta'^{n-m} (1 + \Delta')^m u_x u'_x$$

resolves itself into

$$\Delta^m u_x \times \Delta^{n-m} u'_{x+m},$$

since, as we have seen (352),

$$(1 + \Delta')^m u'_x = u'_{x+m},$$

and thus the expression for $\Delta^n.(u_x u'_x)$ becomes at length

$$u_x \Delta^n u'_x + \frac{n}{1} . \Delta u_x \Delta^{n-1} u'_{x+1} + \&c.$$

The intelligent reader will perceive, without difficulty, the exact parallel which subsists between these processes and those of the Calculus of Generating Functions.

357. It is easy in all cases to find the successive differences of any assigned function of x, but the enquiry taken in a general point of view, without reference to the particular nature of that function, assumes a much more

interesting form, when it is proposed to determine Δ^n in terms of u_x and its differential coefficients. To this end, we have, by Taylor's Theorem, (21)

$$\Delta u_x = u_{x+1} - u_x = \frac{1}{1}\frac{d u_x}{d x} + \frac{1}{1.2}\cdot\frac{d^2 u_x}{dx^2} + \frac{1}{1.2.3}\,\frac{d^3 u_x}{dx^3} + \&c.$$

If we separate the symbols of operation from those of quantity, we get

$$\Delta u_x = \left\{\frac{1}{1}\cdot\frac{d}{dx} + \frac{1}{1.2}\cdot\left(\frac{d}{dx}\right)^2 + \&c.\right\} u_x$$

the d and its powers being immediately referred to the function u_x and affecting that alone. Now, the series within the brackets has for its abbreviated expression,

$$e^{\frac{d}{dx}} - 1,$$

and therefore

$$\Delta u_x = \left(e^{\frac{d}{dx}} - 1\right) u_x.$$

Lagrange first remarked that not only this, but the more general equation

$$\Delta^n u = \left(e^{\frac{d}{dx}} - 1\right)^n u,$$

is universally true; and in fact if we call to mind what has already been said at the latter end of (351) and (354) respecting the symbol Δ, and reflect that the whole is equally applicable to the symbol d, it will scarcely appear to require further demonstration. For, the operation denoted by Δ being equivalent to the series of operations denoted by $e^{\frac{d}{dx}} - 1$ and the repetition of this latter series of operations n times being equivalent to one series of operations having $\left(e^{\frac{d}{dx}} - 1\right)^n$ for its expression, it is evident that the operation denoted by Δ^n is equivalent to that denoted

$\left(e^{\frac{d}{dx}} - 1\right)^n$, and therefore $\Delta^n u_x = \left(e^{\frac{d}{dx}} - 1\right)^n u_x$;
as the discovery of this theorem and its consequences
formed in some respects an epoch in mathematical
ature, and as it seems in general to be regarded as in-
ing a certain degree of obscurity, we shall proceed to a
e particular demonstration of it. Taking then the
ition

$$u_{x+n} = u_x + \frac{n}{1} \cdot \frac{du_x}{dx} + \frac{n^2}{1.2} \cdot \frac{d^2 u_x}{dx^2} + \&c. \text{ (..)}$$

separating the symbols of operation from those of
ntity, we get

$$u_{x+n} = \left\{1 + \frac{nd}{1.dx} + \frac{n^2 d^2}{1.2 dx^2} + \&c.\right\} u_x$$

$$u_{x+n} = e^{n\frac{d}{dx}} u_x.$$

his equation if we write successively $n-1$, $n-2$, &c.
, we find $u_{x+n-1} = e^{(n-1)\frac{d}{dx}} u_x$, $u_{x+n-2} = e^{(n-2)\frac{d}{dx}} u_x$, &c.
w we have shewn (345), that

$$\Delta^n u_x = u_{x+n} - \frac{n}{1} u_{x+n-1} + \&c.$$

writing for u_{x+n}, u_{x+n-1}, &c. their values, we get

$$\Delta^n u_x = e^{n\frac{d}{dx}} u_x - \frac{n}{1} e^{(n-1)\frac{d}{dx}} u_x + \frac{n(n-1)}{1.2} . \&c.$$

again separating the symbols of operation from those of
ntity,

$$\Delta^n u_x = \left\{e^{n\frac{d}{dx}} - \frac{n}{1} e^{(n-1)\frac{d}{dx}} + \frac{n(n-1)}{1.2} e^{(n-2)\frac{d}{dx}} - \&c.\right\} u_x,$$

since $\left(e^{\frac{d}{dx}} - 1\right)^n$ when developed in powers of $\frac{d}{dx}$ must
essarily produce the same series as $e^{n\frac{d}{dx}} - \frac{n}{1} e^{(n-1)\frac{d}{dx}} + \&c.$

would produce, and is therefore an abbreviated expression for the same combination of operations

$$\Delta^n u_x = \left(e^{\frac{d}{dx}} - 1\right)^n u_x *.$$

359. Supposing $f(\Delta)$ to represent any function of Δ developable in powers of Δ, as

$$f(\Delta) = a + b\,\Delta + c\,\Delta^2 + \&c.$$

we have,

$$f(\Delta) u_x = a\,u_x + b\,\Delta\,u_x + \&c.$$

$$= a\,u_x + b.\left(e^{\frac{d}{dx}} - 1\right) u_x + c\left(e^{\frac{d}{dx}} - 1\right)^2 u_x + \&c.$$

or, separating the symbols of operation from those of quantity

$$f(\Delta) u_x = \left\{a + b\left(e^{\frac{d}{dx}} - 1\right) + c\left(e^{\frac{d}{dx}} - 1\right)^2 + \&c.\right\} u_x$$

$$= f\left(e^{\frac{d}{dx}} - 1\right) u_x$$

for it is evident that $f\left(e^{\frac{d}{dx}} - 1\right)$ when developed must produce the same series of powers of $\left(\frac{d}{dx}\right)$ which would result from the developement of the expression

$$a + b.\left(e^{\frac{d}{dx}} - 1\right) + c\left(e^{\frac{d}{dx}} - 1\right)^2 + \&c.$$

which is all that is intended by the expression in question.

---

* This demonstration is in substance the same with that given by Dr. Brinkley in the paper above referred to, *Phil. Trans.* 1807. i, and is the most elegant of any (among the great variety which have appeared of this important theorem) which have come to our knowledge.

This theorem is due to Arbogast, who first devised the method of separating the symbols of operation from those of quantity in researches of this kind.

360. If we suppose the form of the function f to be such that $f(\Delta) = (1+\Delta)^n$, we have

$$f\left(e^{\frac{d}{dx}}-1\right)^n = \left(1+e^{\frac{d}{dx}}-1\right)^n = e^{n\frac{d}{dx}},$$

and consequently,

$$(1+\Delta)^n u_x = e^{n\frac{d}{dx}} u_x = u_{x+n}, \text{ as before.}$$

Again, if $f(\Delta) = \{\log. (1+\Delta)\}^n$, we have

$$\{\log. (1+\Delta)\}^n u_x = \frac{d^n u_x}{dx^n},$$

an equation which expresses the value of any differential coefficient of u_x in terms of the function itself and its successive differences. If, for instance, $n=1$, we have

$$\frac{du_x}{dx} = \frac{\Delta u_x}{1} - \frac{\Delta^2 u_x}{2} + \frac{\Delta^3 u_x}{3} - \&c.$$

361. The developement of the equation

$$\Delta^n u_x = \left(e^{\frac{d}{dx}}-1\right)^n u_x$$

is very easily obtained. It will consist of a series of terms of the form

$$\left\{A_0 + A_1 \cdot \frac{d}{dx} + A_2 \cdot \frac{d^2}{dx^2} + \&c.\right\} u_x$$

that is, $A_0 u_x + A_1 \cdot \dfrac{du_x}{dx} + A_2 \cdot \dfrac{d^2 u_x}{dx^2} + \&c.$

and A_m is evidently the coefficient of t^m in the developement of $(e^t-1)^n$. Now to obtain this, we have,

$$(e^t-1)^n = e^{nt} - \frac{n}{1} \cdot e^{(n-1)t} + \frac{n(n-1)}{1 \cdot 2} e^{(n-2)t} - \&c.$$

but the coefficient of t^m in the developements of e^{nt}, $e^{(n-1)t}$, &c. are respectively

$$\frac{n^m}{1.2\ldots m}, \quad \frac{(n-1)^m}{1.2\ldots m}, \text{ \&c.}$$

and therefore we find, for the coefficient required,

$$A_m = \frac{n^m - \frac{n}{1}(n-1)^m + \frac{n(n-1)}{1.2}(n-2)^m - \text{\&c.}}{1.2.3\ldots m}$$

$$= \frac{\Delta^n o^m}{1.2\ldots m}. \ (350).$$

Now so long as m is less than n, this vanishes, as we have already seen, and since when $n=m$, $\Delta^n o^n = 1.2\ldots n$, the series for $\Delta^n u_x$, becomes

$$\Delta^n u_x = \frac{d^n u_x}{dx^n} + \frac{\Delta^n o^{n+1}}{1.2\ldots(n+1)} \cdot \frac{d^{n+1} u_x}{dx^{n+1}}$$

$$+ \frac{\Delta^n o^{n+2}}{1.2\ldots(n+2)} \cdot \frac{d^{n+2} u_x}{dx^{n+2}} + \text{\&c.*}$$

On the direct Method of Differences, when applied to Functions of two or more Variables.

362. If x and y be any two variables which increase and decrease by units, and if $u_{x,y}$ represent any function whatever of them, the successive substitution of ... 0, 1, 2, ... &c. for each of them, will produce a system of progressions.

....

$$\ldots u_{0,0}, \ u_{0,1}, \ u_{0,2}, \ldots u_{0,y} \ldots$$

$$\ldots u_{1,0}, \ u_{1,1}, \ u_{1,2} \ldots \ldots u_{1,y} \ldots$$

* See Dr. Brinkley's paper, above referred to.

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$\ldots u_{x,0},\ u_{x,1},\ u_{x,2},\ \ldots u_{x,y} \ldots$$

&c. &c. &c. &c.

in each of which, taken horizontally, the general term is a function of one variable, y, alone, the value of x remaining unaltered throughout the whole progression; and if taken vertically, of x alone, y being in that case constant.

Supposing then x the only variable, we have

$$\Delta u_{x,y} = u_{x+1,y} - u_{x,y},$$

which may be termed the partial difference of the proposed function, relative to the variation of x; and in like manner, if by accentuating the characteristic Δ, we denote its reference to the variation of y, we shall have, for the partial difference relative to y,

$$\Delta' u_{x,y} = u_{x,y+1} - u_{x,y}.$$

But if both x and y vary at once, we shall have for the *total difference* of $u_{x,y}$ (which may be denoted* by $\overline{\Delta}\, u_{x,y}$)

$$\overline{\Delta}\, u_{x,y} = u_{x+1,y+1} - u_{x,y}.$$

363. It is sufficiently evident, that if we consider only the partial differences of $u_{x,y}$, the same relations between these and their primitive function, and its successive values will hold, as in the case of one variable, from which the present only differs in appearance.

364. If we consider only the total differences, since x and y are supposed to vary *together, and by the same steps,* we in fact establish a relation between them, viz. $y = x + \text{const.}$ So that $u_{x,y}$ may in fact be regarded as a function of one of them alone, and of course the relations between the

---

* See the *Theorie Analytique des Probabilités,* page 70, where this notation is sanctioned by the authority of Laplace.

function and its total differences must be the same as those above demonstrated. For example,

$$\overline{\Delta}^n u_{x,\,y} = u_{x+n,\,y+n} - \frac{n}{1} \cdot u_{x+n-1,\,y+n-1} + \&c.$$

$$u_{x+n,\,y+n} = u_{x,\,y} + \frac{n}{1}\overline{\Delta}\, u_{x,\,y} + \frac{n(n-1)}{1\,.\,2}\overline{\Delta}^2 u_{x,\,y} + \&c.$$

Again, since $y = x + C$, we have $\frac{dy}{dx} = 1$, and since in general

$$\frac{1}{dx} \cdot du_{x,\,y} = \frac{du_{x,\,y}}{dx} + \frac{du_{x,\,y}}{dy} \cdot \frac{dy}{dx},$$

in this case we have

$$\frac{1}{dx} du_{x,y} = \frac{du_{x,\,y}}{dx} + \frac{du_{x,\,y}}{dy},$$

which we may abbreviate into

$$\frac{1}{dx} du_{x,y} = \left(\frac{d}{dx} + \frac{d}{dy}\right) u_{x,y};$$

consequently the equation of (358.)

$$\Delta^n u_x = \left(e^{\frac{1}{dx} d} - 1\right)^n u_x,$$

when applied to this case, will give

$$\overline{\Delta}^n u_{x,\,y} = \left(e^{\frac{d}{dx} + \frac{d}{dy}} - 1\right)^n u_{x,\,y},$$

and the same may be extended to any number of variables. No confusion can arise from the intermixture of the powers of d, which refer to the variation of x, with those which refer to that of y, since their application is pointed out by the denominators they carry with them, and by the notation of the Differential Calculus, any term such as

$$\left(\frac{d}{dx}\right)^p \cdot \left(\frac{d}{dy}\right)^q u_{x,\,y},$$

meaning nothing more than $\frac{d^{p+q}u_{x,y}}{dx^p.dy^q}$.

365. The developement of this result is worth examining. The coefficient of $\frac{d^{p+q}u_{x,y}}{dx^p.dy^q}$ is that of $t^p.t'^q$ in the developement of $(\varepsilon^{t+t'}-1)^n$. It is evident, that if this function be developed in powers of $t+t'$, the above combination can only result from the term multiplied by $(t+t')^{p+q}$; and therefore since the coefficient of this term is by (361) equal to

$$\frac{\Delta^n o^{p+q}}{1.2.3\ldots(p+q)},$$

and the coefficient of $t^p.t'^q$, in the developement of $(t+t')^{p+q}$, is

$$\frac{(p+q)(p+q-1)\ldots(q+1)}{1.2\ldots p}=\frac{1.2.3\ldots(p+q)}{1.2\ldots p\times 1.2\ldots q},$$

the whole coefficient of $t^p t'^q$ required will be the product of these, or

$$\frac{\Delta^n o^{p+q}}{1.2\ldots p\times 1.2\ldots q},$$

and a similar result will be obtained for any number of variables.*

366. There remains only to be considered the class of relations which involve at once the proposed function, its total, and partial differences. These, however, are too complex, and present too little in the way of interest to detain us. It is sufficient to observe, that the total dif-

---

* See the paper referred to, page 478. (Note), in *Phil. Trans.* 1816, i. "On the Developement of Exponential Functions," where this theorem is deduced from one of much greater generality.

infinitely expressible by means of the partial differences.

Since $\Delta u_{x,y+1} = u_{x+1,y+1} - u_{x,y+1}$

and $\Delta' u_{x+1,y} = u_{x+1,y+1} - u_{x+1,y}$,

If we add these together, and subtract the result from the equation

$$2\bar{\Delta} u_{x,y} = 2u_{x+1,y+1} - 2u_{x,y},$$

we shall find

$$2\bar{\Delta} u_{x,y} - (\Delta u_{x,y+1} + \Delta' u_{x+1,y}) =$$
$$= (u_{x+1,y} - u_{x,y}) + (u_{x,y+1} - u_{x,y})$$
$$= \Delta u_{x,y} + \Delta' u_{x,y};$$

whence we obtain

$$\bar{\Delta} u_{x,y} = \tfrac{1}{2}(\Delta u_{x,y} + \Delta u_{x,y+1}) + \tfrac{1}{2}(\Delta' u_{x,y} + \Delta' u_{x+1,y}),$$

which (as it will hereafter appear) is analogous to the equation

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy,$$

in the Differential Calculus.

On the Inverse Method of Finite Differences.

367. The Direct Method of Differences, of which we have presented a brief outline in the foregoing pages, consists in determining the differences of given functions, and investigating the general relations which exist between these differences (of any orders), and the functions from which they are derived. The inverse method of course has for its object the determination of the primitive function by means of assigned relations between it and its differences; a much more difficult task, and in which it is to

be lamented, that small progress has hitherto been made. These two branches of the Calculus of Finite Differences have precisely the same relation to each other that the two great divisions of the Differential Calculus have, and it will appear hereafter, that the latter Calculus is in fact easily deducible from the former.

In order to investigate the nature of a function between which and its differences and the independent variable given relations shall subsist, the simplest mode of proceeding seems to be, to consider, 1st, The case where the difference is given immediately in functions of the independent variable, or to determine u_x from the equation $\Delta u_x = f(x)$.

2d. When any equation, between u_x the independent variable x, and the differences Δu_x, $\Delta^2 u_x$, &c. is given, such as
$$o = F\left\{x,\ u_x,\ \Delta u_x,\ \Delta^2 u_x, \&c.\right\}$$
which equation, without losing any of its generality, may be also thrown into the form
$$o = F\left\{x,\ u_x,\ u_{x+1},\ u_{x+2}, \ldots u_{x+n}\right\}$$
by means of the expressions $\Delta u_x = u_{x+1} - u_x$, &c. Such an equation is called an equation of differences, and its order is denoted by the highest exponent of Δ, or the interval (n) between the most distant of the successive values of u_x it contains. 3d. When the equation expressing the relation in question involves, at the same time, the differential coefficients of u_x and its differences, or successive values, or those of the differential coefficients, as
$$o = F\left\{x,\ u_x,\ \Delta u_x, \&c.\ \frac{du_x}{dx}, \&c.\ \Delta\frac{du_x}{dx},\ \&c.\right\}$$
Such equations are called equations of mixed differences. Very little is known of their nature, and we shall accordingly consider only the simplest forms of them.

On the Determination of a Function, whose Difference is explicitly given in Terms of the Independent Variable.

368. Since Δu_x is as well the difference of $u_x + C$, as of u_x, if in reascending from the given difference Δu_x to the primitive function u_x, we would give the result all the generality it is susceptible of, it will be necessary to add an arbitrary constant; for since the proposed difference will equally result from $u_x + C$, whatever value we assign to C, provided it remain unchanged, while x varies from x to $x+1$, there is nothing in the nature of the present enquiry to render any one of such values preferable to any other. The reader will observe then that we are at liberty to assign any value whatever to C, should it even be a function of x, provided it remain unchanged, while x changes to $x+1$. This is all that can be understood by a constant in the Calculus of Finite Differences. Suppose C_x to denote such a function of x, that $C_x = C_{x+1}$, or $\Delta\, C_x = 0$; and we shall have

$$\Delta\,(u_x + C_x) = \Delta\, u_x,$$

as if C_x were absolutely independent of x. This condition is satisfied by supposing C_x to represent any function of $\cos 2\,\pi\, x$, for since $\cos 2\,\pi\,(x+1) = \cos\,(2\,\pi + 2\,\pi\, x) = \cos 2\,\pi\, x$, any function of this quantity will possess the property in question. We shall perceive the force of this observation hereafter*, at present we will proceed to the

* So long as we confine our speculations to cases where x has none but integer values (as for example in the summation of series) since $\cos 2\,\pi\, x = 0$, every function of this quantity must be regarded as absolutely invariable, and the arbitrary constant is then of the same nature as in the Differential Calculus.

subject of this section; and first, it will be necessary to explain the notation used in the inverse method.

The characteristic Σ is used to express the operation by which we reascend from the difference Δu_x to the primitive function (or *integral*) u_x + const. so that

$$\Sigma \Delta u_x = u_x + \text{const.}$$

just in the same way, and for the same reason that

$$\int d u_x = u_x + \text{const.}$$

Having given then any function $f(x)$ to find the function of which it is the difference, is the same thing with enquiring the value of the expression $\Sigma f(x)$. We shall proceed to consider the particular forms of $f(x)$, in which this enquiry has been attended with success; and first, it is evident that the *integral* of the sum of any number of functions

$$\Sigma \left\{ f(x) + \phi(x) + \&c. \right\}$$

is equal to the sum of their separate integrals

$$\Sigma f(x) + \Sigma \phi(x) + \&c.$$

for, if we take the difference of this expression, we produce

$$f(x) + \phi(x) + \&c.$$

In the same manner it appears, that $\Sigma a . f(x) = a \Sigma \phi(x)$, a being any constant quantity, so that a constant factor may be brought out from under the integral sign.

369. The first form of $f(x)$, which is directly integrable, is when $f(x)$ = any rational integral function of x. We have already seen, that the difference of any rational function whatever, such as

$$Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots Kx + L,$$

is itself a rational integral function

$$A'x^{m-1} + B'x^{m-2} + \dots K',$$

where $A' = \frac{m}{1} A$

$$B' = \frac{m(m-1)}{1.2} A + \frac{m-1}{1} B$$

$$C' = \frac{m(m-1)(m-2)}{1.2.3} A + \frac{(m-1)(m-2)}{1.2} B + \frac{m-2}{1} C$$

... ...

$$K' = A + B + C + \dots\dots K$$

This expression $A'x^{m-1}$ + &c. on account of the indeterminate values of A, B, C, ... K, and the exponent m, may be made to coincide with any proposed rational integral function, as

$$f(x) = a\,x^n + b\,x^{n-1} + c\,x^{n-2} + \dots k,$$

by making $m-1=n$, or $m=n+1$,

$$a = A' = \frac{m}{1} A,$$

$$b = B' = \frac{m(m-1)}{1.2} A + \frac{(m-1)}{1} B, \text{ \&c.}$$

whence the values of A, B, &c. are obtained as follows:

$$A = \frac{a}{m}$$

$$B = \frac{b}{m-1} - \frac{a}{2}$$

$$C = \frac{c}{m-2} - \frac{b}{2} + \frac{(m-1)a}{12},$$

&c.

The last term L remains indeterminate. It is in fact the arbitrary constant which must be added to complete the integral.

We will take a particular example. Suppose the value of $\Sigma\,(x^2+1)$ were required: we have

$$\Delta(Ax^3+Bx^2+Cx+D)=$$
$$=3Ax^2+(3A+2B)x+(B+C),$$

which will coincide with x^2+1, if we put

$$3A=1,\quad 3A+2B=0,\quad B+C=1,$$

whence $A=\frac{1}{3}$, $B=-\frac{1}{2}$, $C=\frac{3}{2}$, and therefore

$$\Sigma(x^2+1)=\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}+\text{const.}$$

The reader may exercise himself on the following

$$\Sigma 1 = x + C$$
$$\Sigma x = \frac{x^2}{2}-\frac{x}{2}+C$$
$$\Sigma x^2=\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{6}+C$$
$$\Sigma x^3=\frac{x^4}{4}-\frac{x^3}{2}+\frac{x^2}{4}+C$$
$$\Sigma x^4=\frac{x^5}{5}-\frac{x^4}{2}+\frac{x^3}{3}-\frac{x}{30}+C$$
$$\Sigma x^5=\frac{x^6}{6}-\frac{x^5}{2}+\frac{5x^4}{12}-\frac{x^2}{12}+C$$

$$\Sigma x^n=\frac{x^{n+1}}{n+1}-\frac{x^n}{2}+\frac{1}{2}\frac{nx^{n-1}}{1.2}-\frac{1}{6}\frac{n(n-1)(n-2)}{1.2.3.4.5}x^{n-3}+$$
$$+\frac{1}{6}\cdot\frac{n(n-1)\ldots(n-4)}{1.2\ldots\ldots7}x^{n-5}$$
$$-\frac{3}{10}\cdot\frac{n(n-1)\ldots(n-6)}{1.2\ldots\ldots9}x^{n-7}$$
$$+\frac{5}{6}\cdot\frac{n(n-1)\ldots(n-8)}{1.2\ldots\ldots11}x^{n-9}$$
$$-\frac{691}{210}\cdot\frac{n(n-1)\ldots(n-10)}{1.2\ldots\ldots13}x^{n-11}$$

$$+ \quad \frac{35}{2} \cdot \frac{n(n-1)\ldots(n-12)}{1.2\ldots\ldots15} x^{n-13}$$

$$- \quad \frac{3617}{30} \cdot \frac{n(n-1)\ldots(n-14)}{1.2\ldots\ldots17} x^{n-15}$$

$$+ \frac{43867}{42} \cdot \frac{n(n-1)\ldots(n-16)}{1.2\ldots\ldots19} x^{n-17}$$

$$- \frac{1222277}{110} \cdot \frac{n(n-1)\ldots(n-18)}{1.2\ldots\ldots21} x^{n-19}$$

$$+ \quad \&c. + \text{const.}$$

which may be derived by substituting 1 for a, 0 for b, c, &c. and 1, 2, 3,... n, successively, for n, in the general formula above given. The student however who reads for improvement will rarely proceed in this manner, as nothing tends more strongly to impress the principles of an analytical process upon the mind, than following them closely and repeatedly through the details of particular examples. Graphical accuracy can be attained by no other method; the memory is at the same time stored with useful results, and which is by far the most important object, the mind is accustomed to dwell habitually upon the general principle, and to carry on a reasoning process, while the mere algebraic operations are performed in a manner almost mechanical.

With respect to the coefficients expressed in numbers is this last formula, they merit a nearer attention, on account of their perpetual occurrence in the theory of series; we shall take occasion hereafter to develope more particularly their nature and their law.

370. The method delivered above, although generally applicable, is in certain cases superseded in facility and shortness, by the following. We have seen (343), that when u_x is of the form $a + b\,x$, the difference of the function

$$u_x \cdot u_{x+1} \ldots u_{x+n}$$

is itself a function of the same form, having one factor less, and expressed by

$$(n+1)\, b \,.\, u_{x+1} \dots u_{x+n}.$$

Taking therefore the integrals of both, and writing $x-1$ for x, we find

$$\Sigma\, u_x \dots u_{x+n-1} = \frac{u_{x-1} \dots u_{x+n-1}}{(n+1)\, b} + \text{const.}$$

Now $u_x \dots u_{x+n-1}$ is the product of n terms of an arithmetical progression

$$(a+bx)\,(a+b\,.\,\overline{x+1}) \dots (a+b\,.\,\overline{x+n-1}),$$

which, when developed in powers of x is a rational integral function of the nth degree, and to integrate it we have only to *multiply it by the term of the progression immediately preceding, or, to annex one more factor at the beginning, and to divide the result by the common difference, and by the number of factors so increased.* For instance,

$$\Sigma\,(1+x)\,(2+x)\,(3+x) = \frac{x\,(1+x)\,(2+x)\,(3+x)}{4} + \text{const.}$$

$$\Sigma\,(2x+1)\,(2x+3) = \frac{(2x+1)\,(2x+1)\,(2x+3)}{6} + \text{const.}$$

$$\Sigma\left(\frac{x}{2}-3\right)\left(\frac{x}{2}-\frac{5}{2}\right)\left(\frac{x}{2}-2\right) = \frac{(x-7)\,(x-6)\,(x-5)\,(x-4)}{32} + \text{const.}$$

A little practice renders it extremely easy to resolve any proposed rational integral function into one or more *factorials* of this form. An example or two will explain the method.

$$x^3 = x^2\,(x+1) - x^2 = (x-1)\,x\,(x+1) + x\,(x+1) - x^2$$
$$= (x-1)\,x\,(x+1) + x,$$

whence $\Sigma\, x^3 = \dfrac{(x-2)\,(x-1)\,x\,(x+1)}{4} + \dfrac{x\,(x-1)}{2} + C.$

Again, $x^4-3x^3=x^3(x+1)-x^3-3x^3$

$$=x(x-1)x(x+1)+x^2(x+1)-x^3-3x^2$$

$$=\{(x-1)x(x+1)(x+2)-2(x-1)x(x+1)\}$$

$+x^2(x-2)-x^3$

$$=(x-1)x(x+1)(x+2)-2(x-1)x(x+1)-$$

$2\left\{x(x+1)-x\right\}$

which is easily integrated. The principal thing to be attended to in these resolutions is, to keep the numerical coefficients as low as possible by a proper disposition of the preceding and succeeding factors.

In general, any quantity of the form $ax^n+bx^{n-1}+$&c. may be resolved into factorials by the method of indeterminate coefficients; thus, if ax^2+bx+c be assumed equal to

$$A(x+1)(x+2)+B(x+1)+C$$

$$=Ax^2+(3A+B)x+(2A+B+C),$$

the comparison of terms will give

$$A=a,\ B=b-3a,\ C=c-2a-(b-3a)=c-b+a.$$

If one or more factors be deficient in a factorial of this kind, it may be supplied as in the following examples:

$$(x+1)(x+2)(x+4)(x+5)=(x+3-2)(x+2)(x+4)(x+5)$$

$$=(x+2)(2+3)(x+4)(x+5)-2.(x+2)(x+4)(x+5).$$

Again,

$$(x+2)(x+4)(x+5)=(x+3)(x+4)(x+5)-(x+4)(x+5),$$

so that the proposed function becomes

$$(x+2)\ldots(x+5)-2.(x+3)\ldots(x+5)+2.(x+4)(x+5).$$

Again,

$$(2x+3)(2x+7)=(2x+5)(2x+7)-2.(2x+7).$$

A variety of trifling artifices of this kind will suggest themselves to the intelligent reader, which will tend at least to abridge his labour, if they do not much increase his knowledge.

371. The next form of $f(x)$ to be examined is that of a rational fraction; but few cases present themselves here in which the integration can be accomplished. They are comprehended with exceptions of no moment, in the expression

$$\frac{1}{u_x \,.\, u_{x+1} \ldots u_{x+n-1}}, \text{ where } u_x = a + b\,x,$$

and some others reducible to this form; for if we take the difference of

$$\frac{-1}{(n-1)\,b\,.\,u_x \ldots u_{x+n-2}} + \text{const.}$$

we shall find for our result the function proposed, of which this is consequently the integral: hence, to find the integral of a fraction, whose denominator is the product of n terms in arithmetical progression, *the last term must be effaced, and the result divided by the number of terms remaining, and by their common difference, and affected with a negative sign, is the integral, a constant being added.* Thus

$$\Sigma \frac{1}{x\,(x+1)\,(x+2)} = \text{const.} - \frac{1}{2} \,.\, \frac{1}{x\,(x+1)}$$

$$\Sigma \frac{1}{(3\,x+1)\,(3\,x+4)} = \text{const.} - \frac{1}{3} \,.\, \frac{1}{3\,x+1}\,.$$

372. Should any of the factors in the denominator be deficient, they may be supplied by multiplying both numerator and denominator by them, and reducing the numerator of the resulting fraction to the sums or differences of factorials consisting of the first, or last terms only, of the denominator, and by this means resolving the fraction into several whose denominators shall be complete factorials of the kind above treated. An example will render this clearer. Suppose the fraction to be

$$\frac{1}{u_x \,.\, u_{x+1} \,.\, u_{x+3}}, \text{ where } u_x = a + b\,x$$

by multiplying and dividing by $u_{x+2} = (a + 2b) + bx$ it becomes

$$\frac{a + 2b + bx}{u \cdot u_{x+1} \cdot u_{x+2} \cdot u_{x+3}}$$

the numerator of which is equal to $a + bx + 2b = u_x + 2b$, or else to $u_{x+3} - b$: thus the fraction is resolved into two, in either of the two following ways

$$\frac{1}{u_{x+1} \cdot u_{x+2} \cdot u_{x+3}} + \frac{2b}{u_x \ldots\ldots u_{x+3}}$$

$$\frac{1}{u_x \cdot u_{x+1} \cdot u_{x+2}} - \frac{b}{u_x \ldots\ldots u_{x+3}}$$

each of which is integrable by the preceding No.

In like manner,

$$\frac{1}{(x+1)(x+3)(x+4)} = \frac{x+2}{(x+1)\ldots(x+4)} =$$

$$= \frac{1}{(x+2)\ldots(x+4)} + \frac{1}{(x+1)\ldots(x+4)}$$

$$\frac{1}{(x-1)(x+1)} = \frac{x}{(x-1)x(x+1)} = \frac{1}{x(x+1)} + \frac{1}{(x+1)\,x.(x+1)}.$$

In general, (u_x being still of the form $a + bx$) if the proposed fraction be

$$\frac{Ax^n + Bx^{n-1} + \ldots\ldots K}{u_x \cdot u_{x+1} \ldots\ldots u_{x+n+1}}$$

where the degree of the numerator is at least higher by two units than that of the denominator; assume

$$Ax^n + Bx^{n-1} + \ldots\ldots K =$$

$$= a + bu_x + cu_x \cdot u_{x+1} + \ldots\ldots k\,u_x \ldots u_{x+n-1},$$

and, developing this last in powers of x, the comparison of terms will give $n+1$ equations for determining $a, b, c, \ldots k$ and the fraction resolves itself into the following, each of which is integrable.

$$\frac{a}{u_x\ldots\ldots u_{x+n+1}} + \frac{b}{u_{x+1}\ldots\ldots u_{x+n+1}} + \ldots\ldots \frac{k}{u_{x+n}\cdot u_{x+n+1}}$$

Thus, for example, if we propose to integrate

$$\frac{(3x+1)(3x+4)}{x(x+1)(x+2)(x+3)}$$

we assume

$$(3x+1)(3x+4) = a + bx + cx(x+1)$$

$$\text{or,} \quad 4 + 15x + 9x^2 = a + (b+c)x + cx^2$$

whence,

$$a = 4, \ c = 9, \ b = 15 - 9 = 6, \ \text{so that}$$

$$\frac{(3x+1)(3x+4)}{x(x+1)(x+2)(x+3)} = \frac{4}{x\ldots(x+3)} + \frac{6}{(x+1)\ldots(x+3)} + \frac{9}{(x+2)(x+3)}$$

and the integral required is

$$C - \frac{4}{3x(x+1)(x+2)} - \frac{3}{(x+1)(x+2)} - \frac{9}{x+2}.$$

373. The next form of $f(x)$ which comes to be examined is $(f) = a^x$; now we have seen that

$$\Delta a^x = (a-1) \cdot a^x, \text{ whence } \Sigma a^x = \frac{a^x}{a-1} + \text{const.}$$

To this we may add the functions sin. $x\theta$ and cos. $x\theta$, which are combinations dependent on this form; for since (164)

$$\cos. x\theta = \frac{1}{2}\left\{\left(e^{\theta\sqrt{-1}}\right)^x + \left(e^{-\theta\sqrt{-1}}\right)^x\right\}$$

$$\sin. x\theta = \frac{1}{2\sqrt{-1}}\left\{\left(e^{\theta\sqrt{-1}}\right)^x - \left(e^{-\theta\sqrt{-1}}\right)^x\right\}.$$

If we integrate these by the above formula, and reduce the results as much as possible by the help of the equations

$$\cos.(x-1)\theta - \cos. x\theta = 2\sin.\left(\frac{\theta}{2}\right).\sin.\left(\frac{2x-1}{2}\theta\right)$$

$$1 - \cos.\theta = 2.\left(\sin.\frac{\theta}{2}\right)^2,$$

and

$$\sin. x\theta - \sin. (x-1)\theta = 2 . \sin. \left(\frac{\theta}{2}\right) . \cos. \left(\frac{2x-1}{2}\theta\right),$$

we shall obtain

$$\Sigma \cos. (x\theta) = \frac{\sin. \left(\frac{2x-1}{2}\theta\right)}{2 \sin. \left(\frac{\theta}{2}\right)} + \text{const.}$$

$$\Sigma \sin. (x\theta) = -\frac{\cos. \left(\frac{2x-1}{2}\theta\right)}{2 \sin. \left(\frac{\theta}{2}\right)} + \text{const.}$$

and in the same manner may the integrals of the functions

$$(\sin. x\theta)^m . (\cos. x\phi)^n, \quad a^x . \sin. x\theta, \quad a^x . \cos. x\theta, \text{ \&c.}$$

be obtained.

374. We have seen (343), that

$$\Delta u_x \ldots\ldots u_{x+n} = u_{x+1} \ldots u_{x+n} (u_{x+n+1} - u_x),$$

and if we suppose $u_x = pa^x + q$, we shall find by writing $x-1$ for x,

$$\Sigma a^x (pa^x + q)(pa^{x+1} + q) \ldots (pa^{x+n-1} + q) =$$

$$= a . \frac{(pa^{x-1} + q)(pa^x + q) \ldots (pa^{x+n-1} + q)}{p.(a^{n+1} - 1)} + \text{const.}$$

and in like manner we obtain,

$$\Sigma \frac{a^x}{(pa^x + q) \ldots\ldots (pa^{x+n-1} + q)} =$$

$$= \text{const.} - \frac{1}{p.(a^{n-1} - 1)} . \frac{1}{(pa^x + q) \ldots\ldots (pa^{x+n-2} + q)} .$$

375. Since $\Delta (u_x . v_x) = u_x \Delta v_x + v_{x+1} \Delta u_x$, if we integrate both sides, we find

$$\Sigma u_x \Delta v_x = u_x v_x - \Sigma v_{x+1} \Delta u_x$$

a formula analogous to $\int y\, dx = yx - \int x\, dy$, and which enables us to find the integrals of a great variety of func-

tions. Suppose, for instance, that the functions proposed were

$$f(x) = u_x \,.\, a^x$$

where u_x is a rational integral function of the nth degree, or

$$u_x = A x^n + B x^{n-1} + \dots K$$

we have then, $\Delta v_x = a^x$, $v_x = \dfrac{a^x}{a-1}$, whence

$$\Sigma\, u_x a^x = \frac{a^x}{a-1} u_x - \Sigma \frac{a}{a-1} \,.\, a^x \Delta u_x,$$

but by writing Δu_x for u_x, we find

$$\Sigma a^x \Delta u_x = \frac{a^x}{a-1} \Delta u_x - \Sigma \frac{a}{a-1} a^x \Delta^2 u_x,$$

and so on, to

$$\Sigma a^x \Delta^{n-1} u_x = \frac{a^x}{a-1} \Delta^{n-1} u_x - \Sigma \frac{a}{a-1} a^x \,.\, \Delta^n u_x$$

which last integral, since $\Delta^n u_x$ is constant, is found equal to

$$\frac{a}{(a-1)^2} a^x \Delta^n u_x + \text{const.}$$

and consequently,

$$\Sigma\, u_x a^x = \frac{a^x u_x}{a-1} - \frac{a^{x+1} \Delta u_x}{(a-1)^2} +$$

$$\frac{a^{x+2} \Delta^2 u_x}{(a-1)^3} \dots\dots \pm \frac{a^{x+n} \Delta^n u_x}{(a-1)^{n+1}} + \text{const.}$$

376. In general, u_x and v_x being any functions of x, we have

$$\begin{aligned}\Sigma (u_x \,.\, v_x) &= u_x \Sigma v_x - \Sigma (\Delta u_x \Sigma v_{x+1}) \\ &= u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Sigma (\Delta^2 u_x \Sigma^2 v_{x+2}) \\ &= \&c.\end{aligned}$$

$$= u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_x \Sigma^3 v_{x+2} \dots \pm \Delta^n u_x \Sigma^{n+1} v_{x+n}$$
$$\mp \Sigma (\Delta^{n+1} u_x \Sigma^{n+1} v_{x+n+1}); \qquad (a).$$

v i nay be any integer number we please; and if u_x ional integral function of the nth degree, it

$$v_x) \quad {}_x\Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \ldots\ldots \pm \Delta^n u_x \Sigma^{n+1} v_{x+n}.$$

reader may take as examples to this formula the following

$$\Sigma . \quad \cos. x\theta, \; \Sigma \frac{x}{(2x+1)(2x+3)(2x+5)},$$

$\Sigma x(x+1).(\sin. x\theta)^3$.

377. To derive a general formula of the same kind for $\Sigma^2(u_x, v_x)$, $\Sigma^3(u_x . v_x)$, &c., we have only to consider that, by writing in the equation (a) of (376) Σv_x for v_x, we get

$$\Sigma(u_x \Sigma v_x) = u_x \Sigma^2 v_x - \Delta u_x \Sigma^3 v_{x+1} \ldots \pm \Delta^n u_x \Sigma^{n+2} v_{x+n}$$
$$\mp \Sigma(\Delta^{n+1} u_x \Sigma^{n+2} v_{x+n+1}).$$

Again, in the same equation writing Δu_x for u_x, $\Sigma^2 v_{x+1}$ for v_x, and $n-1$ for n, we find

$$\Sigma(\Delta u_x \Sigma^2 v_{x+1}) = \Delta u_x \Sigma^3 v_{x+1} - \ldots \mp \Delta^n u_x \Sigma^{n+2} v_{x+n}$$
$$\mp \Sigma(\Delta^{n+1} u_x \Sigma^{n+2} v_{x+n+1}),$$

and so on, till we come to

$$\Sigma(\Delta^n u_x \Sigma^{n+1} v_{x+n}) = \Delta^n u_x \Sigma^{n+2} v_{x+n}$$
$$- \Sigma(\Delta^{n+1} u_x \Sigma^{n+2} v_{x+n+1}).$$

Now, taking the integral of (a), we find

$$\Sigma^2(u_x v_x) = \Sigma(u_x \Sigma v_x) - \Sigma(\Delta u_x \Sigma^2 v_{x+1}) + \ldots \pm \Sigma(\Delta^n u_x \Sigma^{n+1} v_{x+n})$$
$$\mp \Sigma^2(\Delta^{n+1} u_x \Sigma^{n+1} v_{x+n+1})$$

in which, substituting the values given by the above equations, we find at length

$$\Sigma^2(u_x v_x) = u_x \Sigma^2 v_x - 2\Delta u_x \Sigma^3 v_{x+1} + 3\Delta^2 u_x \Sigma^4 v_{x+2} \ldots$$
$$\pm (n+1)\, \Delta^n u_x \Sigma^{n+2} v_{x+n}$$
$$\mp (n+1)\Sigma(\Delta^{n+1} u_x \Sigma^{n+2} v_{x+n+1}) \mp \Sigma^2(\Delta^{n+1} u_x \Sigma^{n+1} v_{x+n+1}).$$

If we again integrate this equation, and for each term write its value derived in the same manner from the equation (a) of (376), we get

$$\Sigma^3(u_x v_x) = u_x \Sigma^3 v_x - 3\,\Delta u_x \Sigma^4 v_{x+1} + \dots$$
$$\pm \frac{(n+1)(n+2)}{1.2}\Delta^n u_x \Sigma^{n+3} v_{x+n}$$
$$\mp \frac{(n+1)(n+2)}{1.2}\Sigma(\Delta^{n+1}u_x \Sigma^{n+3}v_{x+n+1})$$
$$\mp \frac{n+1}{1}\Sigma^2(\Delta^{n+1}u_x\Sigma^{n+2}v_{x+n+1}) \mp \Sigma^3(\Delta^{n+1}u_x\Sigma^{n+1}v_{x+n+1}),$$

and so on, to

$$\Sigma^r(u_x\,.\,v_x) = u_x\Sigma^r v_x - \frac{r}{1}\Delta u_x \Sigma^{r+1} v_{x+1}$$
$$+ \frac{r(r+1)}{1.2}\Delta^2 u_x \Sigma^{r+2} v_{x+2} - \dots$$
$$\mp \frac{(n+1)(n+2)\dots(n+r-1)}{1.2\dots(r-1)}\Delta^n u_x \Sigma^{r+n} v_{x+n}$$
$$\mp \Big\{ \Sigma^r(\Delta^{n+1}u_x\,.\,\Sigma^{n+1}v_{x+n+1})$$
$$+ \frac{n+1}{1}\Sigma^{r-1}(\Delta^{n+1}u_x\Sigma^{n+2}v_{x+n+1}) + \dots\dots$$
$$\dots + \frac{(n+1)(n+2)\dots(n+r-1)}{1.2\dots\dots(r-1)}\Sigma(\Delta^{n+1}u_x\Sigma^{n+r}v_{x+n+1})\Big\}$$

378. This leads us to consider, more extensively, the general relations which subsist between any functions, of whatever form, and their integrals of any order. Now, since the performance of the operation Σ upon any series of terms, as

$$A\,\Delta^m u_x + B\Delta^n u_x + \&c.$$

reduces it to

$$A\,\Delta^{m-1}u_x + B\,\Delta^{n-1}u_x + \&c.$$

it appears, that prefixing Σ to $(A\Delta^m + B\Delta^n + \&c.)\,u_x$ has the same effect as prefixing Δ^{-1}; in other words, Σ is equivalent to Δ^{-1}, and in like manner since integrating $\Delta^m u_x$ n times reduces it to $\Delta^{m-n}u_x$, Σ^n must be equivalent to Δ^{-n}. The same reasoning is applicable to the symbols $\int^n$, and $\left(\frac{d}{dx}\right)^{-n}$. When-

ever therefore, in separating symbols of operation from those of quantity, as in the expression $F(\Delta)u_x, f\left(\frac{d}{dx}\right)u_x$ terms containing negative powers of Δ or $\frac{d}{dx}$ occur, they are understood to be replaced by the corresponding positive powers of Σ and $\int$. Thus, for example,

$$(a\Delta^{-1} + b + c\Delta)u_x = a \,.\, \Sigma u_x + b\, u_x + c\Delta u_x.$$

This being premised, we proceed to shew that the equation

$$\Delta^n u_x = \left(e^{\frac{d}{dx}} - 1\right)^n u_x,$$

which was shewn in (357) to hold good for positive values of n, is also true for negative, the negative powers of Δ and d which occur, being understood as above explained, or, that

$$\Sigma^n u_x = \left(e^{\frac{d}{dx}} - 1\right)^{-n} u_x;$$

for, since

$$\Delta^n \Sigma^n u_x = u_x,$$

we must have

$$\left(e^{\frac{d}{dx}} - 1\right)^n \Sigma^n u_x = u_x.$$

Σ^n must therefore represent such an operation, or series of operations, as is capable of exactly counteracting that represented by $\left(e^{\frac{d}{dx}} - 1\right)^n$. Now it is evident, that $\left(e^{\frac{d}{dx}} - 1\right)^{-n}$ does in fact represent such a series; for since

$$\left(e^{\frac{d}{dx}} - 1\right)^n \times \left(e^{\frac{d}{dx}} - 1\right)^{-n} = 1;$$

this equation must also hold good when both factors are developed in powers of d, and consequently the developement of the latter must be such as exactly to destroy all the powers of d in that of the former, and reduce it to unity, which prefixed to u_x, gives simply u_x. If then Σ^n be at all capable of representation by any series of operations denoted by d and its positive and negative powers, that series

can be no other than the developement of $\left(e^{\frac{d}{dx}}-1\right)^{-n}$. Now it is easily shewn, that *it is* capable of being so represented. Suppose for instance $n=1$, and since

$$u_x=\frac{1}{1}\frac{du_x}{dx}+\frac{1}{1.2}\frac{d^2u_x}{dx^2}+\text{\&c.}$$

if we write Σu_x for u_x, we get

$$u_x=\frac{1}{1}\frac{d\Sigma u_x}{dx}+\frac{1}{1.2}\frac{d^2\Sigma u_x}{dx^2}+\text{\&c.}$$

which equation is evidently capable of being satisfied by such an expression of Σu_x as the following:

$$\Sigma u_x=A\int u_x dx+Bu_x+C\frac{du_x}{dx}+\text{\&c.}$$

$$=\left\{A\left(\frac{d}{dx}\right)^{-1}+B+C.\frac{d}{dx}+\text{\&c.}\right\}u_x$$

provided A, B, C, &c. are properly assumed. Again, since we have

$$\Delta^2 u_x=a.\frac{d^2u_x}{dx^2}+b.\frac{d^3u_x}{dx^3}+\text{\&c.}$$

where a, b, c, &c. are the coefficients of the developement of $\left(e^{\frac{d}{dx}}-1\right)^2$, if we put $\Sigma^2 u_x$ for u_x, we get

$$u_x=a.\frac{d^2\Sigma^2 u_x}{dx^2}+b.\frac{d^3\Sigma^2 u_x}{dx^3}+c.\text{\&c.}$$

which in like manner is capable of being satisfied by

$$\Sigma^2u_x=\left\{A\left(\frac{d}{dx}\right)^{-2}+B\left(\frac{d}{dx}\right)^{-1}+C+D.\frac{d}{dx}+\text{\&c.}\right\}u_x$$

and so on in general.

If the reader should not immediately see the force of the above reasoning, we would recommend him to actually perform the operation of determining A, B, C, &c. in the expression

$$\Sigma u_x=A\int u_x dx+Bu_x+C.\frac{du_x}{dx}+\text{\&c.}$$

by substitution in the equation

$$u_x = \frac{1}{1}\frac{d\Sigma u_x}{dx} + \&c.$$

which he will see is precisely the same as in the determination of a *quantity* Σ *in powers of another*, d, from the equation

$$1 = \frac{d}{1}\Sigma + \frac{d^2}{1.2}\Sigma + \&c.$$

by assuming for Σ a series with indeterminate coefficients,

$$\Sigma = A.\frac{1}{d} + B + Cd + \&c.$$

and substitution. We will accompany him one or two steps in the process

$$u_x = \frac{1}{1}\left(Au_x + B\frac{du_x}{dx} + C.\&c.\right.$$

$$\left.+\frac{1}{1.2}\left(A\frac{du_x}{dx} + B.\&c.\right)\right.$$

whence

$$\frac{A}{1} = 1,^* \; A = 1, \; \frac{B}{1} + \frac{A}{1.2} = 0, \; B = -\frac{1}{2}, \&c.$$

and the substitution of $\Sigma = Ad^{-1} + B + Cd + \&c.$ in the above equation between Σ and d gives

---

* It is easily proved, that if $Au_x + B.\frac{du_x}{dx} + C.\frac{d^2u_x}{dx^2} + \&c.$ $= au_x + b.\frac{du_x}{dx} + c.\frac{d^2u_x}{dx^2} + \&c.$ whatever be the form of the function u_x, then $A=a$, $B=b$, $C=c$, &c. for since the equation is true independent of any particular form of u_x, let $u_x = e^{zx}$, and dividing both sides of the resulting equation by e^{zx}, we get $A + Bz + Cz^2 + \&c. = a + bz + cz^2 + \&c.$ which being true independent of any particular value of z, gives $A=a$, $B=b$, &c.

$$1 = \frac{1}{1}(A + Bd + \&c.)$$
$$+ \frac{1}{1.2}(Ad + \&c.)$$
$$+ \&c.$$

which plainly leads to the same equations. Now this latter operation is the same with finding Σ from the equation

$$1 = \Sigma(e^d - 1), \text{ or } \Sigma = \frac{1}{e^d - 1},$$

and the coefficients A, B, &c. are therefore those of the powers of d in the developement of $(e^d - 1)^{-1}$. The like reasoning will apply to the general equation.

Lagrange's theorem is therefore proved to hold good for negative as well as positive values of n. When $n = -1$, we have

$$\Sigma u_x = \frac{1}{e^{\frac{d}{dx}} - 1} u_x.$$

Let $(e^t - 1)^{-1}$ be developed in powers of t, and a series will be found as follows:

$$(e^t - 1)^{-1} = t^{-1} - \frac{1}{2} + \frac{1}{2} \cdot \frac{t}{1.2.3} - \frac{1}{6} \cdot \frac{t^3}{1.2.3.4.5} + \frac{1}{6} \cdot \frac{t^5}{1...7} -$$

the numerical coefficients being the same as in the expression for $\Sigma . x^n$. (369), and we therefore have

$$\Sigma u_x = \int u_x dx - \frac{u_x}{2} + \frac{1}{2} \cdot \frac{du_x}{1.2.3.dx} - \frac{1}{6} \cdot \frac{d^3 u_x}{1...5.dx^3} + \&c.$$

a formula of the most extensive use (as we shall hereafter see) in the numerical computation of certain functions of very high numbers.

On the Integration of Equations of Differences.

379. Having discussed the case when the difference of a function is immediately given in terms of x, we now proceed to consider those in which a certain relation only between x, the function sought, and one or more of its differences, or successive values (367) is given, and which, as we have there remarked, may be expressed in general by an equation of the form

$$0 = F(x, u_x, u_{x+1}, \dots u_{x+n}); \dots\dots\dots\dots (a)$$

n being the order of the equation.

Suppose u_x to contain besides x any number n of constants, $a, b, c, \dots k$, so that

$$u_x = \phi\left\{x, a, b, c, \dots k\right\}$$

shall be its expression. If from this we form the values of $u_{x+1}, u_{x+2}, \dots u_{x+n}$, we shall produce the equations

$$u_{x+1} = \phi\left\{x+1, a, b, \dots k\right\}$$
$$u_{x+2} = \phi\left\{x+2, a, b, \dots k\right\}$$
$$\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots$$
$$u_{x+n} = \phi\left\{x+n, a, b, \dots k\right\}$$

Now as these equations hold good simultaneously, and their number is $n+1$, we may eliminate from them the n quantities $a, b, \dots k$, and our final equation, which will be of the form (a), will be totally independent of them, and this without assigning to them any particular values whatever. As far then as this equation is concerned, these values are perfectly arbitrary, and consequently, in reascending from it to the expression of u_x, it appears, that to give the latter all its generality, or to obtain the complete inte-

gral, n arbitrary constants must be introduced as in the theory of differential equations.

380. In this latter theory, the particular solutions of any proposed equation necessarily contains fewer arbitrary constants than the complete integral (294). This, however, as Charles has shewn, is not the case with equations of differences, solutions of which may, in certain cases, be derived (by a process analogous to that which gives the particular solutions of a differential equation) containing as many arbitrary constants as the complete integral itself, from which they were deduced.

This will appear, for the first order, by considering that provided the equations

$$u_x=\phi(x,a),\quad u_{x+1}=\phi(x+1,a)$$

hold good at once, the elimination of a will produce the same equation,

$$0=F(x,u_x,u_{x+1}),$$

whether we suppose a constant, or a function of x (as a_x); now, on this latter supposition, since $u_x=\phi(x,a_x)$, and therefore $u_{x+1}=\phi(x+1,a_{x+1})$ we ought to have

$$\phi(x+1,a_{x+1})=\phi(x+1,a_x), \text{ or}$$

$$\phi(x+1,a_{x+1})-\phi(x+1,a_x)=0. \qquad (a)$$

This equation is satisfied by the supposition that $a_{x+1}=a_x$, or $a_x=$const. so that $a_{x+1}-a_x$, or some power of this, will necessarily be one factor of $\phi(x+1,a_{x+1})-\phi(x+1,a_x)$: but, besides this, there may be others of the form $f(x,a_x,a_{x+1})$ which do not vanish by the supposition of $a_x=$const. Now any one of these factors being put equal to zero, will satisfy the equation (a) as well as $a_{x+1}-a_x=0$. Suppose then

$$f(x,a_x,a_{x+1})=0,$$

and as this is an equation of differences of the first order

for determining a_x, the complete expression of this function *must contain one arbitrary constant*, b; so that

$$a_x=\psi(x, b) \text{ and } u_x=\phi\big(x, \psi(x, b)\big),$$

which is essentially different from the former $\phi(x, a)$ and contains an arbitrary constant; and the same may be said of all the other factors into which the equation (a) can be resolved.

Let us take, for instance, the equation

$$u_x=x(u_{x+1}-u_x)+(1-u_{x+1}+u_x)^2,$$

one complete integral of which (as substitution will prove) is

$$u_x=x(1-a)+a^2,$$

a being an arbitrary constant. If now we suppose a (or a_x) a function of x, such that

$$(x+1)(1-a_x)+a_x^2=(x+1)(1-a_{x+1})+a^2_{x+1},$$

we shall find by reduction

$$0=(a_{x+1}-a_x)(a_{x+1}+a_x-x-1),$$

in which the first factor being made equal to zero, we get $\Delta a_x=0$, or a_x=const. but, if we suppose the other to vanish, we have

$$a_{x+1}+a_x-(x+1)=0,$$

an equation which is satisfied by the following:

$$a_x=\frac{x}{2}+\frac{1}{4}+b.(-1)^x,$$

b being any arbitrary constant. Let this be substituted for a in the expression above given for u_x, and it becomes

$$u_x=\frac{1}{16}+x-\frac{x^2}{4}+b^2+\frac{b}{2}(-1)^x,$$

which value, upon trial, satisfies the proposed equation, and contains an arbitrary constant. There is no reason therefore why it should not be regarded as another complete inte-

gral of the proposed equation (although the analogy between the method by which it was obtained, and that by which particular solutions are found in the Differential Calculus may perhaps cause some to prefer the latter name), especially since this solution, treated in the same manner, by supposing b a function of x, *brings us back again to the original expression for* u_x, *from which we set out.* Thus there is the same reason for considering that expression in the light of a particular solution, as the other. The subject is an inseresting one; but we cannot, in an essay like the present, pursue it farther.

381. We proceed to the integration of equations of differences; and first,

$$u_{x+1} - A_x u_x = B_x,$$

the general equation of the first order and degree, is completely integrable, A_x and B_x being any given functions of x, by supposing

$$u_x = v_x \,.\, A_1 \,.\, A_2 \,...\, A_{x-1}.$$

As the product of all the successive values of a function A_x, beginning with some fixed term, as A_1, or more generally, A_n (n being independent of x) occurs very often in the theory of equations of differences, it will be necessary to use a particular notation to designate it, and for this purpose we shall employ the capital P, thus

$$A_1 \,.\, A_2 \,...\, A_{x-1} \,.\, A_x = P\,A_x$$

$$A_1 \,.\, A_2 \,...\,...\, A_{x-1} = P\,A_{x-1}, \text{ \&c.}$$

PA_x being considered as a function of x, derived according to this particular law from A_x, and having PA for its characteristic. If A_x be constant, and equal to A, we have

$$PA_x = A \,.\, A \,.\, A \,... \text{(to } x \text{ terms)} = A^x.$$

To return now to our equation of differences: the supposition

$$u_x = v_x \cdot PA_{x-1},$$

gives

$$v_{x+1} \cdot PA_x - v_x \cdot A_x \cdot PA_{x-1} = B_x;$$

but $A_x \cdot PA_{x-1} = PA_x$, by the definition above given, and thus it becomes

$$(v_{x+1} - v_x) \cdot PA_x = B_x,$$

whence

$$v_{x+1} - v_x \text{ (or } \Delta v_x) = \frac{B_x}{PA_x},$$

and we have, therefore,

$$v_x = C + \Sigma \frac{B_x}{PA_x}$$

$$u_x = \left\{ C + \Sigma \frac{B_x}{PA_x} \right\} \cdot PA_{x-1},$$

C being an arbitrary constant.

General Theory of Equations of the first Degree.

382. The general Theory of Equations of the first Degree and any order, bears a striking analogy to that of Differential Equations of the same description. Suppose

$$u_{x+n} - {}^1A_x \cdot u_{x+n-1} + {}^2A_x u_{x+n-2} \dots (\pm)^n A_x u_x = B_x, \dots (a)$$

to be any such equation. We shall first demonstrate, *that its integration is reducible to that of the same equation, deprived of its last term* B_x, in such a manner, that if V_x be the complete expression for u_x in the equation

$$u_{x+n} - {}^1A_x u_{x+n-1} + \dots \pm {}^nA_x u_x = 0, \dots\dots (b)$$

its general expression in (a) may be obtained by adding to V_x a certain function of x, which can always be determined, provided (b) is integrable: and, secondly, *that the equation* (b) *so deprived of its last term is always integrable, provided we*

can by any means obtain n − 1 *particular integrals, or functions of* x, *which satisfy it, essentially different from each other.* In demonstrating these propositions, we will suppose $n=3$ to avoid the great complexity of the expressions which an indeterminate value of n would induce; but every step of the process is equally applicable to any value of n.

Suppose then ${}^1\alpha_x$, ${}^2\alpha_x$, ${}^3\alpha_x$, to be n functions of x, whose form is at present unknown, and we may suppose the equation

$$u_{x+3} - {}^1A_x u_{x+2} + {}^2A_x u_{x+1} - {}^3A_x = B_x\,,$$

to have arisen from the elimination of two $(n-1)$ functions, ${}^1u_x$ and ${}^2u_x$ between three equations,

$$u_{x+1} - {}^1\alpha_x u_x = {}^1u_x \,\ldots\ldots (1)$$

$$ {}^1u_{x+1} - {}^2\alpha_x {}^1u_x = {}^2u_x \,\ldots\ldots (2)$$

$$ {}^2u_{x+1} - {}^3\alpha_x {}^2u_x = B_x \,\ldots\ldots (3 = n)$$

for, if we substitute in (2) the values of ${}^1u_x$ and ${}^1u_{x+1}$, deduced from (1), we find

$$ {}^2u_x = u_{x+2} - ({}^1\alpha_{x+1} + {}^2\alpha_x)\, u_{x+1} + {}^1\alpha_x\, {}^2\alpha_x,$$

and again substituting this for ${}^2u_x$, in the equation (3), and for ${}^2u_{x+1}$, its value obtained by writing $x+1$ for x, it becomes $u_{x+3} - \left\{ {}^1\alpha_{x+2} + {}^2\alpha_{x+1} + {}^3\alpha_x \right\} u_{x+2}$

$$+ \left\{ {}^1\alpha_{x+1} \cdot {}^2\alpha_{x+1} + {}^1\alpha_{x+1} \cdot {}^3\alpha_x + {}^2\alpha_x \cdot {}^3\alpha_x \right\} u_{x+1}$$

$$- {}^1\alpha_x \cdot {}^2\alpha_x \cdot {}^3\alpha_x \cdot u_x = B_x,$$

which is of the same form with (a), and will coincide with it altogether, if ${}^1\alpha_x$, ${}^2\alpha_x$, ${}^3\alpha_x$ are determined; so that

$$\left.\begin{aligned} {}^1A_x &= {}^1\alpha_{x+2} + {}^2\alpha_{x+1} + {}^3\alpha_x \\ {}^2A_x &= {}^1\alpha_{x+1} \cdot {}^2\alpha_{x+1} + {}^1\alpha_{x+1} \cdot {}^3\alpha_x + {}^2\alpha_x \cdot {}^3\alpha_x \\ {}^3A_x &= {}^1\alpha_x \cdot {}^2\alpha_x \cdot {}^3\alpha_x\,. \end{aligned}\right\}$$

Now, first, as these equations do not contain B_x, the determination of ${}^1\alpha_x$, &c. from them, is the same whatever be the form of B_x, and therefore the same as if $B_x = 0$, or

the equation (a) were deprived of its last term; but, secondly, when ${}^1\alpha_x$, &c. are once obtained, the equation (a) is readily integrated; for we have, by integrating the equations (1), (2), (3), by the last No.

$$u_x = P\,{}^1\alpha_{x-1} \cdot \left\{ {}^1C + \Sigma \frac{{}^1u_x}{P\,{}^1\alpha_x} \right\}$$

$${}^1u_x = P\,{}^2\alpha_{x-1} \cdot \left\{ {}^2C + \Sigma \frac{{}^2u_x}{P\,{}^2\alpha_x} \right\}$$

$${}^2u_x = P\,{}^3\alpha_{x-1} \cdot \left\{ {}^3C + \Sigma \frac{B_x}{P\,{}^3\alpha_x} \right\}$$

${}^1C$, ${}^2C$, ${}^3C$ being three arbitrary constants, and by substituting first for ${}^1u_x$ its value given in the second of these equations, and then for ${}^2u_x$ its value given in the third, we find a result of the following form:

$$u_x = {}^1U_x \,.\, {}^1C + {}^2U_x \,.\, {}^2C + {}^3U_x \,.\, {}^3C + Q_x,$$

where (as is easily seen, by executing the operations)

$$\left.\begin{aligned} {}^1U_x &= P\,{}^1\alpha_{x-1} \\ {}^2U_x &= P\,{}^1\alpha_{x-1}\, \Sigma \frac{P\,{}^2\alpha_{x-1}}{P\,{}^1\alpha_x} \\ {}^3U_x &= P\,{}^1\alpha_{x-1}\, \Sigma \frac{P\,{}^2\alpha_{x-1}}{P\,{}^1\alpha_x}\, \Sigma \frac{P\,{}^3\alpha_{x-1}}{P\,{}^2\alpha_x} \end{aligned}\right\}, \quad (c)$$

$$Q_x = P\,{}^1\alpha_{x-1}\, \Sigma \,.\, \frac{P\,{}^2\alpha_{x-1}}{P\,{}^1\alpha_x}\, \Sigma \,.\, \frac{P\,{}^3\alpha_{x-1}}{P\,{}^2\alpha_x}\; \Sigma\, \frac{B_x}{P\,{}^3\alpha_x}, \quad (d)$$

the full point after the Σ extending its operation over all which follows it. The whole difficulty of determining the complete integral of (a) is therefore reduced to the discovery of these functions, ${}^1\alpha_x$, ${}^2\alpha_x$, ${}^3\alpha_x$, to effect which, let us suppose $B_x = 0$, which, as we have seen, does not influence their values. This gives $Q_x = 0$, and

$$u_x = {}^1U_x \,.\, {}^1C + {}^2U_x \,.\, {}^2C + {}^3U_x \,.\, {}^3C,$$

which is the complete integral of (b), and where the values of ${}^1U_x$ &c. are the same as before.

Suppose now that the equation (b)

$$0=u_{x+3}-{}^1A_x.u_{x+2}+{}^2A_x.u_{x+1}-{}^3A_xu_x$$

is integrable; then the functions ${}^1U_x$, ${}^2U_x$, ${}^3U_x$, which multiply the three arbitrary constants in its integral, are known; and from these ${}^1\alpha_x$, ${}^2\alpha_x$, ${}^3\alpha_x$, are at once determined by the equations (c), which give

$${}^1\alpha_x=\frac{{}^1U_{x+1}}{{}^1U_x},$$

$${}^2\alpha_x=\frac{{}^1U_{x+2}}{{}^1U_{x+1}}\cdot\frac{\Delta\left\{\frac{{}^2U_{x+1}}{{}^1U_{x+1}}\right\}}{\Delta\left\{\frac{{}^2U_x}{{}^1U_x}\right\}},$$

and a similar expression may be obtained for ${}^3\alpha_x$, more complicated, however, for which reason we have omitted to set it down.

Thus ${}^1\alpha_x$, ${}^2\alpha_x$, ${}^3\alpha_x$, are obtained, and these being substituted in Q_x, give the value of that function, in terms of x, which being added to the complete integral of (b), gives that of (a).

But if by any means we can find $n-1$ of the functions ${}^1U_x$, &c, (that is, $n-1$ particular integrals of (b), as is evident, since when ${}^2C$, ${}^3C$, &c. $=0$, and ${}^1C=1$, the general integral reduces itself to ${}^1U_x$), the nth may be derived from them, and the equation (a), in consequence integrated, as follows: having, by means of ${}^1U_x$ and ${}^2U_x$ $(={}^{n-1}U_x)$, which are known, determined the values of ${}^1\alpha_x$ and ${}^2\alpha_x$, as above; we may employ either of the equations

$${}^1A_x={}^1\alpha_{x+2}+{}^2\alpha_{x+1}+{}^3\alpha_x,\quad {}^3A_x={}^1\alpha_x.{}^2\alpha_x.{}^3\alpha_x,$$

to give that of ${}^3\alpha_x$, and having this, it is only requisite to substitute these expressions of ${}^1\alpha_x$, ${}^2\alpha_x$, ${}^3\alpha_x$, in the last of the equations (c), and in (d), to get the values of ${}^3U_x$ and Q_x, which substituted in the expression

$$^1U_x\,.\,^1C+{}^2U_x\,.\,^2C+{}^3U_x\,.\,^3C+Q_x,$$

give the complete integral of (a), as before.

383. Suppose all the coefficients of the proposed equation constant, and $B_x=0$, or let

$$0=u_{x+3}-{}^1Au_{x+2}+{}^2Au_{x+1}-{}^3Au_x,$$

and it is evident, that the equations (e) will be satisfied by supposing $^1\alpha_x$, $^2\alpha_x$, $^3\alpha_x$ constant, which gives

$$^1A={}^1\alpha+{}^2\alpha+{}^3\alpha,\quad {}^2A={}^1\alpha\,.\,^2\alpha+{}^1\alpha\,.\,^3\alpha+{}^2\alpha\,.\,^3\alpha,\quad {}^3A={}^1\alpha\,.\,^2\alpha\,.\,^3\alpha;$$

hence these quantities are given at once, being the roots of the equation

$$0=u^3-{}^1A\,u^2+{}^2A\,u-{}^3A,$$

and since $P^1\alpha_{x-1}$ in this case $={}^1\alpha^{x-1}$, we find by (c)

$$^1U_x={}^1\alpha^{x-1},\quad {}^2U_x=\frac{^2\alpha^{x}-1}{(^2\alpha-{}^1\alpha)},\quad {}^3U_x=\frac{^3\alpha^{x}-1}{(^3\alpha-{}^2\alpha)(^3\alpha-{}^1\alpha)};$$

but, as ^1C, ^2C, ^3C, are arbitrary, we are at liberty to write instead of them, the following:

$$^1\alpha\,.\,^1C,\quad {}^2\alpha\,(^2\alpha-{}^1\alpha)\,{}^2C,\quad {}^3\alpha\,(^3\alpha-{}^2\alpha)\,(^3\alpha-{}^1\alpha)\,{}^3C,$$

and we have, for the value of u_x,

$$u_x={}^1C\,.\,^1\alpha^x+{}^2C\,.\,^2\alpha^x+{}^3C\,.\,^3\alpha^x,$$

and the same may be shewn for the general equation of the nth order, with constant coefficients. Thus, if

$$0={}_{x+n}-{}^1A\,.\,u_{x+n-1}\,\ldots\,\pm\,{}^nA\,.\,u_x,$$

we shall have

$$u_x={}^1C\,.\,^1\alpha^x+{}^2C\,.\,^2\alpha^x+\,.\,.\,.\,{}^nC\,.\,^n\alpha^x,$$

where $^1\alpha$, &c. are the n roots of

$$0=u^n-{}^1A\,.\,u^{n-1}+\,\ldots\,\pm{}^nA.$$

384. Should this equation have equal roots, suppose $^2\alpha={}^1\alpha+k$, and we get

$$u_x=(^1C+{}^2C)\,{}^1\alpha^x+{}^2C\,(k\,x\,.\,^1\alpha^{x-1}+k^2\,.\,\&c.)+{}^3C,\ \&c.$$

or, writing ${}^1C$ for ${}^1C+{}^2C$, and ${}^2C$ for $\frac{k\,{}^2C}{{}^1\alpha}$, (since ${}^1C$ and ${}^2C$ are arbitrary), and then making k vanish,

$$u_x=({}^1C+{}^2C\,x)\,.\,{}^1\alpha^x+{}^3C\,.\,{}^3\alpha^x+\&c.$$

and if three roots are equal, we shall, in like manner, find

$$u_x=({}^1C+{}^2C\,x+{}^3C\,x^2)\,.\,{}^1\alpha^x+\&c.$$

just as in the Differential Calculus. See (281). The proper number of arbitrary constants is thus restored, and the expression for u_x so found is of course the complete integral.

385. The above equation, as well as the more general one

$$u_{x+n}-{}^1A\,.\,u_{x+n-1}\,.\,.\,.\,\pm{}^nA\,.\,u_x=B,$$

may also be integrated by assuming

$$u_x=\alpha^x+K.$$

For this, by substitution, gives

$$\left\{\alpha^{x+n}-\,.\,.\,.\,\pm{}^nA\,.\,\alpha^x\right\}+K\,(1-{}^1A+\,.\,.\,.\,\pm{}^nA)-B=0.$$

Suppose now

$$K=\frac{B}{1-{}^1A+\,.\,.\,.\,\pm{}^nA},$$

and the constant part disappears of itself. The remaining part being divided by α^x, gives

$$\alpha^n-{}^1A\,.\,\alpha^{n-1}+\,...\,\pm{}^nA=0,$$

an equation of the nth degree, whose roots being called α, β, ... μ, it is evident that each of the expressions α^x, β^x, ... μ^x, satisfies the equation

$$u_{x+n}-{}^1A\,.\,u_{x+n-1}+\,.....\,\pm{}^nA\,.\,u^x=0.$$

These are, in consequence, the n particular integrals of that equation, and of course

$${}^1C\,.\,\alpha^x+{}^2C\,.\,\beta^x+\,.....\,{}^nC\,.\,\mu^x$$

is its complete integral. Consequently,

$$^1C \,.\, \alpha^x + \ldots {}^nC \,.\, \mu^x + \frac{B}{1 - {}^1A + \ldots \pm {}^nA}$$

is the complete integral of the proposed equation; for, the quantity to be added to $^1C \,.\, \alpha^x + \ldots {}^nC \,.\, \mu^x$ being, as we have shewn, independent of the arbitrary constants, must be the same as when $^1C = 1, {}^2C = {}^3C = \&c. = 0$, and therefore the same as we have denoted by K, and this result is easily verified by actual substitution.

The equation of the first degree and second order is generally integrable, provided we admit as given the evaluation of a continued fraction whose terms follow any given law, their number being variable *.

386. Of equations of differences beyond the first degree very little is known. The equation with constant coefficients

$$0 = u_{x+1}\, u_x - a\, u_{x+1} + b\, u_x + c$$

is integrable, by supposing

$$u_x = \frac{v_{x+1}}{v_x} + K,$$

which being substituted, and the result multiplied by v_x, gives

$$0 = v_{x+2} + (K + b)\, v_{x+1} + (K^2 - Ka + Kb + c)\, v_x$$
$$+ (K - a) \,.\, \frac{v_{x+2}}{v_{x+1}}\, v_x,$$

and if we take $K - a = 0$, or $K = a$, we get

* The English reader may consult a paper "on Equations of Finite Differences" by the Author of this Appendix, published in the Memoirs of the Analytical Society for 1813, where the theory of this case, founded on a process delivered by Laplace in the Mécanique Céleste, is detailed at length.

$$0 = v_{z+2} + (a+b)\, v_{z+1} + (ab+c)\, v_z,$$

an equation integrable by (385). Let $p\,.\,\alpha^z + q\,.\,\beta^z$ be its complete integral, p and q being arbitrary constants, and we have

$$u_z = \frac{\alpha^{z+1} + \left(\frac{q}{p}\right)\beta^{z+1}}{\alpha^z + \left(\frac{q}{p}\right)\beta^z} + a;$$

or, since $\frac{q}{p}$ is equivalent only to one arbitrary constant C,

$$u_z = \frac{\alpha^{z+1} + C\,.\,\beta^{z+1}}{\alpha^z + C\,.\,\beta^z} + a.$$

On Equations of mixed Differences.

387. Equations of mixed differences, in which the successive values and their differential coefficients rise no higher than the first degree, and where the coefficients of the several terms are constant, are integrable by the same artifice as we employed in (385). Their general form is

$$0 = u_z + A\,u_{z+1} + B u_{z+2} + \&c. + K$$
$$+ \frac{d\{C u_z + D\,u_{z+1} + \&c.\}}{dx} + \frac{d^2\{E u_z + \&c.\}}{dx^2} + \&c.$$

To take, however, a simple case, suppose

$$0 = u_z + A u_{z+1} + B\frac{du_z}{dx} + C\frac{du_{z+1}}{dx} + K.$$

The substitution of $v_z + H$ for u_z first gives a result, which by supposing

$$K + H(1+A) = 0, \text{ or } H = -\frac{K}{1+A}$$

reduces itself to

$$0 = v_z + A v_{z+1} + B\frac{dv_z}{dx} + C\frac{dv_{z+1}}{dx},$$

which is satisfied by supposing $v_x = e^{kx}$, since this being substituted produces (after dividing the whole by e^{kx})

$$o = 1 + A e^k + B k + C k e^k.$$

Let α, β, &c. be the values of k, or the roots of this equation, and it is easily seen that the expression

$$v_x = {}^1C . \alpha^x + {}^2C . \beta^x + \&c.$$

satisfies the condition, whence it appears that

$$u_x = {}^1C . \alpha^x + {}^2C . \beta^x + \&c. - \frac{K}{1 + A}.$$

388. Certain equations, in which the coefficients are variable, may be reduced to the above form by a very simple substitution.

Thus, if in the equation

$$o = u_x - (1 + a . e^x) \frac{d u_x}{d x} + b (1 + a . e^{x+1}) u_{x+1} + c,$$

in which e is the number whose logarithm is unity, if for u_x we put $X_x . v_x$, we have

$$o = v_x \left\{ X_x - (1 + a e^x) \frac{d X_x}{d x} \right\} - (1 + a e^x) X_x \frac{d v_x}{d x}$$

$$+ b (1 + a e^{x+1}) X_{x+1} v_{x+1} + c.$$

Suppose now X_x determined by the differential equation

$$X_x - (1 + a e^x) \frac{d X_x}{d x} = 1,$$

or, separating the variables,

$$\frac{d X_x}{X_x - 1} = \frac{d x}{1 + a e^x}:$$

a particular integral of this is

$$X_x = \frac{1}{1 + a e^x},$$

which, substituted in the transformed equation, gives

$$o = v_x - \frac{d v_x}{d x} + b v_{x+1} + c,$$

which is of the form above considered.*

On the Application of the Calculus of Differences to the Summation of Series.

389. If the sum of the x first terms of any progression

$$u_1, \quad u_2, \quad u_3, \ldots u_x$$

be required, two cases present themselves, one where the general term is explicitly given in functions of the index x; the other where only certain relations between the consecutive terms, or these and their indices are expressed. In the first case, if we suppose

$$S_x = u_1 + u_2 + \ldots\ldots u_x,$$

we have

$$S_{x+1} = u_1 + u_2 + \ldots u_x + u_{x+1},$$

and subtracting,

$$S_{x+1} - S_x (= \Delta S_x) = u_{x+1};$$

whence we find

$$S_x = \Sigma u_{x+1} + \text{const.}$$

Now, in this, making $x = o$, we find

$$S_0 = o = \Sigma u_1 + \text{const.}$$

Σu_1 denoting the value which the function Σu_{x+1} takes when $x = o$, and subtracting this from the former,

$$S_x = \Sigma u_{x+1} - \Sigma u_1.$$

* On the subject of equations of differences, involving more than one independent variable, the reader is referred to a paper by Laplace, in the Mem. des. Savans Etrangers, 1773.

If the sum be required between the nth and xth terms, we have, in like manner,

$$S_x - S_n = \Sigma u_{x+1} - \Sigma u_{n+1},$$

for the value of the series

$$u_{n+1} + u_{n+2} + \ldots\ldots u_x.$$

Ex. 1. In the arithmetical progression

$$a, \quad a+b, \quad a+2b, \ldots a+(x-1)b,$$

we have $u_{x+1} = a + xb$, and $\Sigma u_{x+1} = xa + \frac{x(x-1)}{1.2} b + C$,

whence $\Sigma u_1 = 0 + C$, and the sum of the x first terms is therefore

$$\Sigma u_{x+1} - \Sigma u_1 = xa + \frac{x(x-1)}{2} b.$$

Ex. 2. The sum of the x first terms of any progression of figurative numbers being required: the first order consists of the series $1+1+1+$ &c. in which $u_x = 1$, and $\Sigma u_{x+1} - \Sigma u_1 = x$, the sum of its x first terms, which, by the construction of these numbers is the general term, or the value of u_x in the second order,

$$1+2+3+ \ldots x.$$

Consequently, for this order, $u_{x+1} = x+1$,

$$\Sigma u_{x+1} = \frac{x(x+1)}{1.2} + C, \quad \Sigma u_1 = 0 + C,$$

$$S_x = \Sigma u_{x+1} - \Sigma u_1 = \frac{x(x+1)}{1.2},$$

which is consequently the expression for u_x in the third order, and therefore

$$u_{x+1} = \frac{(x+1)(x+2)}{1.2},$$

which gives for S_x in the third order

$$\Sigma u_{x+1} - \Sigma u_1 = \frac{x(x+1)(x+2)}{1.2.3},$$

and so on.

Ex. 3. The series proposed being the geometrical progression

$$a + ab + \dots ab^{x+1},$$

we have

$$u_{x+1} = ab^x, \quad \Sigma u_{x+1} = a \cdot \frac{b^x}{b-1} + C, \text{ and therefore}$$

$$\Sigma u_{x+1} - \Sigma u_1, \text{ or } S_x = a \cdot \frac{b^x - 1}{b-1},$$

the sum required.

Ex. 4. Let $S_x = 1^3 + 2^3 + 3^3 + \dots x^3$.

Here,

$$\Sigma u_{x+1} = \Sigma (x+1)^3 = \Sigma \left\{ (x+1)^2 \cdot x + (x+1)^2 \right\} =$$

$$= \Sigma \left\{ x(x+1)(x+2) - x(x+1) + x(x+1) + (x+1) \right\}$$

$$= \frac{(x-1)x(x+1)(x+2)}{4} - \frac{x(x+1)}{2} + C,$$

and since $\Sigma u_1 = 0 + C$, the sum required, or

$$\Sigma u_{x+1} - \Sigma u_1 = \left\{ \frac{x(x+1)}{2} \right\}^2.$$

Thus it appears, that

$$1^3 + 2^3 + \dots x^3 = (1 + 2 + \dots x)^2,$$

which is by no means an inelegant property of this series.

Ex. 5.

$$S_x = 1^2 - 2^2 + 3^2 - 4^2 \dots \pm x^2.$$

In this case,

$$\Sigma u_{x+1} = \Sigma (-1)^x \cdot (x+1)^2,$$

and we must use the formula

$$\Sigma v_x \cdot a^x = v_x \cdot \frac{a^x}{a-1} - \Delta v_x \cdot \frac{a^{x+1}}{(a-1)^2} + \Delta^2 v_x \cdot \frac{a^{x+2}}{(a-1)^3} - \&c.$$

where $v_x = (x+1)^2$, $\Delta v_x = 2x+3$, $\Delta^2 v_x = 2$, $a = -1$.
We find then, after all reductions,

$$\Sigma u_{x+1} = (-1)^{x+1} \cdot \frac{x(x+1)}{2} + C, \quad \Sigma u_1 = 0 + C,$$

and therefore

$$S_x = (-1)^{x+1} \cdot \frac{x(x+1)}{2}.$$

Ex. 6. If $S\,x = \frac{1}{1.2} + \frac{1}{2.3} + \ldots \frac{1}{x(x+1)}$,
we have,

$$\Sigma u_{x+1} = \Sigma \frac{1}{(x+1)(x+2)} = C - \frac{1}{x+1}$$

$$\Sigma u_1 = \qquad C - \frac{1}{1},$$

and therefore

$$S_x = 1 - \frac{1}{x+1} = \frac{x}{x+1}.$$

Ex. 7. Suppose

$$S_x = \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \&c.$$

Here

$$\Sigma u_{x+1} = \Sigma \frac{1}{(x+1)(x+3)} = \frac{(x+3)-1}{(x+1)(x+2)(x+3)} =$$

$$= \Sigma \left\{ \frac{1}{(x+1)(x+2)} - \frac{1}{(x+1)(x+2)(x+3)} \right\} =$$

$$= C - \frac{1}{x+1} + \frac{1}{2(x+1)(x+2)};$$

whence we obtain, after all reductions,

$$S_x = \frac{x(3x+5)}{4(x+1)(x+2)}.$$

Ex. 8. Let

$$S_x = \frac{2}{1.3.7} + \frac{4}{3.7.15} + \frac{8}{7.15.31} + \&c.$$

to x terms, and we have, in this case,

$$\Sigma u_{x+1} = \Sigma \frac{2^{x+1}}{(2^{x+1}-1)(2^{x+2}-1)(2^{x+3}-1)}$$

$$= C - \frac{1}{3(2^{x+1}-1)(2^{x+2}-1)},$$

and

$$\Sigma u_1 = C - \frac{1}{9};$$

whence, subtracting,

$$S_x = \frac{1}{9} - \frac{1}{3(2^{x+1}-1)(2^{x+2}-1)}.$$

Ex. 9. If

$$S_x = \frac{1^2}{2.3.4.6} + \frac{2^2}{3.4.5.7} + \text{\&c. } (x \text{ terms}).$$

In this case,

$$\Sigma u_{x+1} = \Sigma \frac{(x+1)^2}{(x+2)(x+3)(x+4)(x+6)}.$$

Now, $(x+1)^2 = (x+4)(x+6) - (8x+23)$;

also, $8x+23 = 8(x+6) - 25$;

whence $(x+1)^2 = (x+4)(x+6) - 8(x+6) + 25$,

which, substituted, gives

$$\Sigma u_{x+1} = \Sigma \frac{1}{(x+2)(x+3)} - 8 . \Sigma \frac{1}{(x+2)(x+3)(x+4)} +$$

$$+ 25 . \Sigma \frac{x+5}{(x+2) \ldots \ldots \ldots (x+6)},$$

the last term of which resolves itself into two integrable fractions, by substituting $(x+6)-1$ for $x+5$, in the numerator, and we get

$$S_x = \frac{1}{1}\left\{\frac{1}{2} - \frac{1}{x+2}\right\} - \frac{8}{2}\left\{\text{\&c.}\right\} + \text{\&c.}$$

On recurring Series.

390. The general term of a series may, as we have already observed (389), be either given explicitly, in functions of the index; or it may be determined implicitly by some relations capable of being expressed by an equation, in the same way as the nature of a curve is definable, either by at once expressing its ordinate in functions of its abscissa, or by assigning some equation, either algebraic, differential, or other, between them. We propose, in this article, to consider the nature of series in which an equation of the first degree, with constant coefficients, holds good between a certain definite number of consecutive terms, in whatsoever part of the series they may be taken. Such series are called *recurring series*, because any term, such as u_{x+n}, being equal to the sum of a certain number (n) of those immediately preceding it, each affected (in their order) with a constant coefficient; if we wish to form the successive terms, by means of this property, we are obliged perpetually to recur to those already determined. For example, in the recurring series

$$2+3+5+9+17+33+\&c.$$

we have $5=3.3-2.2$, $9=3.5-2.3$, $17=3.9-2.5$, &c. and in general,

$$u_{x+2}=3\,u_{x+1}-2\,u_x.$$

The general equation of every recurring series is

$$0=u_{x+n}+a\,u_{x+n-1}+b\,u_{x+n-2}+\ldots k\,u_x, \qquad (a)$$

and from this, having given the n first terms, all the rest may be produced in their order; since, if $x=1$,

$$u_{n+1}=-(a\,u_n+b\,u_{n-1}+\ldots k\,u_1),$$

and so on, for all the succeeding terms. Those preceding

u_1 (if the series be continued backward) may also be produced from n terms so given, since, making $x=0$, we have

$$u_0 = -\frac{1}{k}(u_n + a\,u_{n-1} + \ldots + j\,u_1).$$

It is evident, then, that so far as the equation (a) is concerned, these terms (or any other set of n consecutive terms we may fix upon) may be regarded as perfectly arbitrary and independent. The complete integral of (a) will, as we have seen, involve n arbitrary constants, and as u_x is restricted in the present case, to denote the general term of the particular series u_1, u_2, &c. whose first n terms are given, if we make the values which this integral takes, when $1, 2, \ldots n$ are put for x, coincide with these assigned values of $u_1 \ldots u_n$, we get n equations for determining the constants. To exemplify this in the instance above given, the equation $u_{x+2} - 3\,u_{x+1} + 2\,u_x = 0$, corresponds to an infinite variety of series besides $2+3+5+$&c. for if we make $u_1 = a$, $u_2 = b$, we get $u_3 = 3\,b - 2\,a$, &c. and $u_0 = \frac{-b+3\,a}{2}$, &c. Now, if we integrate the proposed equation, we get

$$u_x = C_1 + C_2 \,.\, 2^x,$$

the roots of $u^2 - 3\,u + 2 = 0$, being 1 and 2, and if we assume

$$u_1 = C_1 + 2\,C_2 = 2, \quad u_2 = C_1 + 4\,C_2 = 3,$$

we get $\qquad C_1 = 1, \quad C_2 = \frac{1}{2}, \quad u_x = 1 + 2^{x-1},$

for the general term of the particular series under consideration.*

---

* To particularize any proposed recurring series, it is sometimes usual to express the coefficients which connect any term with the preceding ones under the name "Scale of Relation." Thus $f+g$ is the scale of relation of the series whose equation is $u_{x+2} = f\,u_{x+1} + g\,u_x$.

391. To investigate the general term of a recurring series, we have therefore only to integrate the equation expressing the relation between its successive terms,* and to determine the arbitrary constants in the integral, by making the expression so found, coincide successively with a sufficient number of the first, or any other terms which may happen to be given: and hence a remarkable consequence follows, that (except in certain particular cases) a recurring series may be resolved into one or more geometric progressions. This will easily appear, if we consider that (a) being the equation of the series, if we call the roots of

$$o = u^n + a\,u^{n-1} + b\,u^{n-2} + \dots k \dots\dots\dots (b)$$

$\alpha, \beta, \gamma, \delta$, &c. the complete value of u_x will be

$$C_1 \,.\, \alpha^x + C_2 \,.\, \beta^x + C_3 \,.\, \gamma^x \dots + C_n \,.\, \mu^x,$$

$C_1, C_2, \dots C_n$ being n arbitrary constants. Now, $C_1\alpha^x$, $C_2\beta^x$, &c. are the general terms of the respective geometric series,

$$C_1\alpha + C_1\alpha^2 + \text{\&c.} \quad C_2\beta + C_2\beta^2 + \text{\&c.}$$

so that the recurring series $u_1 + u_2 +$ &c. is made up of the several geometric ones,

$$\begin{aligned} &C_1(\alpha + \alpha^2 + \dots \alpha^x) \\ &+ C_2(\beta + \beta^2 + \dots \beta^x) + \text{\&c.} \end{aligned}$$

The only exception is in the case of equal roots, when the integral of the equation (a) (390), changes its form. Suppose two roots α, β, equal; then the part of the integral depending on these roots changes its form, and becomes

$$(C_1 + C_2\,x)\,\alpha^x,$$

and the series $C_2(1 \,.\, \alpha + 2 \,.\, \alpha^2 + \dots x\,\alpha^x)$ enters into the

---

* For brevity's sake, this will in future be called the equation of the series, as in the theory of curves we say the equation of a curve.

composition of the recurring series, instead of the series $C_2(\beta+\beta^2+\&c.)$ If three roots α, β, γ, be equal, the series $C_3(\gamma+\gamma^2+\&c.)$ must, in like manner, be replaced, either by

$$C_3(1^2\alpha+2^2\alpha^2+3^2\alpha^3+\&c.$$

or by $$C_3(1.2.\alpha+2.3.\alpha^2+\&c.$$

The constants C_1, C_2, &c. are easily determined by the equations

$$\left.\begin{aligned} C_1\alpha+C_2\beta+C_3\gamma+\&c. &= u_1 \\ C_1\alpha^2+C_2\beta^2+C_3\gamma^2+\&c. &= u_2 \\ C_1\alpha^3+C_2\beta^3+C_3\gamma^3+\&c. &= u^3 \end{aligned}\right\} \ldots\ldots\ldots (c)$$
$$\&c.$$

or in the case of equal roots, by the equations

$$(C_1+C_2.1)\alpha+C_3\gamma+\&c.=u_1, \&c.$$

392. The sum of any recurring series may be obtained either by summing the geometric or other series, of which it is composed, or by the following process. If in the equation of the series (a) (390), we put $u_{x+n-1}+\Delta u_{x+n-1}$ for u_{x+n}, it becomes

$$o=\Delta u_{x+n-1}+(1+a)u_{x+n-1}+bu_{x+n-2}+\ldots k u_x.$$

In this again, writing $u_{x+n-2}+\Delta u_{x+n-2}$ for its equal, u_{x+n-1}, in the second term, we get

$$o=\Delta u_{x+n-1}+(1+a)\Delta u_{x+n-2}+(1+a+b)u_{x+n-2}+\ldots ku_x$$

and so on, to

$$o=\Delta u_{x+n-1}+(1+a)\Delta u_{x+n-2}+\ldots(1+a+b+\ldots j)\Delta u_x$$
$$+(1+a+\ldots j+k)u_x,$$

and integrating, we find, after writing $x+1$ for x,

$$\Sigma u_{x+1}=\frac{-(u_{x+n}+(1+a)u_{x+n-1}+\&c.)}{1+a+b+\ldots k}+C,$$

which gives for the sum of the first x terms, (or $\Sigma u_{x+1}-\Sigma u_1$)

$$-\frac{(u_{x+n}-u_n)+(1+a)(u_{x+n-1}-u_{n-1})+\&c.}{1+a+b+\ldots k}.$$

393. It is proper to remark, that, although the roots of the equation $u^n+a\,u^{n-1}+\ldots k=0$ be not obtainable by any known method, still both the general term, and the sum of the recurring series may be completely exhibited in an algebraic formula, free from radicals, and involving only x, a, b, ... k; for since the equations (c) (391), for determining C_1, C_2, &c. involve these quantities so combined with α, β, &c. that if in any, or all of them, two of the latter quantities (as α and β) be transposed, the equations remain unaltered, provided C_1 and C_2 be transposed in the same way: if therefore C_1 be obtained from these equations in functions of α, β, γ, &c. in the form

$$C_1=f(\alpha,\beta,\gamma,\&c.)$$

it will follow, that we shall have

$$C_2=f(\beta,\alpha,\gamma,\&c.)$$
$$C_3=f(\gamma,\beta,\alpha,\&c.),\ \&c.$$

Again, since the interchange of C_2 for C_3, and β for γ, makes no alteration in the value of C_1, we must also have

$$C_1=f(\alpha,\gamma,\beta,\&c.)$$

and in like manner

$$C_1=f(\alpha,\beta,\delta,\gamma,\&c.)$$

and so on, which can only take place on the supposition that $f(\alpha,\beta,\gamma,\&c.)$ is symmetrical with respect to all the roots, except α. Hence it is evident, that $C_1.\alpha^x$, or $\alpha^x.f(\alpha,\beta,\gamma,\delta,\ldots)$ must involve α in the same manner that $C_2.\beta^x$ or $\beta^x.f(\beta,\alpha,\gamma,\delta,\ldots)$ involves β, and so on; and therefore

$$C_1.\alpha^x+C_2.\beta^x+\&c.\ \text{or}\ u_x,$$

must be a symmetrical algebraic function of the roots α,

β, &c. and is therefore capable of expression, in finite rational functions of the coefficients ;* whence it also follows, that S_x (as we have exhibited its value in the last article), is likewise capable of being so expressed. The *resolution* of the above equation, or the actual determination of the several geometric series of which a recurring series is composed, is therefore neither necessary for its summation (in the usual sense of the word), nor for the investigation of its general term.

In the case of equal roots, the symmetry above-mentioned takes place only among the unequal roots, and the constants connected with them. The foregoing reasoning therefore extends only to these roots and constants; but as the equal roots of an equation may always be determined, and the equation depressed to one which shall contain only the unequal ones, any symmetrical function of these latter may be determined, without knowing their separate values. Now, suppose α and β the two equal roots; then by transposing the terms affected with α, in the equations (d) (391), we get

$$C_3\,\gamma + C_4\,\delta + \&c. = u_1 - (C_1 - C_2 \,.\, 1)\,\alpha$$
$$C_3\,\gamma^2 + C_4\,\delta^2 + \&c. = u_2 - \&c.$$

and so on, to which the preceding reasoning applies (α being considered as a known quantity), and by which it appears, that

$$(C_1 \,.\, \alpha^x + C_2\, x \,.\, \alpha^x) + C_3\,\gamma^x + C_4\,\delta^x + \&c. = u_x,$$

is a symmetrical function of γ, δ, &c. the unequal roots.

394. The consideration of generating functions is well adapted to exhibit the theory of recurring series in a clear light. Let us suppose $\varphi.(t)$ to be the generating function of u_x, so that

* Meditationes Algebraicæ, Cap. 1.

$$\varphi.(t)=u_1 t+u_2 t^2+\ldots u_x t^x.$$

Then will the generating function of

$$u_{x+n}+a u_{x+n-1}+\ldots k u_x \ldots\ldots\ldots (e)$$

be

$$\varphi.(t).\left\{\frac{1}{t^n}+\frac{a}{t^{n-1}}+\ldots k\right\}\ldots(f).$$

Now, since we consider only positive values of x greater than zero, this may be supposed equal to any function of the form

$$\frac{1}{t^i}\left\{At+Bt^2+\ldots Kt^i\right\}$$

where A, B, ... K are arbitrary, for the coefficients of t^x in this is evidently zero. Consequently

$$\varphi.(t)=\frac{1}{t^{i-n}}.\quad\frac{At+Bt^2+\ldots Kt^i}{1+at+\ldots kt^n};$$

but since $\varphi.(t)=u_1+u_2 t^2+$ &c. this must contain only positive powers of t greater than zero, and therefore $i-n=0$, or $i=n$, and

$$\varphi.(t)=\frac{At+Bt^2+\ldots Kt^n}{1+at+\ldots kt^n},$$

u_x is therefore the coefficient of t^x in the developement of this fraction. Now (151), this may always be resolved into n fractions,

$$\frac{C_1 t}{1-\alpha t}+\frac{C_2 t}{1-\beta t}+\ldots\frac{C_n t}{1-\mu t},$$

α, β, ... μ being the n unequal roots of the equation

$$u^n+a u^{n-1}+\ldots k=0;$$

and each of these fractions being developed in powers of t, we find, for the coefficient of t^x, or the value of u_x,

$$C_1.\alpha^x+C_2.\beta^x+\ldots C_n.\mu^x,$$

as before.

395. We will take one example of the summation of a recurring series. Let us suppose

$$S_x = a + 5\,a^2 + 19\,a^3 + 65\,a^4 + \ldots\, u_x,$$

the equation of this series is

$$u_{x+2} - 5\,a\,u_{x+1} + 6\,a^2\,u_x = 0.$$

Now, the roots of the equation $u^2 - 5\,a\,u + 6\,a^2 = 0$, are $2\,a$ and $3\,a$; whence

$$u_x = C_1\,.\,(2\,a)^x + C_2\,.\,(3\,a)^x;$$

and by taking

$$\left.\begin{array}{l} C_1\,.\,2\,a + C_2\,.\,3\,a = a \\ C_1\,.\,4\,a^2 + C_2\,.\,9\,a^2 = 5\,a^2 \end{array}\right\}$$

we find $C_1 = -1$, and $C_2 = 1$, whence $u_x = (3^x - 2^x)\,a^x$ for the general term. Again, throwing the equation of the series into the form

$$\Delta\,u_{x+1} + (1 - 5\,a)\,\Delta\,u_x + (1 - 5\,a + 6\,a^2)\,u_x = 0,$$

if we write $x+1$ for x, and integrate, we obtain

$$u_{x+2} + (1 - 5\,a)\,u_{x+1} + (1 - 5\,a + 6\,a^2)\,\Sigma\,u_{x+1} = 0;$$

whence it is easy to see, that

$$\Sigma\,u_{x+1} - \Sigma\,u_1 = -\,\frac{(u_{x+2} - u_2) + (1 - 5\,a)\,(u_{x+1} - u_1)}{1 - 5\,a + 6\,a^2},$$

which, by reduction, appears to be precisely what would result from the immediate integration of $(3^{x+1} - 2^{x+1})a^{x+1}$.

On the Connexion between the Differential Calculus and that of Differences.

396. The Differential Calculus, and that of Differences, although forming two distinct branches of analytical investigation, have still a very near relation to each other; and when the former is considered in the light in

which Leibnitz presented it, or as depending on the theory of limits, it becomes a particular case of the latter. This must have been already noticed by the attentive reader of the former part of this work; and, in confirmation of it, we shall now deduce Taylor's theorem from the equation

$$u_{x+n} = u_x + \frac{n}{1} \Delta u_x + \frac{n(n-1)}{1.2} \Delta^2 u_x + \text{\&c.}$$

To this end, suppose $u_x = f(hx) = u$, or, writing t for hx, $u_x = f(t)$. Also, let $u_{x+n} = f(xh + nh) = f(t+k)$, or $nh = k$. The above equation may then be written as follows:

$$f(t+k) = f(t) + \frac{k}{1} \cdot \frac{\Delta f(t)}{h} + \frac{k^2}{1.2} \cdot \frac{\Delta^2 f(t)}{h^2} \left(1 - \frac{1}{n}\right)$$
$$+ \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) . \text{\&c.}$$

Now, the successive values of $f(t)$, or u_x, u_{x+1}, &c. may be considered as forming a series,

$$f(t),\ f(t+h),\ f(t+2h), \ldots f(t+k),$$

the number of whose terms is $n+1$; and it is evident, that by increasing this number, and proportionably diminishing h, the quantity by which t increases at every step, the intermediate terms will approach nearer and nearer to each other in magnitude, and their differences, or $\Delta f(t)$, $\Delta f(t+h)$ &c. will continually approach to zero as their limit; and for the same reason the differences of these, or $\Delta^2 f(t)$, &c. and so on, will likewise approach to zero as their limit. Hence, if we consider the limits only $\frac{\Delta f(t)}{h}$, or $\frac{f(t+h) - f(t)}{h}$ will be expressed by $\frac{df(t)}{dt}$, (5) and in like manner, $\frac{\Delta^2 f(t)}{h^2}$, or $\frac{1}{h}\left\{\frac{\Delta f(t+h)}{dt} - \frac{\Delta f(t)}{dt}\right\}$ has $\frac{d^2 f(t)}{dt^2}$ for its limit, and so on. Again, the factors $1 - \frac{1}{n}$, $1 - \frac{2}{n}$,

&c. have unity for their limit; so that the limit of the above equation relative to the increase of n, and consequent diminution of h or $\frac{k}{n}$, is

$$f(t+k) = f(t) + \frac{k}{1} \cdot \frac{d\,f(t)}{d\,t} + \frac{k^2}{1\,.\,2} \cdot \frac{d^2 f(t)}{d\,t^2} + \text{\&c.}$$

Having once arrived at Taylor's theorem, the analytical theory of the Differential Calculus no longer presents any difficulty, and accordingly we may perceive by this process in what manner this calculus results from that of differences.

The general relations above investigated between functions and their differences, when we take the limits, will furnish, as is evident, analogous equations in the Differential Calculus. For example, if we set out from the equation

$$\Delta^n (u_x \,.\, u'_x \,.\, \text{\&c.}) = \{ (1+\Delta)(1+\Delta') \text{\&c.} - 1 \}^n u_x u'_x \ldots$$

In taking the limits, we have only to consider u_x, u'_x, &c. as functions of $h\,x$ or t, x being supposed to vary by successive units, and to be increased in proportion as h diminishes, so that $h\,x$ or t shall remain finite. Thus, the limit of $\frac{\Delta^n (u_x \,.\, \text{\&c.})}{h^n}$ will be $\frac{d^n (u_x \,.\, \text{\&c.})}{d\,t^n}$, or (if for u_x, u'_x, &c. we write u, u', &c. and regard these as functions of t) $\frac{d^n (u\,u' \text{\&c.})}{d\,t^n}$. Again, $\frac{\{(1+\Delta)(1+\Delta')\text{\&c.} - 1\}^n u\,u' \text{\&c.}}{h^n}$

$$= \left\{ \left(\frac{\Delta}{h} + \frac{\Delta'}{h} + \text{\&c.} \right) + \frac{\Delta\,\Delta'}{h} + \text{\&c.} \right\}^n u\,u' \,.\, \text{\&c.}$$

and if this be developed, every term except

$$\left\{ \frac{\Delta}{h} + \frac{\Delta'}{h} + \text{\&c.} \right\}^n u\,u' \text{\&c.}$$

will have the sum of the exponents of Δ, Δ', &c. in the

numerator higher than that of h in the denominator; and consequently approach to zero, as its limit; also, if this function be developed, any term, as $\left(\frac{\Delta}{h}\right)^p \cdot \left(\frac{\Delta'}{h}\right)^q$. &c. $u\,u'$ &c. will have $\frac{d^p \,.\, d'^q \,.\, \&c.\; u\,u'\; \&c.}{d\,t^{p+q+\&c.}}$ for its limit, and the result of this operation will therefore coincide with the developement of

$$\frac{\{d+d'+\&c.\}^n\, u\,u'\; \&c.}{d\,t^n}.$$

Thus we have at length

$$d^n(u \,.\, u' \,.\, \&c.) = (d+d'+\&c.)^n\, u\,u'\; \&c.$$

where, in the developement of the second member, the accents affixed to the d's point out their application, as already explained in respect to the symbol Δ* (354).

Let us again consider the equation of (366),

$$\overline{\Delta}\, u_{x,\,y} = \tfrac{1}{2}\left\{\Delta u_{x,\,y} + \Delta u_{x,\,y+1}\right\} + \tfrac{1}{2}\left\{\Delta' u_{x,y} + \Delta' u_{x+1,y}\right\}$$

relative to two variables. If we here make $x\,h=t$, $y\,k=t'$, and suppose $u_{x,\,y}=f(t,\,t')$ we shall have (considering only the limits)

$$\frac{\overline{\Delta}\, u_{x,\,y}}{h} = \frac{f(t+h,\, t'+k) - f(t,\, t')}{h} = \frac{1}{d\,t}\, d f\ (t,\, t').$$

---

* Equations of this kind, elegant in their form, and extremely convenient in their application, are well adapted to set the advantage of Leibnitz's notation in a conspicuous light. It would be next to impracticable, without circumlocution, to express this result in the fluxional notation. A corresponding remark applies to the equation

$$\Delta^n(u_x\; u'_x \ldots \&c.) = \left\{(1+\Delta)(1+\Delta') \,.\, \&c. - 1\right\}^n u_x\, u'_x,\; \&c.$$

to the expression of which the English notation of increments, as delivered by Emerson and others, is equally inadequate. It is, in fact, in every respect the most inconvenient and obscure system which could well have been proposed.

Again, the limits of $\frac{\Delta u_{x,\,y+1}}{h}$, and of $\frac{\Delta^1 u_{x+1,\,y}}{h}$, that is, of

$$\frac{f(t+h,\,t'+k)-f(t,\,t'+k)}{h}, \text{ and } \frac{f(t+h,\,t'+k)-f(t+h,\,t')}{h}$$

which, when k and h are diminished *ad infinitum*, approach to

$$\frac{d\,f(t,\,t')}{d\,t} \text{ and } \frac{d\,f(t,\,t')}{d\,t'} \cdot \frac{d\,t'}{d\,t},$$

as their limits, which are also the respective limits of $\frac{\Delta u_{x,\,y}}{h}$ and $\frac{\Delta' u_{x,\,y}}{h}$. The equation therefore becomes

$$\frac{1}{d\,t}\,d\,f(t,\,t')=\frac{d\,f(t,\,t')}{d\,t}+\frac{d\,f(t,\,t')}{d\,t'}\cdot\frac{d\,t'}{d\,t},$$

or, multiplying by $d\,t$, and writing u for $f(t,\,t')$,—

$$d\,u=\frac{d\,u}{d\,t}\,d\,t+\frac{d\,u}{d\,t'}\,d\,t'. \qquad (123)$$

397. If we regard u_x as a function of $a+b\,x$, since $\Delta(a+b\,x)=b$, this will be the same as to suppose it a function of another variable t, whose difference is b, and which is itself dependent on x, by the equation $a+b\,x=t$. In the same manner, if we suppose $f(x)=t$, we get $\Delta\,t=f(x+1)-f(x)$; eliminating x between these, an equation will result between t and $\Delta\,t$, and u_x will become a function of a variable t, between which and its difference a certain relation is established. It is, however, in all cases unnecessary to introduce this consideration, it being far simpler to regard the independent variable, as having its difference essentially unity, and in the light we have considered the calculus of differences, no other hypothesis respecting its value can, in strictness, be admitted. Whenever, therefore, temporary reasons render it convenient to assign any other value to it, its dependence on some other variable x, whose difference is unity, ought never to be left out of view.

On the Determination of Functions from given Conditions.

398. We have already seen (320), that the arbitrary functions which enter into the integrals of partial differential equations, must be determined so as to satisfy certain assigned relations between the functions, and the variables they involve, and two simple instances of this determination have been given. We propose here to consider the subject in a rather more general point of view, and explain the treatment of cases where the function to be determined enters under various forms, and in different combinations. The integration of equations of differences is only a particular case of this kind; the equation $u_{x+1} - 2\,u_x$, or $\varphi\,(x+1) - 2\,\varphi\,(x) = 0$ denoting only that such a function of x is required, that when $x+1$ is substituted for x, the result shall be twice the original function. Suppose, instead of $x+1$ we had some other function of x, as $a+bx$, and our equation would be

$$\varphi\,(a+b\,x) - 2\,\varphi\,(x) = 0,$$

which is of a different kind from any we have treated. The following process delivered by Laplace, applies to every equation of the form

$$\varphi\,\{\alpha\,(x)\} + A\,(x)\,.\,\varphi\,(x) + B\,(x) = 0,$$

when $\alpha\,(x)$, $A\,(x)$, $B\,(x)$ denote functions of x, and $\varphi\,(x)$ the function to be found. Suppose z a certain function of x, and u_z a function of z, of a certain form, to be determined, and let the conditions be

$$u_z = x, \quad u_{z+1} = \alpha\,(x),$$

or that when x is changed to $\alpha\,(x)$, z shall change to $z+1$, which if z were an integer, would come to the same as supposing x to be the zth term of a series of functions

&c. $x,\ \alpha(x),\ \alpha(\alpha(x)),\ \alpha(\alpha(\alpha(x)))$; &c.

These equations give

$$u_{z+1}=\alpha(u_z),$$

which, since the form of the function α is known, is an equation of differences of the first order, the independent variable being z. Let this be integrated, and the form of the function u_z becomes thereby known. Hence, in the equation $u_z=x$, the manner in which z enters being known, z may be determined in functions of x. Let now u_z be written for x in the given equation, and it becomes (replacing $\alpha(x)$ by (u_{z+1})

$$\phi(u_{z+1})+A(u_z)\,.\,\phi(u_z)+B(u_z)=0. \qquad (a)$$

It is evident, that $\phi(u_{z+1})$ is the same function of $z+1$, that $\phi(u_z)$ is of z. If then we suppose for an instant $\phi(u_z)=v_z$, we have $\phi(u_{z+1})=v_{z+1}$, and

$$v_{z+1}+A(u_z)\,.\,v_z+B(u_z)=0;$$

and since $A(u_z)$ and $B(u_z)$ are known functions of z, this is also an equation of differences of the first order for the determination of v_z. Let its integral be found, and it will be of the form

$$v_z=F(z,C).$$

Now, if from the equation $u_z=x$, the value of z be deduced as above, and the result substituted for z in this integral, the second member will become a known function of x; but $v_z=\phi(u_z)=\phi(x)$; and thus at length the form of $\phi(x)$ becomes known.

The constant C introduced by the integration of (a), may, as we have before remarked, be an arbitrary function of $\cos 2\pi z$, or $=\chi(\cos 2\pi z)$; thus our equation for v_z becomes

$$\phi(x)=v_z=F(z,\ \chi(\cos 2\pi z)),$$

in which, if the value of z in terms of x be substituted,

we obtain, by assigning the form of the function χ in any way we please, an indefinite variety of forms of the function re[illegible]ed.

[illegible]e integral of the equation $u_{z+1}=\alpha(u_z)$ also involves [illegible] constant, which may in like manner be re[illegible]n arbitrary function of $\cos 2\pi z$; but this for [illegible]art leads to transcendental equations of very [illegible]lexity.

[illegible]example, let us take the above equation,

$$\varphi(a+bx)-2[illegible](x)=0.$$

[illegible]sing here $a+bx=u_{z+1}$, $x=u_z$, we get

$$u_{z+1}=a+bu_z,$$

[illegible]

$$u_z=cb^z-\frac{a}{b-1}=x, \qquad (b)$$

c being an arbitrary constant.

The proposed equation now becomes

$$\varphi(u_{z+1})-2\varphi(u_z)=0, \text{ or } v_{z+1}-2v_z=0,$$

if $\varphi(u_z)$ be made equal to v_z: consequently, integrating,

$$v_z=\varphi(x)=2^z \cdot \chi(\cos 2\pi z),$$

$\chi(\cos 2\pi z)$ being an arbitrary function of $\cos 2\pi z$. In this, for z write its value, deduced from the equation (b), and we find the general expression for $\varphi(x)$, which, in the simplest case, when $c=1$, and $\chi(\cos 2\pi z)=1$, will appear to be

$$\varphi(x)=\left\{x+\frac{a}{b-1}\right\}^{\frac{\log 2}{\log b}}.$$

399. The same artifice applies to the more general equation

$$0=F\left\{x, \varphi(\alpha(x)), \varphi(\beta(x))\right\},$$

which it must be remarked is reducible to the more simple form

$$o=F\{x, \phi(x), \phi(\alpha(x))\},$$

by the substitution of t for $\alpha(x)$, or by substituting for x the *inverse function* of $\alpha(x)$, by which is understood that function which written instead of x in the expression of $\alpha(x)$ produces x, as the final result. Thus, if we substitute $\frac{a-x}{cx-b}$ instead of x in the expression $\frac{a+bx}{1+cx}$, the result is x, and the former is therefore the inverse function of the latter. In like manner e^x and $\log x$ are inverse functions of each other, since $\log(e^x)=x$.

The artifice so ingeniously employed by Laplace in the case of the above equation, is however inapplicable, when the function to be determined enters under more than two forms, as in the equation

$$o=F\{x, \phi(x), \phi\alpha(x), \phi\beta(x)\},$$

F, α, β, being the characteristics of known functions; the reason of which is evident. If we suppose

$$x=u_z, \quad \alpha(x)=u_{z+1}, \quad \text{or } u_{z+1}=\alpha(u_z),$$

we are not at liberty to suppose $\beta(x)=u_{z+2}$, or u_{z-1}, &c. (without which the resulting equation cannot be treated as one of differences), for such a supposition would give

$$\beta(x)=u_{z+2}=\alpha(u_{z+1})=\alpha\alpha(u_z)=\alpha\alpha(x),$$

and thus establish a relation between the functions $\alpha(x)$ and $\beta(x)$, which may not hold good. If, however, it should so happen, as for example, in the equations

$$o=F\{x, \phi(x), \phi(x^2), \phi(x^4)\}$$

$$o=F\left\{x, \phi(x), \phi\left(\frac{1}{x+1}\right), \phi\left(\frac{x+1}{x+2}\right)\right\}$$

the process may be applied, whatever number of terms of this kind the equation may contain. The discovery of a direct method of treating equations of this kind *in general* is, however, still a desideratum.

400. The necessity of supposing the constants introduced by the integration of the equations of differences, at which we arrive, to be arbitrary functions of $\cos 2\pi z$, is here evident, as z is in no part of the process supposed to be an integer number, but merely to increase or decrease by units. Geometrical considerations will still far-
FIG. ther elucidate this point.* If on the line AR (fig. 56),
56. drawn parallel to the axis of the abscissæ OX, at any distance from it, and divided into parts AA', $A'A''$, $A''A'''$, &c. equal to each other, and to unity, we construct curves

AB, $A'B'$, $A''B''$, $A'''S$, AC, $A'C'$, $A''C''$, $A'''T$. AD, $A'D'$, $A''D''$, $A'''U$, &c. consisting, between these points, A, A', A'', &c. through which they pass, of equal and similar parts, these curves will satisfy the equation $\Delta y = 0$. This is evident at once for the points A, A', A'' A''', &c. and it is also plain, that if we take $AP = x$, $AP' = x + 1$, the arcs AL and $A'L'$, AM and $A'M'$, AN and $A'N'$ being equal and similar, the ordinates LP and $L'P'$, MP and $M'P'$, NP and $N'P'$, &c. will be also respectively equal, and we have consequently for each curve $\Delta y = 0$.

Since the continuity of the resulting curves does not follow as a necessary consequence from the equation $\Delta y = 0$, the curves AS, AT, AU, &c. are not subject to this law. The polygon $EF\,E'F'\,E''F''\,E'''\dots V$, consisting of similar parts EFE', $E'F'E''$, &c. in like manner gives $\Delta y = 0$, at intervals denoted by unity: the same would be true for a series of equal and similar arcs of any curve whatever put together discontinuously, as in the case of the arcs GH, $G'H'$, $G''H''$, &c.

It is evident, that the equation

---

* The rest of this article is translated from the French edition, with slight modifications, suited to the light in which we have considered the subject.

$$y=\varphi(\cos 2\,\pi\,x)$$

gives rise to lines which satisfy the above condition.

The determination of the arbitrary functions which enter into the integrals of equations of differences, cannot be accomplished by making these integrals coincide with a limited number of given values; for it is evident, that every arbitrary function comprehends implicitly an infinite number of arbitrary values. Suppose, for instance,

$$u_x=X_x\,.\,\varphi(\cos 2\,\pi\,x),$$

from which a certain number of values, a, a', a'', &c. of u_x are to result. If these are to correspond to

$$x=0,\quad x=1,\quad x=2,\ \text{\&c.}$$

the first only of the conditions can be generally satisfied; since, having assigned to $\varphi(\cos 2\,\pi\,x)$ an initial determinate value, such that $u_0=a$, the same value will occur again for the indices $x=1$, $x=2$, &c. from which it follows that the values of u_x, corresponding to these indices, are also determined: a', a'', &c. must therefore necessarily correspond to intermediate indices. If, however, instead of assigning a limited number of insulated values, independent of each other, we suppose that u_x in the interval comprised between $x=0$ and $x=1$, shall give always the same values as a certain given equation $u_x=f(x)$, the question becomes determinate. In fact, if we would determine the value of u_x, corresponding to an index equal to a number m, *plus* a fraction n, whether commensurable or not, the value of u_n must first be found by the equation $u_x=f(x)$, and the result, $u_n=f(n)$, compared with

$$u_n=X_n\,.\,\varphi(\cos 2\,\pi\,n)$$

gives the value which $\varphi(\cos 2\,\pi\,n)$ must have in this case, and which is the same with $\varphi\left(\cos 2\,\pi\,(m+n)\right)$. The value of u_{m+n} then becomes

$$u_{m+n} = X_{m+n} \cdot \varphi(\cos 2\pi n),$$

and is entirely determined. The only condition to which the equation $u_x = f(x)$ is subject, is, that the same values of u_0 and u_1 must result from it as from the equation

$$u_x = X_x \cdot \varphi(\cos 2\pi x).$$

The geometrical construction of equations of differences accords perfectly with this theory. Suppose $\Delta u_x = F(x, u_x)$ any equation of this kind of the first order. After assuming, or determining, by the conditions of the problem, the
FIG. first point B (fig. 57) of the curve required, since the pro-
57. posed equation affords no information respecting the points corresponding to the portion of the abscissa $AA' = 1$; but only gives the ordinate $A'B' - u_1$, we are at liberty to draw through BB' a portion of any curve whatever. This being done, to obtain the portion corresponding to the abscissa $A'A''$, we take any point P' in it, and in the direction $P'A$ set off $P'P$ equal to $AA' = 1$, and erecting the ordinate PM, draw MD' parallel to PP'; then, having deduced from the equation $\Delta u_x = F(x_1\ u_x)$ the value of Δu_x for the abscissa AP, this will give the line $D'M'$, which, added to $P'D' = PM$, will determine the point M'. In the same manner must all the points of the arc $B'B''$ be found, and this arc employed in its turn, like the arc BB' will give those of the third arc $B''B'''$, and so on. It is evident, that by the same process we may continue the curve backwards from A, and it will still satisfy the proposed equation, since the values of the differences $\Delta u_x = D'M'$ will have been deduced from this equation. The reader is left to apply similar considerations to equations of the second, and higher orders.

On Interpolations.

401. One of the principal uses of the Calculus of Differences consists in the *Interpolation of Series*, or the insertion of terms intermediate between those of a given series, which shall be subject to the same law as the rest. For this purpose, the terms of this series must be considered as particular values of the function which expresses its general term, corresponding to a given regular succession of indices. When this general term is given, we may deduce from it as many particular values as we please; but when this is not the case, as for instance, when only a limited number of the first terms is given, it is the business of interpolation to discover that general term, or at least to assign some function of the index which shall represent the proposed series of values.

The problem in this state is plainly indeterminate, as in fact it requires us to derive the analytical expression of a function from a limited number of its numerical values, and therefore admits of an infinite variety of answers. It comes to the same, as forming the equation of a curve which shall pass through a limited number of points, whose abscissæ represent the values of the independent variable, and the ordinates those of the function, and whatever number of such points we assign, must be insufficient to determine the curve, unless its species were given. When, however, (as is almost always the case in practice), we propose to interpolate a series within very narrow limits, we may conceive the expression of its general term developed in a series of positive powers of its index (or of that index *plus* some constant quantity), and neglect all the powers beyond a certain limit; and by this means the function which in that case is a rational integral one, becomes known.

402. Suppose then, that the given values

$$u_0, u_1, u_2, \ldots\ldots u_{n-1},$$

of a certain unknown function u_x are found to correspond to the given values

$$z_0, z_1, \ldots\ldots z_{n-1}$$

of a certain function z_x, whose form is either given or not; it is required to assign the value of u_x, which shall correspond to any proposed value z_x of the latter function, in which x lies either between the limits 0, $n-1$, or not far beyond them on either side, and varies but little.

Since the function is supposed to vary continuously (which is essential to interpolation), there must be some one of the given values of z_x, as z_i, for instance, which differs but little from z_x, so that $z_x - z_i$ is small, and its powers after the $(n-1)$th may be neglected. Now, since u_x is an implicit function of z_x, and therefore of $z_x - z_i$, it may be expressed in a series

$$\alpha + \beta (z_x - z_i) + \ldots \mu (z_x - z_i)^{n-1} + \&c.$$

which, if we stop at the nth term, and then develope each term in powers of z_x, takes the form

$$u_x = A + B z_x + C z_x^2 + \ldots M . z_x^{n-1},$$

A, B, C, &c. being certain functions of z_i. Thus it appears, that provided x vary but little, so that $z_x - z_i$ shall remain small, it is permitted to assign such a form to u_x as the above (the coefficients A, B, &c. being constant, though at present unknown) although z_x should happen not to be of small magnitude.

To determine these coefficients we have only to make the above expression of u_x coincide with the n given values it assumes, when $x = 0, 1, 2, \ldots n-1$, which gives n equations for that purpose (which is the reason why we stopped at the $(n-1)$th power of $z_x - z_i$: had we stopped sooner, we should have a condition too much; if later, too little). Thus,

$$u_0 = A + B z_0 + C z_0^2 + \ldots M z_0^{n-1}$$
$$u_1 = A + B z_1 + C z_1^2 + \ldots M z_1^{n-1}$$
....
$$u_{n-1} = A + B z_{n-1} + C z^2_{n-1} + \ldots \left(M z^{n-1}_{n-1}\right)$$

If we subtract successively the first of these equations from the second, this from the third, and so on, and divide the results by $z_1 - z_0$, $z_2 - z_1$, &c. we shall find, (putting for the sake of brevity

$$\frac{u_1 - u_0}{z_1 - z_0} = U_0, \ \frac{u_2 - u_1}{z_2 - z_1} = U_1, \ \frac{u_3 - u_2}{z_3 - z_2} = U_2, \text{ \&c.})$$

$$U_0 = B + C(z_0 + z_1) + D(z_0^2 + z_0 z_1 + z_1^2) + \text{\&c.}$$
$$U_1 = B + C(z_1 + z_2) + D(z_1^2 + z_1 z_2 + z_2^2) + \text{\&c.}$$
$$U_2 = B + C(z_2 + z_3) + D(z_2^2 + z_2 z_3 + z_3^2) + \text{\&c.}$$
&c.

Again, subtracting U_0 from U_1, U_1 from U_2, and so on, and denoting the quantities $\frac{U_1 - U_0}{z_2 - z_0}$, $\frac{U_2 - U_1}{z_3 - z_1}$, &c. by U_0', U_1', &c., we find

$$U_0' = C + D(z_2 + z_1 + z_0) + \text{\&c.}$$
$$U_1' = C + D(z_3 + z_2 + z_1) + \text{\&c.}$$
&c.

whence we derive

$$\frac{U_1' - U_0'}{z_3 - z_0} = D + \text{\&c.} = U_0''; \text{ \&c.}$$

Now, if for the sake of keeping our ideas distinct, we suppose only four terms in the expression for u_x, the operation will terminate here, and we shall have the following values of A, B, C, D,

$$D = U_0''$$
$$C = U_0' - U_0''(z_2 + z_1 + z_0)$$

$$B = U_0 - U_0'(z_1 + z_0) + U_0''(z_2 z_1 + z_2 z_0 + z_1 z_0)$$

$$A = u_0 - U_0 . z_0 + U'_0 z_0 z_1 - U_0'' . z_0 z_1 z_2 .$$

403. If we substitute these values in the expression for u_z, we find

$$u_z = u_0 + U_0(z_z - z_0) + U_0'\{z^2_z - z_z(z_1 + z_0) + z_1 z_0\}$$
$$+ U_0''\{z^3_z - z^2_z(z_2 + z_1 + z_0) + z_z(z_2 z_1 + z_2 z_0 + z_1 z_0) - z_2 z_1 z_0\}$$

It is easy to see that the coefficients of U_0, U_0', &c. are decomposable into simple factors, and that u_z may be written as follows:

$$u_z = u_0 + U_0 . (z_z - z_0) + U_0'(z_z - z_0)(z_z - z_1) +$$
$$+ U_0''(z_z - z_0)(z_z - z_1)(z_z - z_2).$$

If we pursue this process farther, we still obtain a formula analogous to the preceding; so that in general,

$$u_z = u_0 + U_0(z_z - z_0) + U_0'(z_z - z_0)(z_z - z_1) +$$
$$+ U_0''(z_z - z_0)(z_z - z_1)(z_z - z_2) + \&c.$$

where we suppose

$$U_0 = \frac{u_1 - u_0}{z_1 - z_0}, \quad U_1 = \frac{u_2 - u_1}{z_2 - z_1}, \quad U_2 = \frac{u_3 - u_2}{z_3 - z_2}, \ \&c.$$

$$U_0' = \frac{U_1 - U_0}{z_2 - z_0}, \quad U_1' = \frac{U_2 - U_1}{z_3 - z_1}, \ \&c.$$

$$U_0'' = \frac{U_1' - U_0'}{z_3 - z_0}, \ \&c.$$

&c.

404. We have thus obtained two expressions for u_z, or rather the same in two different forms; but it is yet susceptible of another very elegant form, due to Lagrange, and which has the peculiar advantage of being well adapted to logarithmic computation. If in the expression for U_0' we write the values of U_1 and U_0, we find

$$U_0' = \frac{u_2(z_1 - z_0) - u_1(z_2 - z_0) + u_0(z_2 - z_1)}{(z_1 - z_0)(z_2 - z_0)(z_2 - z_1)},$$

which is of the form

$$\alpha u_0+\beta u_1+\gamma u_2;$$

in like manner, U_1', U_2', &c. are found to be of the form

$$\alpha' u_1+\beta' u_2+\gamma' u_3,\ \alpha'' u_2+\beta'' u_3+\gamma'' u_4,\ \&c.$$

Again, if these results be written in the expression for U_0'', it will take the form

$$\alpha u_0+\beta u_1+\gamma u_2+\delta u_3,$$

and so on, α, β, γ, δ, &c. being certain coefficients independent on u_0, u_1, &c. and consisting of various combinations of z_0, z_1, &c. If now these values be written for U_0, U_0', &c. in the expression for u_x, it will take the form

$$u_x=a\,u_0+b\,u_1+c\,u_2+\ \ldots\ k\,u_{n-1},$$

where a, b, c, ... k are coefficients dependent on z_0, z_1 ... z_{n-1}, z_x. These may easily be determined in any particular case, by pursuing the above process of substitution; but their values may be seen as it were by inspection, if we recollect, that when $x=0$ (or $z_x=z_0$) u_x becomes u_0, and therefore in that case $b=0$, $c=0$, ... $k=0$, and $a=1$; thus z_x-z_0 is a factor of all the coefficients but a; and in like manner z_x-z_1 of all but b, and so on; thus we must have

$$a=A\times(z_x-z_1)(z_x-z_2)\ldots(z_x-z_{n-1});$$

and making $x=0$,

$$1=A(z_0-z_1)(z_0-z_2)\ldots(z_0-z_{n-1});$$

whence

$$A=\frac{1}{(z_0-z_1)(z_0-z_2)\ldots(z_0-z_{n-1})},$$

and

$$a=\frac{(z_x-z_1)(z_x-z_2)\ldots(z_x-z_{n-1})}{(z_0-z_1)(z_0-z_2)\ldots(z_0-z_{n-1})};$$

in like manner,

$$b=\frac{(z_x-z_0)(z_x-z_2)\ldots(z_x-z_{n-1})}{(z_1-z_0)(z_1-z_2)\ldots(z_1-z_{n-1})},$$

&c.

for none of the coefficients can contain any power of z_x higher than $n-1$. Substituting therefore these values, we get

$$u_x = \frac{(z_x-z_1)(z_x-z_2)\ldots(z_x-z_{n-1})}{(z_0-z_1)(z_0-z_2)\ldots(z_0-z_{n-1})}\,u_0$$

$$+\frac{(z_x-z_0)(z_x-z_2)\ldots(z_x-z_{n-1})}{(z_1-z_0)(z_1-z_2)\ldots(z_1-z_n}\,u_1$$

$$+\ \&c.$$

each term of which m ly calculated by means of logarithms.

405. In the case wh lifferences of the terms z_0, z_1, &c. are constant, c $+xh$, we have

$$U_0=\frac{\Delta u_0}{h},\quad U_1=\frac{\Delta u_1}{h},\ \&c.$$

whence we find

$$U_0'=\frac{\Delta^2 u_0}{1.2.h^2},\quad U_1=\frac{\Delta^2 u_1}{1.2.h^2},\ \&c.\quad U_0''=\frac{\Delta^3 u_0}{1.2.3.h^3},\ \&c.$$

and consequently

$$u_x=u_0+\frac{z_x-z_0}{h}\,\Delta u_0+\frac{(z_x-z_0)(z_x-z_1)}{1.2.h^2}\,\Delta^2 u_0+\ \&c.$$

which, when $p+xh$ is substituted for z_x, reduces itself to

$$u_x=u_0+\frac{x}{1}\,\Delta u_0+\frac{x(x-1)}{1.2}\,\Delta^2 u_0+\ \&c.$$

and therefore might have been immediately derived from the equation so often mentioned,

$$u_{x+n}=u_x+\frac{n}{1}\,\Delta u_x+\ \&c.$$

by writing 0 for x, and x for n.

406. It is sometimes necessary to calculate the differential coefficients of a function from similar data, that is,

when only some of its values are assigned. In this case, we must have recourse to the equation

$$\frac{d^n u_x}{d x^n} = \left\{ \log (1 + \Delta) \right\}^n u_x .$$

Thus, since

$$\frac{d u_x}{d z_x} = \frac{d u_x}{d x} \cdot \frac{d x}{d z_x},$$

we have

$$\frac{du_x}{d z_x} = \left(\frac{d z_x}{d x}\right)^{-1} \left\{ \frac{\Delta u_x}{1} - \frac{\Delta^2 u_x}{2} + \&c. \right\}$$

and if x increase uniformly, or nearly so, $\frac{d z_x}{d x} = h$, and

$$\frac{d u_x}{d z_x} = \frac{1}{h} \left\{ \frac{\Delta u_x}{1} - \&c. \right\}$$

which gives, for the value of $\frac{d u_0}{d z_0}$, the following expression:

$$\frac{1}{h} \left\{ \frac{\Delta u_0}{1} - \frac{\Delta^2 u_0}{2} + \&c. \right\}$$

If the increase or decrease of z_x be not uniform, but follow any assigned law, z_x will be a given function of x. Thus, if z_x should have its second differences equal, $\Delta^2 z_x = 2 k$, $z_x = j + h x + k x^2$; whence we find

$$\frac{d z_x}{d x} = h + 2 k x, \text{ and}$$

$$\frac{d u_x}{d z_x} = \frac{1}{h + 2 k x} \left\{ \frac{\Delta u_x}{1} - \frac{\Delta^2 u_x}{2} + \&c. \right\}$$

407. The most ordinary cases where interpolation is required, are those where the given differences Δu_0, $\Delta^2 u_0$, &c. form a series, the terms of which converge to zero, as in the following example, taken from a table of logarithms. Suppose it were required to find the common logarithm of 3·14159̇26536 by means of a table, containing

the logarithms of numbers from 1 to 1000, to ten decimals. The logarithms in the tables must be regarded as particular values of u_z, and the numbers, as those of z;
and if we take

$$z_0=3{\cdot}14,\ z_1=3{\cdot}15,\ z_2=3{\cdot}16,\ z_3=3{\cdot}17,\ z_4=3{\cdot}18,$$

we shall find the following values for u_0, u_1, &c. and their differences

	u_z	Δu_z	$\Delta^2 u_z$	$\Delta^3 u_z$	$\Delta^4 u_z$
$u_0=$	0·4969296481				
		13809057			
$u_1=$	0·4983105538		−43769		
		13765288		+277	
$u_2=$	0·4996870826		−43492		−3
		13721796		+274	
$u_3=$	0·5010592622		−43218		
		13678578			
$u_4=$	0·5024271200				

Now, $z_z=3{\cdot}1415926536$, and since $z_1-z^0=h$, $h=0{\cdot}01$,

$$z_z-z_0=0{\cdot}0015926536,\quad z_z-z_1=-0{\cdot}0084073464,$$

$$z_z-z_2=-0{\cdot}0184073464,\quad z_z-z_3=-0{\cdot}0284073464,$$

which values, substituted in the equation

$$u_z=u_0+\frac{z_z-z_0}{1\,.\,h}\Delta u_0+\frac{(z_z-z_0)(z_z-z_1)}{1\,.\,2\,.\,h^2}\Delta^2 u_0$$

$$+\frac{(z_z-z_0)(z_z-z_1)(z_z-z_2)}{1\,.\,2\,.\,3\,.\,h^3}\Delta^3 u_0$$

$$+\frac{(z_z-z_0)(z_z-z_1)(z_z-z_2)(z_z-z_3)}{1\,.\,2\,.\,3\,.\,4\,h^4}\Delta^4 u_0,$$

after executing the arithmetical operations, gives

$$u_z=0{\cdot}4971498726.$$

On the Numbers of Bernouilli.

408. If we develope the function $\frac{t}{e^t-1}$ in powers of t, we shall obtain a series, the coefficients of which, as we have already proved, are the same with those of $\int u_x dx$, u_x, $\frac{du_x}{dx}$, &c. in the general expression above given for Σu_x. It is evident that this developement will contain no odd powers of t higher than the first; for, if we suppose $\frac{t}{e^t-1}=f(t)$, we have

$$f(-t)=\frac{-t}{e^{-t}-1}=\frac{te^t}{e^t-1},$$

and consequently

$$f(t)-f(-t)=\frac{t(1-e^t)}{e^t-1}=-t.$$

Now, if $f(t)$ be supposed equal to a series,

$$p+qt+rt^2+st^3+\text{\&c.}$$

we have $\quad f(-t)=p-qt+rt^2-st^3+\text{\&c.}$

whence, $\quad f(t)-f(-t)=2qt+2st^3+\text{\&c.}$

which, compared with $-t$, gives $q=-\frac{1}{2}$, $s=0$, &c.; consequently $f(t)$ may be expressed in a series of the following form:

$$\frac{t}{e^t-1}=1-\frac{t}{2}+B_1.\frac{t^2}{1.2}-B_3.\frac{t^4}{1.2.3.4}+\text{\&c.}$$

B_1, B_3, B_5, &c. being certain numbers (called the *Numbers of* BERNOUILLI), whose law we propose to determine.

If we take the differential coefficients of the function $f(t)$, we shall find, that the supposition $t=0$ renders them all $\frac{0}{0}$, so that the usual method of developement is inapplicable. An ingenious artifice, due to Laplace (from

whom the whole of this very instructive process is taken*), enables us to elude this difficulty. If we resolve $f(t)$ into two fractions, by means of the identical equation

$$\frac{t}{x^2-1} = \frac{\frac{t}{2}}{x-1} - \frac{\frac{t}{2}}{x+1},$$

where $x = e^{\frac{t}{2}}$, we find

$$f(t) = f\left(\frac{t}{2}\right) - \frac{\frac{t}{2}}{e^{\frac{t}{2}}+1},$$

and therefore the coefficient of t^{2x}, in the developement of $f\left(\frac{t}{2}\right) f - (t)$ must equal that of the same power in the developement of $\frac{\frac{t}{2}}{e^{\frac{t}{2}}+1}$, that is, to the same coefficient in the developement of $\frac{t}{e^t+1}$, multiplied by $\left(\frac{1}{2}\right)^{2x}$. This last coefficient we will, for the present, call a_{2x}, and if we denote by b_{2x} the coefficient of t^{2x} in $f(t)$, we shall have $b_{2x}\left(\frac{1}{2^{-x}}-1\right)$ for its coefficient in $f\left(\frac{t}{2}\right) - f(t)$; consequently,

$$b_{2x}\left(\frac{1}{2^{2x}}-1\right) = \frac{a_{2x}}{2^{2x}},$$

which gives

$$b_{2x} = -\frac{a_{2x}}{2^{2x}-1};$$

but we have also

$$b_{2x} = (-1)^{x+1} \cdot \frac{B_{2x-1}}{1 \cdot 2 \ldots (2x)};$$

whence we find

$$B_{2x-1} = (-1)^x \cdot \frac{1 \cdot 2 \ldots\ldots (2x)}{2^{2x}-1} \cdot a_{2x}. \qquad (a)$$

* Mem. de L'Acad. 1779. sur l'usage du calc. des diff. partielles daus la theorie des suites.

It only remains now to determine the value of a_{2x}; and since the same difficulty does not occur in the differential coefficients of $\frac{t}{e^t+1}$, we may here employ the method explained in (19). Let u represent the function $\frac{1}{e^t+1}$, and a_{2x} will equal the coefficient of t^{2x-1}, in the developement of u, or

$$a_{2x} = \frac{1}{1 \cdot 2 \ldots (2x-1)} \cdot \frac{d^{2x-1} u}{dt^{2x-1}}, \qquad (b)$$

where $t=o$, or $e^t=1$, after the differentiations.

Now, a few successive differentiations will convince us that $\frac{d^{2x-1}u}{dt^{2x-1}}$ must always be of the form

$$\frac{d^{2x-1}u}{dt^{2x-1}} = \frac{a+be^t+ce^{2t}+\ldots ke^{(2x-1)t}}{(1+e^t)^{2x}},$$

and the numerator of the second number may be determined at once, if we consider that this equation, being identical, if we multiply both sides by $(1+e^t)$ the first, however developed, can contain only positive powers of e^t. For the sake of brevity, let $e^t=v$, whence we get $\frac{dv}{dt} = e^t = v$, and

$$(v+1)^{2x} \cdot \frac{d^{2x-1} \cdot (v+1)^{-1}}{dt^{2x-1}} = a+bv+\ldots kv^{2x-1}.$$

Now, $\qquad (v+1)^{-1} = v^{-1} - v^{-2} + v^{-3} - \&c.$

$$\frac{d(v+1)^{-1}}{dt} = -1 \cdot v^{-1} + 2 \cdot v^{-2} - 3 \cdot v^{-3} + \&c.$$

and in general

$$\frac{d^{2x-1} \cdot (v+1)^{-1}}{dt^{2x-1}} =$$

$$-\left\{ 1^{2x-1} \cdot v^{-1} - 2^{2x-1} \cdot v^{-2} + 3^{2x-1} \cdot v^{-3} - \&c. \right\}$$

and multiplying this by the series

$$v^{2x}+\frac{2x}{1}v^{2x-1}+\frac{2x(2x-1)}{1.2}v^{2x-2}+\&c.$$

the developement of $(v+1)^{2x}$, all the negative powers of v must necessarily destroy each other, and may therefore be left out of consideration, when we find for our result

$$-1^{2x-1}v^{2x-1}+\left(2^{2x-1}-\frac{2x}{1}1^{2x-1}\right)v^{2x-2}$$

$$-\left(3^{2x-1}-\frac{2x}{1}2^{2x-1}+\frac{2x(2x-1)}{1.2}1^{2x-1}\right)v^{2x-3}+\&c.$$

expressing the value of $a+bv+\ldots kv^{2x-1}$; which, making $v=1$, and consequently $(1+v)^{2x}=2^{2x}$, gives us the value of $\frac{d^{2x-1}u}{dt^{2x-1}}$, in that case equal to

$$\frac{1}{2^{2x}}\left\{-1^{2x-1}+\left(2^{2x-1}-\frac{2x}{1}1^{2x-1}\right)-\left(3^{2x-1}-\&c.\right)+\&c.\right\}$$

The substitution of this in the equation (b) gives the value of a_{2x}, which again substituted in the expression for B_{2x-1}, at length affords the following general formula for the xth number of Bernouilli.

$$B_{2x-1}=\frac{2x(-1)^{x+1}}{2^{2x}(2^{2x}-1)}\left\{1^{2x-1}-\left(2^{2x-1}-\frac{2x}{1}1^{2x-1}\right)\right.$$

$$\left.+\left(3^{2x-1}-\&c.\right)-\&c.\right\}$$

Thus we find, $B_1=\frac{1}{6}$, $B_3=\frac{1}{30}$, $B_5=\frac{1}{42}$.

409. These numbers are of the most extensive utility in the theory of series, and to notice all their applications would be endless; we shall content ourselves with presenting a few of the most simple.

The numbers of Bernouilli occur in the developement of the function $\tan\theta$; for, since by dividing the exponential expression for $\sin\theta$ by that for $\cos\theta$ (164), we have

$$\tan\theta = \frac{1}{\sqrt{-1}}\left\{\frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}\right\} =$$

$$= \frac{1}{\sqrt{-1}}\left\{1 - \frac{2}{1+e^{2\theta\sqrt{-1}}}\right\}$$

the coefficient of θ^{2x-1} will be equal to that of the same power of t, in the developement of $\frac{1}{1+e^t}$, multiplied by $-\frac{2}{\sqrt{-1}} \cdot (2\sqrt{-1})^{2x-1}$, or to that of t^{2x}, in the developement of $\frac{t}{1+e^t}$, multiplied by $(2\sqrt{-1})^{2x}$, or $(-1)^x \cdot 2^{2x}$, and is therefore represented by $(-1)^x \cdot 2^{2x} \cdot a_{2x}$, in which expression, if we substitute the value of a_{2x}, deduced from the equation (a) of the last article, it becomes

$$\frac{2^{2x}(2^{2x}-1)}{1 \cdot 2 \dots (2x)} B_{2x-1},$$

so that giving x all its values from o to ∞, we find

$$\tan\theta = \frac{4 \cdot 3}{1 \cdot 2} B_1\,\theta + \frac{16 \cdot 15}{1.2.3.4} B_3\,\theta^3 + \frac{64 \cdot 63}{1.2.3.4.5.6} B^5\,\theta^5 + \&c.$$

for it is easy to see that tan θ can contain none, but odd powers of θ.

The developement of cotan θ is also easily obtained; for, since cotan $\theta = \frac{1}{\tan\theta}$, we have

$$\text{cotan}\,\theta = \sqrt{-1} \cdot \frac{e^{\theta\sqrt{-1}} + e^{\theta\sqrt{-1}}}{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}} =$$

$$= \sqrt{-1}\left\{1 + \frac{2}{e^{2\theta\sqrt{-1}} - 1}\right\}$$

Now, since

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{B_1}{1.2}\,t - \frac{B_3}{1.2.3.4}\,t^3 + \&c.$$

if for t we write $2\,\theta\,\sqrt{-1}$, we find

$$\cot\theta = \frac{1}{\theta} - \frac{2^2}{1.2} B_1 \theta - \frac{2^4}{1.2.3.4} B_3 \theta^3 - \&c.$$

410. These numbers also occur in the expressions for the sums of series of the form

$$\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \&c. \text{ (to } \infty)$$

$$\frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \&c.$$

But in order to shew this, it will be necessary to resolve the functions $\sin\theta$ and $\cos\theta$ into their factors. Now, since $\sin\theta$ vanishes on the supposition, that $\theta = 0$, or $\theta = \pm\pi$, $\theta = \pm 2\pi$, &c. it follows, that θ, $1 + \frac{\theta}{\pi}$, $1 - \frac{\theta}{\pi}$, $1 + \frac{\theta}{2\pi}$, $1 - \frac{\theta}{2\pi}$, &c. must be factors of $\sin\theta$. Again, since these are the only values which render $\sin\theta$ zero, its expression can admit no other factors, functions of θ, and therefore we must have

$$\sin\theta = A.\theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right), \&c. \text{ (ad inf.)}$$

Now as θ diminishes, $\frac{\sin\theta}{\theta}$ approaches unity as its limit, and therefore we must have $A = 1$, and

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right) \&c.$$

Exactly in the same way, since $\cos\theta$ vanishes upon the several suppositions

$$\theta = \pm\frac{\pi}{2}, \theta = \pm\frac{3\pi}{2}, \&c. \text{ or } 1 - \left(\frac{2}{\pi}\theta\right)^2 = 0, \&c.$$

and approaches unity as its limit, when θ decreases to zero, we must have

$$\cos\theta = \left(1 - \frac{\theta^2}{1^2\left(\frac{\pi}{2}\right)^2}\right)\left(1 - \frac{\theta^2}{3^2\left(\frac{\pi}{2}\right)^2}\right). \&c.$$

These equations, writing in them respectively $\pi\theta$ and $\frac{\pi}{2}\theta$ for θ, give

$$\sin \pi\theta = \pi\theta \,.\left(1-\frac{\theta^2}{1^2}\right)\left(1-\frac{\theta^2}{2^2}\right) \&c.$$

$$\cos \frac{\pi}{2}\theta = \left(1-\frac{\theta^2}{1^2}\right)\left(1-\frac{\theta^2}{3^2}\right) \&c.$$

If we take the logarithmic differential of the first of these, we find

$$\pi \,.\, \text{cotan}\, \pi\theta = \frac{1}{\theta} - \frac{2\theta}{1^2} \,.\, \frac{1}{1-\left(\frac{\theta}{1}\right)^2}$$

$$- \frac{2\theta}{2^2} \,.\, \frac{1}{1-\left(\frac{\theta}{2}\right)^2} - \&c,$$

If each term of the second member of this be developed in powers of θ, we shall find for the coefficient of θ^{2x-1} in their aggregate

$$-2\left(\frac{1}{1^{2x}} + \frac{1}{2^{2x}} + \frac{1}{3^{2x}} + \&c.\right)$$

but, by the last article, the coefficient of θ^{2x-1} in π cotan $\pi\theta$ is found to be

$$-\frac{2^{2x}\pi}{1.\,2\ldots\ldots 2\,x} B_{2x-1} \times \pi^{2x-1},$$

and equating these, we get

$$\frac{1}{1^{2x}} + \frac{1}{2^{2x}} + \frac{1}{3^{2x}} + \&c. = \frac{2^{2x-1}\pi^{2x}}{1\,.\,2\ldots 2\,x} B_{2x-1};$$

thus,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c. = \frac{\pi^2}{6}$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \&c. = \frac{\pi^4}{90}$$

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \&c. = \frac{\pi^6}{945}.$$

If we treat the expression for $\cos \frac{\pi}{2}\theta$ in the same manner,

and compare the result with the developement of $-\frac{\pi}{2}\cdot\tan\frac{\pi}{2}\theta$, we shall find

$$\frac{1}{1^{2x}}+\frac{1}{3^{2x}}+\frac{1}{5^{2x}}+\&c.=\frac{1}{2}\frac{(2^{2x}-1)\pi^{2x}}{1.2\ldots 2x}B_{2x-1};$$

thus we have

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\&c.=\frac{\pi^2}{8}$$

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\&c.=\frac{\pi^4}{96}.$$

Again, since the series

$$\frac{1}{1^{2x}}-\frac{1}{2^{2x}}+\frac{1}{3^{2x}}-\&c.$$

is composed of the two series

$$\frac{1}{1^{2x}}+\frac{1}{3^{2x}}+\frac{1}{5^{2x}}+\&c.$$

and $$-\left(\frac{1}{2^{2x}}+\frac{1}{4^{2x}}+\&c.\right)$$

the latter of which is equal to $-\frac{1}{2^{2x}}\left(\frac{1}{1^{2x}}+\frac{1}{2^{2x}}+\&c.\right)$

it is evident, that we must have

$$\frac{1}{1^{2x}}-\frac{1}{2^{2x}}+\frac{1}{3^{2x}}-\&c.=\frac{(2^{2x-1}-1)\pi^{2x}}{1.2\ldots 2x}B_{2x-1}:$$

thus, $$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\&c.=\frac{\pi^2}{12}.$$

411. The equation

$$\Sigma u_x=\int u_x\,dx-\frac{u_x}{2}+\frac{B_1}{1.2}\cdot\frac{du_x}{dx}-\frac{B_3}{1.2.3.4}\frac{d^3u_x}{dx^3}+\&c.$$

is likewise very useful in determining, by approximation, the values of various analytical expressions, which can

scarcely be obtained in any other manner. Suppose, for instance, it were required to find the product of the natural numbers from 1 to x, x being a very high number, as 1000. If then, we take $u_x = \log x$, we have

$$\Sigma \log x = \int d x \,.\, \log x - \frac{\log x}{2} + \frac{B_1}{1\,.\,2} \,.\, \frac{1}{x} - \frac{B_3}{3\,.\,4} \,.\, \frac{1}{x^2} + \&c.$$

$$= C + (x - \tfrac{1}{2}) \log x - x + \frac{B_1}{1\,.\,2} \,.\, \frac{1}{x} - \&c.$$

and therefore, since $\log (1\,.\,2 \ldots x) = \log x + \Sigma \log x$,

$$\log (1\,.\,2 \ldots x) = C + (x + \tfrac{1}{2}) \log x - x + \&c.$$

To determine the constant C we must consider the limit, when x becomes infinite, in which case we have

$$\log (1\,.\,2 \ldots x) = C + (x + \tfrac{1}{2}) \log x - x\,;$$

whence

$$\log (1\,.\,2 \ldots 2\,x) = C + (2\,x + \tfrac{1}{2}) \log 2\,x - 2\,x,$$

and,

$$\log (2\,.\,4 \ldots 2\,x) = \log (2^x\,.\,1\,.\,2 \ldots x) = x \,.\, \log 2$$
$$+ C + (x + \tfrac{1}{2}) \log x - x.$$

From these three equations C may be found; for if we subtract the last of them from the second, we get

$$\log (1\,.\,3\,.\,5 \ldots \overline{2\,x - 1}) = x \,.\, \log x + (x + \tfrac{1}{2}) \log 2 - x\,; \qquad (a)$$

and again, subtracting this from the third,

$$\log \frac{2\,.\,4 \ldots 2\,x}{1\,.\,3 \ldots (2\,x - 1)} = C + \tfrac{1}{2} \log (2\,x) - \log 2\,;$$

whence

$$2\,C - 2 \log 2 = \log \frac{2^2\,.\,4^2 \ldots (2\,x - 2)^2\,.\,(2\,x)^2}{1^2\,.\,3^2 \ldots\ldots (2\,x - 1)^2\,.\,2\,x}$$

$$= \log \left\{ \frac{2\,.\,2}{1\,.\,3} \times \frac{4\,.\,4}{3\,.\,5} \times \ldots \frac{2\,x\,.\,2\,x}{(2\,x - 1)(2\,x + 1)} \right\}$$

Since, x being infinite, the limit of $\dfrac{1}{2\,x + 1}$ is the same with

$\frac{1}{2x}$. Now, the second member of this equation may be readily obtained from the expression for $\sin \pi\theta$, above given; for, if we make $\theta = \frac{1}{2}$, we get (410),

$$\sin\frac{\pi}{2} = 1 = \frac{\pi}{2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right) \&c.$$

$$= \frac{\pi}{2} \cdot \frac{2^2-1}{2^2} \cdot \frac{4^2-1}{4^2} \cdot \&c.$$

whence

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \times \frac{4 \cdot 4}{3 \cdot 5} \times \&c.$$

Thus we have

$$2\,C - 2\log 2 = \log\left(\frac{\pi}{2}\right);$$

whence $$C = \log \sqrt{2\pi},$$

and

$$\log(1.2\ldots x) = \log\sqrt{2\pi} + (x + \tfrac{1}{2})\log x - x + \frac{B_1}{1.2} \cdot \frac{1}{x} - \&c.$$

and passing from logarithms to numbers,

$$1 \cdot 2 \ldots x = \sqrt{2\pi} \cdot e^{-x} \cdot x^{x+\frac{1}{2}} \times e^{\frac{1}{12x} - \frac{1}{360x^3} + \&c.}$$

$$= \sqrt{2\pi x} \cdot \left(\frac{x}{e}\right)^x \cdot \left\{1 + \frac{1}{12x} + \frac{1}{288x^2} + \&c.\right\}$$

If x be considerable, the series within the brackets may be regarded as unity, and we have

$$1 \cdot 2 \ldots x = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x:$$

thus, if $x = 1000$, this result approaches within about a 12000th part of the truth; and if we calculate its value by a common table of logarithms, we find

$$
\begin{array}{rl}
\text{Log}\,\sqrt{2000\,\pi} = & 1\cdot 8990899 \\
1000\,.\,\text{Log}\,1000 = & 3000\cdot 0000000 \\
\hline
& 3001\cdot 8990899 \\
1000\,.\,\text{Log}\,.\,e = & 434\cdot 2944819 \\
\hline
& 2567\cdot 6046080
\end{array}
$$

consequently $1\,.\,2 \ldots 1000$, a number whose direct arithmetical calculation would, in all probability, be impracticable to human industry, consists of 2568 places of figures, of which the five first to the left are 40238, and this number, followed by 2563 ciphers, expresses the result within about one 12000th part of the whole; a degree of accuracy abundantly sufficient in many researches, where the ratios only of high numbers are required, as in most problems of chances.

The product of the successive odd numbers, $1, 3, \ldots (2x-1)$ is, in like manner, when x is considerable, expressed by the function

$$\sqrt{2}\,.\left(\frac{2x}{e}\right)^x,$$

as appears from the equation (a) of this article.

Application of the Integral Calculus to the Theory of Series.

412. There exists a great variety of series, the sums of which, to infinity, or to a limited number (x) of terms may be represented by an integral, taken within certain limits, and as analysis affords the means of ascertaining either accurately, or at least by approximation, the value of any proposed integral, such expressions may be fre-

quently of great utility. The following method, due to Euler, applies to a very large class of series, comprising all which proceed according to the powers of some quantity t, affected with coefficients, consisting of factors in arithmetical progression, either in the numerator or denominator. To begin with a simple instance, let

$$s = \frac{t}{2} + \frac{2}{3}t^2 + \frac{3}{4}t^3 + \&c.$$

Multiplying both sides by t, we find

$$ts = 1 \cdot \frac{t^2}{2} + 2 \cdot \frac{t^3}{3} + 3 \cdot \frac{t^4}{4} + \&c.$$

whence

$$\frac{d(ts)}{dt} = 1 \cdot t + 2 \cdot t^2 + 3t^3 + \&c.$$

and thus the denominators are taken away by differentiation. Again, if we consider that $\int x t^{x-1}\,dt = t^x$, we shall readily see, that the numerators, 1, 2, &c. may be taken away by integration; thus,

$$\frac{d(ts)}{t} = 1 \cdot dt + 2t\,dt + 3t^2\,dt + \&c.$$

and

$$\int \frac{d(ts)}{t} = t + t^2 + t^3 + \&c. = \frac{1}{1-t};$$

consequently,

$$s = \frac{1}{t}\int t \cdot d\,\frac{1}{1-t} = \frac{1}{t}\int \frac{t\,dt}{(1-t)^2}$$

$$= \frac{1}{t}\left(C + \log(1-t) + \frac{1}{1-t}\right).$$

Now, by the supposition, s vanishes when $t = o$, and therefore $C + 1 = o$, or $C = -1$; whence

$$s = \frac{\log(1-t)}{t} + \frac{1}{1-t}.$$

More generally, suppose

$$s=\frac{\alpha+\beta}{a+b}t+\frac{\alpha+2\beta}{a+2b}t^2+\ldots\frac{\alpha+x\beta}{a+xb}.t^x.$$

If we multiply both sides by $p\,t^r$ and then differentiate, it becomes

$$p.\frac{d\,(s\,t^r)}{d\,t}=(\alpha+\beta).\frac{p\,r+p}{a+b}t^r+\ldots(\alpha+x\beta).\frac{p\,r+x\,p}{a+x\,b}t^{r+x-1},$$

in which the factor $\frac{p\,r+p\,x}{a+b\,x}$, in the general term, will go out, if we take $p=b$, and $p\,r=a$, or $r=\frac{a}{b}$, when it becomes

$$b.\frac{d(s\,t^{\frac{a}{b}})}{d\,t}=(\alpha+\beta)t^{\frac{a}{b}}+\ldots(\alpha+x\beta).t^{x+\frac{a}{b}-1}.$$

To take away the factor $\alpha+\beta x$ we must multiply this again by $p\,t^r\,d\,t$, and integrate, which gives

$$p\,b\int t^r d\,(s\,t^{\frac{a}{b}})=\frac{p\,\alpha+p\,\beta}{\frac{a}{b}+r+1}t^{\frac{a}{b}+r+1}+\ldots\frac{p\,\alpha+p\,\beta\,x}{\frac{a}{b}+r+x}t^{\frac{a}{b}+r+x},$$

in which, if we make $p\beta=1$, or $p=\frac{1}{\beta}$, $p\,\alpha=\frac{a}{b}+r$, or $r=\frac{\alpha}{\beta}-\frac{a}{b}$, we shall have

$$\frac{b}{\beta}\int t^{\frac{\alpha}{\beta}-\frac{a}{b}}d\left(s\,t^{\frac{a}{b}}\right)=t^{\frac{\alpha}{\beta}}(t+t^2+\ldots t^x)=\frac{t-t^{x+1}}{1-t}t^{\frac{\alpha}{\beta}},$$

and consequently

$$s=\frac{\beta}{b}t^{-\frac{a}{b}}\int t^{\frac{a}{b}-\frac{\alpha}{\beta}}d\left\{\frac{1-t^x}{1-t}t^{\frac{\alpha}{\beta}+1}\right\}$$

the integral being taken from $t=0$ to $t=t$, provided $1+\frac{a}{b}$ be positive. Thus, if x be infinite, the sum to infinity, (which we will call S) is

$$S=\frac{\beta}{b}t^{-\frac{a}{b}}\int t^{\frac{a}{b}-\frac{\alpha}{\beta}}d\left\{\frac{t^{\frac{\alpha}{\beta}+1}}{1-t}\right\}$$

and $S-s$, or the sum of all the terms after the xth, is

$$\frac{\beta}{b} t^{-\frac{a}{b}} \int t^{\frac{a}{b}-\frac{\alpha}{\beta}} d \left\{ \frac{t^{x+\frac{\alpha}{\beta}+1}}{1-t} \right\}$$

413. If there be more factors in the numerators or denominators, they may be taken away in the same manner: for instance, if

$$s = \frac{t}{1^2} - \frac{t^2}{2^2} + \frac{t^3}{3^2} - \&c.$$

we have

$$t\frac{ds}{dt} = \frac{t}{1} - \frac{t^2}{2} + \frac{t^3}{3} - \&c.$$

$$\frac{t}{dt} d \frac{t}{dt} ds = t - t^2 + \&c. = \frac{t}{1+t};$$

whence

$$s = \int \frac{dt}{t} \int \frac{dt}{1+t},$$

the integrals being taken between the limits o and t. If we make $\frac{dt}{t} = dv$, or $t = e^v$, the limits of v will be $-\infty$, and $\log t$; and thus we find

$$s = \iint \frac{e^v dv^2}{1+e^v},$$

between the limits $v = -\infty$, and $v = \log t$.

In the same manner we may obtain an expression for the series

$$s = \frac{t}{1^n} \pm \frac{t^2}{2^n} + \frac{t^3}{3^n} \pm \&c.$$

which will be found to be

$$s = \int^n \frac{e^v}{1 \pm e^v} dv^n$$

the integrations being all executed between the same limits. From these expressions may be deduced all the properties of the functions which have these series for their develope-

ments, and to which the name of "Logarithmic Transcendents" have been given by Mr. Spence, who, in an excellent Essay,* to which the reader is referred, has treated the subject with very considerable success, and discovered a variety of remarkably elegant properties, which they possess.

414. The series

$$1 . t - 1 . 2 t^2 + 1 . 2 . 3 t^3 - \&c.$$

which, however small a value we attribute to t, must always diverge after a certain number of terms, being proposed, we have

$$\frac{dt}{t} = 1. dt - 1 . 2 t dt + 1 . 2 . 3 t^2 dt - \&c.$$

$$\int^{\prime} \frac{s dt}{t} = t - 1 . t^2 + 1 . 2 t^3 - \&c.$$

$$= t - st;$$

and therefore

$$\frac{s dt}{t} = dt (1 - s) - t ds,$$

or,

$$\frac{ds}{dt} + \frac{1+t}{t^2} . s = \frac{1}{t};$$

whence we find (257)

$$s = \frac{1}{t} e^{\frac{1}{t}} \int^{\prime} e^{-\frac{1}{t}} dt,$$

the limits of the integral being o and t. If $t = 1$, or the above integral be taken from $t =$ o to $t = 1$, we have the expression for the value of the series

$$1 - 1 . 2 + 1 . 2 . 3 - \&c.$$

and a similar method of treatment applies to the series

* Essay on the various orders of Logarithmic Transcendents, 1809. This work also contains tables of their numerical values, of some extent.

$$s = \frac{\alpha+\beta}{a+b} \cdot t + \frac{(\alpha+\beta)(\alpha+2\beta)}{(a+b)(a+2b)} t^2 + \ldots$$

$$+ \frac{(\alpha+\beta)(\alpha+2\beta)\ldots(\alpha+x\beta)}{(a+b)(a+2b)\ldots(a+xb)} t^x;$$

but this example will suffice to indicate the mode of proceeding in more complicated cases, observing that each integration affords a means of taking away a factor from the numerator, and each differentiation from the denominator of the general term. These series, in which the number of factors increases from term to term, have been designated by Euler, under the name of *hypergeometrical* series. Their summation is by these means always reducible to the integration of a differential equation, and the subsequent evaluation of an integral between given limits.

415. Now it very frequently happens, that the value of an integral between certain limits, is assignable, although its general expression cannot be obtained. For instance, the integral

$$\int^{2} \frac{e^v}{1+e^v} dv^2, \text{ or } \int dv \cdot \log(1+e^v)$$

taken from $v = -\infty$ to $v = 0$ is equal to $\frac{\pi^2}{12}$. In fact, this integral, as we have seen, is nothing more than the expression for

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \&c. = \frac{\pi^2}{12},$$

and, had we arrived at this result by any other means, it would have afforded us a legitimate summation of the above series. Exactly in the same manner, we have

$$\int^{2x-1} dv^{2x-1} \cdot \log(1+e^v) = \frac{(2^{2x-1}-1)\pi^{2x}}{1.2\ldots 2x} B_{2x-1},$$

as appears from the expression in (410) for the series

$$\frac{1}{1^{2x}} - \frac{1}{2^{2x}} + \&c.$$

the integrals being taken from $v = -\infty$ to $v = 0$.

The theory of these *definite integrals*, and their application to a multitude of important objects, forms one of the most interesting, but at the same time the most abstruse branches of Analysis. In many cases the forms they assume are peculiarly simple. Thus, if the integrals comprised in the form $\int \frac{x^{m-1}\,dx}{\sqrt{1-x^2}}$ (173) be taken between the limits $x=0$, $x=1$, the arc A becomes $\frac{\pi}{2}$, and we have

$$\int \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \qquad \int \frac{x\,dx}{\sqrt{1-x^2}} = 1$$

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\left(\frac{\pi}{2}\right) \qquad \int \frac{x^3\,dx}{\sqrt{1-x^2}} = \frac{2}{1.3}$$

$$\int \frac{x^4\,dx}{\sqrt{1-x^2}} = \frac{1\,.\,3}{2\,.\,4}\left(\frac{\pi}{2}\right) \qquad \int \frac{x^5\,dx}{\sqrt{1-x^2}} = \frac{2\,.\,4}{1\,.\,3\,.\,5},$$

&c. &c.

and, in general

$$\int \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{2\,.\,4\,.\,6\,\ldots\,(2n)}\,.\left(\frac{\pi}{2}\right)$$

$$\int \frac{x^{2n-1}\,dx}{\sqrt{1-x^2}} = \frac{2\,.\,4\,.\,6\,\ldots\,(2n-2)}{1\,.\,3\,.\,5\,\ldots\,(2n-1)}.$$

In like manner, if the integral (154)

$$\int \frac{dx}{(1+x^2)^n}$$

be taken between the limits $x = 0$ and $x = \infty$, since $\frac{x}{(1+x^2)^{q-1}}$, $\frac{x}{(1+x^2)^{q-2}}$, ... $\frac{x}{1+x^2}$, vanish at both the

limits, and $\int \frac{dx}{1+x^2}$ between the same limits, is equal to $\frac{\pi}{2}$, we shall find, by the expressions given in that article,

$$\int \frac{dx}{(1+x^2)^n} = \frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)} \cdot \left(\frac{\pi}{2}\right).$$

416.* Definite integrals furnish also the means of representing portions of the series

$$u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c.$$

beginning with any term. The following is D'Alembert's process for obtaining this result, by which he at the same time demonstrates Taylor's theorem (*Recherches sur differens points importans du systeme du Monde, tom.* I. *p.* 50.)

Let u' be the value of u, which results from the change of x to $x+h$; supposing then

$$u' = u + P,$$

and, differentiating with respect to h, which does not enter into u, we get

$$\frac{du'}{h} = \frac{dP}{dh}, \text{ whence } P = \int \frac{du'}{dh} dh,$$

$$u' = u + \int \frac{du'}{dh} dh.$$

Again, let

$$\frac{du'}{ah} = \frac{du}{dx} + Q;$$

and differentiating once more, with respect to h, we have

$$\frac{d^2u'}{dh^2} = \frac{dQ}{dh}, \text{ whence } Q = \int \frac{d^2u'}{dh^2} dh,$$

* Translated from the French edition.

$$\frac{du'}{dh} = \frac{du}{dx} + \int \frac{d^2u'}{dh^2}\,dh, \int \frac{du'}{dh}\,dh = \frac{du}{dx}\,\frac{h}{1} + \iint \frac{d^2u'}{dh^2}\,dh^2,$$

$$u' = u + \frac{du}{dx}\,\frac{h}{1} + \iint \frac{d^2u'}{dh^2}\,dh^2.$$

Again, making

$$\frac{d^2u'}{dh^2} = \frac{d^2u}{dx^2} + R,$$

we find

$$\frac{d^3u'}{dh^3} = \frac{dR}{dh}, \text{ whence } R = \int \frac{d^3u'}{dh^3}\,dh,$$

$$\frac{d^2u'}{dh^2} = \frac{d^2u}{dx^2} + \int \frac{d^3u'}{dh^3}\,dh,$$

$$u' = u + \frac{du}{dx}\,\frac{h}{1} + \frac{d^2u}{dx^2}\,\frac{h^2}{1.2} + \iiint \frac{d^3u'}{dh^3}\,dh^3.$$

Continuing this process, we should arrive at the following equation:

$$u' = u + \frac{du}{dx}\,\frac{h}{1} + \frac{d^2u}{dx^2}\,\frac{h^2}{1.2}\,\ldots\ldots\ldots\ldots$$

$$+ \frac{d^{n-1}u}{dx^{n-1}}\,\frac{h^{n-1}}{1.2\ldots(n-1)} + \int^n \frac{d^nu'}{dh^n}\,dh^n,$$

the integrals being taken so as to vanish when $h = o$.

If we suppose, for brevity's sake, $\frac{d^nu'}{dh^n} = H$, we have (220),

$$\int^n H\,dh^n =$$

$$\frac{1}{1.2\ldots(n-1)}\left\{h^{n-1}\int H\,dh - \frac{(n-1)\,h^{n-2}}{1}\int Hh\,dh\right.$$

$$\left. + \frac{(n-1)(n-2)\,h^{n-3}}{1.2}\int Hh^2\,dh - \&c.\right\};$$

and it is easy to see that we may substitute, instead of the above series, the expression

$$\frac{1}{1.2\ldots(n-1)}\int H(t-h)^{n-1}\,dh,$$

taken from the limit $h=o$, provided that after the integration, we change t to h; for, if we develope this expression, and after transferring from under the sign $\int$ the powers of t, which multiply its successive terms, make $t=h$, we arrive at the above series.

Hence it follows, that

$$u'=u+\frac{du}{dx}\frac{h}{1}+\frac{d^2u}{dx^2}\frac{h^2}{1.2}\cdots\cdots\cdots\cdots$$

$$+\frac{d^{n-1}u}{dx^{n-1}}\frac{h^{n-1}}{1.2\ldots(n-1)}+\frac{1}{1.2\ldots(n-1)}\int\frac{d^nu'}{dh^n}(t-h)^{n-1}dh,$$

provided we take the integral so as to vanish when $h=o$, and then change t to h.

In this formula we may change $\frac{d^nu'}{dh^n}$ into $\frac{d^nu'}{dx^n}$ (21);

and if, under the integral sign, we make $t-h=zt$, or $h=t(1-z)$, we shall have

$$dh=-t\,dz,\int\frac{d^nu'}{dx^n}(t-h)^{n-1}dh=-\int\frac{d^nu'}{dx^n}t^n z^{n-1}dz.$$

The limits of the integral being $z=1$, $z=o$; we may take away the negative sign by reversing the order of the limits, that is to say, by taking the integral from $z=o$ to $z=1$. Lastly, if we transfer t^n from under the integral sign, and write h for t, the last term of the above formula will become

$$\frac{h^n}{1.2\ldots(n-1)}\int\frac{d^nu'}{dx^n}z^{n-1}dz.$$

This lattertheorem is due to Lagrange, who gave it in a different manner in his "*Theorie des Fonctions Analytiques,*" Nos. 47, &c. He employs it to prove, that we can always render the sum of all the terms of Taylor's Series,

commencing with any given term, less than the term immediately preceding them. If M and m denote the greatest and least of the values which $\frac{d^n u'}{d x^n}$, has in the interval from x to $x + h$, we may easily convince ourselves that

$$\int \frac{d^n u'}{d x^n} z^{n-1} d z < \int M z^{n-1} d z, \text{ and } > \int m z^{n-1} d z,$$

provided the differential coefficient $\frac{d^n u'}{d x^n}$ do not change its sign or become infinite in this interval (211). These last integrals between the given limits are $\frac{M}{n}$ and $\frac{m}{n}$; and by taking h small enough, the quantity $\frac{h^n}{1.2 \ldots n} M$ may be rendered as small as we please, in comparison with $\frac{h^{n-1}}{1.2 \ldots (n-1)}$.

NOTES.

Note (A).

A limit, according to the notions of the ancients, is some fixed quantity, to which another of variable magnitude can never become equal, though in the course of its variation it may approach nearer to it than any difference that can be assigned; always supposing that the change which the variable quantity undergoes, is one of continued increase, or continued diminution. In this sense, we may consider the area of a circle as the limit of the areas of the circumscribed and inscribed polygons; for by increasing the number of sides of these figures, their difference may be made less than any assigned area, however small; and since the circle is necessarily less than the first, and greater than the second, it must differ from either of them by a quantity less than that by which they differ from each other: it will thus answer all the conditions of a limit, which are included in the definition we have just given.

We must not, however, conceive, that a limit, though defined to be a fixed quantity, is therefore essentially constant and invariable: we consider it as determinate, with reference to the variable quantity only, which is affected by causes, of which the limit is independent.

The consideration of limit was first introduced by the ancient geometers, in order to discover such properties of incommensurable quantities, and of circles and other curves, as were beyond the reach of direct investigation:

of which description are all the properties of circles, which are not dependent upon their intersection with straight lines. In all enquiries of this kind, they made use of their *Method of Exhaustions*, one of the most refined inventions of antiquity; and though its processes are more laborious and less comprehensive than those of many methods of later invention, in the conclusive accuracy of its reasoning, and in the clearness of its evidence, it is probably superior to them all. Thus, in order to demonstrate that different circles are to each other in the duplicate ratio of their radii,* they imagined regular polygons to be inscribed in them, each consisting of the same number of sides; and since the areas of these polygons are to each other as the squares of the radii of the respective circles, and as the same proportion must hold also, when the number of sides is so much increased, that the differences of the circles and inscribed figures may become less than any that can be assigned; they inferred from analogy, that the proportion subsisting between the areas of the polygons, must prevail likewise between the circles themselves: but this was not sufficient; it was necessary to demonstrate rigorously, that this must be the case; which was effected by shewing, that any supposition to the contrary, must necessarily lead to an absurdity. They proceeded in a similar way, in their investigations concerning the surfaces and volumes of solid bodies: they conceived bodies terminated by plane surfaces to be inscribed or circumscribed, and by increasing the number of these planes, they demonstrated rigorously, by a *reductio ad absurdum*, that the curve surfaces, or volumes of the bodies in question, were the true limits of the surfaces or volumes of the auxiliary bodies, which were thus introduced: they were then enabled, by the law of continuity, which is essential, at least in geometry, to

* Euclid, Book xii. prop. 2.

our notion of a limit, to transfer the properties of the auxiliary body to the limit itself.

It was in this manner, that Archimedes demonstrated that the convex surface of a right cone is equal to the area of a circle, whose radius is a mean proportional between the side of the cone, and the radius of the circle, which constitutes its base: that the surface of a sphere is quadruple the area of one of its great circles, and also, that the convex surface of any one of its zones, is equal to the area of a rectangle, whose adjacent sides are equal to the circumference of a great circle of the sphere, and the altitude of the zone: besides many other important propositions, both in geometry and mechanics, which it is not necessary to enumerate here.

We are indebted likewise to the same illustrious geometer, for the first summation of a geometric series, and for its application to the quadrature of a parabolic area: both these investigations are intimately connected with the theory of limits: they also constituted the first examples of the employment of series in the quadrature of curvilinear spaces, a principle which being amplified in the hands of later geometers, led finally, through a succession of important discoveries, to that of the Differential and Integral Calculus itself.

The rigorous verification of the conditions of a limit, which is required by the method of Exhaustions, must, however, encumber every demonstration in which it is employed: even the clearness of its evidence, and its visible dependence on first principles, form but a slight recommendation to it, when applied to propositions in the higher departments of mathematical science, where the mind is unable to comprehend at one view all the steps of the process, by which the conclusion is derived; the great excellence of every method, indeed, must be the union of brevity, with strictness of demonstration: and this was the

object which was attempted to be attained by various modern geometers, and in which they finally succeeded.

The first attempt of this kind was made by Cavalerius, in his Geometria Indivisibilium, in which he considers lines, planes, and solid bodies, as composed of an infinite number of indivisible or infinitely small elements: thus, lines are supposed to be made up of an infinite number of points, planes of an infinite number of lines, and solid bodies of an infinite number of planes: suppositions which are all equally contrary to the first principles of geometry. We shall probably best explain the nature and spirit of this method by a simple example.

If we conceive a triangle to be made up of an infinite number of lines drawn parallel to its base, these lines will form an arithmetic series, of which the first term is equal to zero, and the last to the base; and if a perpendicular be conceived to be drawn from the vertex to the base, the lines of which the triangle is composed, will intersect this perpendicular in an infinite number of points, which will together compose this line, which may therefore be taken as the representative of their number. Now, the sum of a series of the nature above-mentioned, is equal to the product of the last term, and half the number of terms, or in other words, the area of the triangle is equal to the rectangle contained by the base, and half the perpendicular.

We may apply this method likewise, with considerable, though partial success, to the quadrature of curvilinear spaces, and the cubature of solid bodies; and in nearly all cases, the problem will reduce itself to the summation of a series. We thus see how nearly, in practice at least, this method approaches to many of the processes of Algebra, and of the Integral Calculus, in investigations of this nature.

Pascal, who made use of this method in the discovery and demonstration of some properties of the cycloid,

endeavoured to shew, that the objections to it, were merely verbal, and would be completely removed by some little additional explanation of its first principles: thus, when we speak of a plane as the sum of an infinite number of lines, we ought tacitly to suppose, that they severally form rectangles with the equal, and infinitely small portions of some given line; and we may explain, in a similar manner, the expressions which assert, that a line is the sum of an infinite number of points, and a solid body the sum of an infinite number of planes. The method of indivisibles, is reduced, by this statement, to the method of infinitesimals, in which areas are considered as the sums of an infinite number of inscribed rectangles, and solid bodies, as the sums of an infinite number of inscribed solids, of infinitely small altitude, and two of whose surfaces are equal and parallel planes: the small areas and portions of the solid bodies corresponding to each inscribed rectangle, or inscribed solid, which these hypotheses neglect, form a series of infinitesimals of the second order, bearing no assignable ratio whatever to the infinitesimal rectangles and solids, of which the original areas or bodies are conceived to be composed. We confine our reasonings to the figures and solids inscribed, and assume, as a first principle, that all the properties which can be proved to belong to them, may be transferred to the original figures or bodies themselves. We consequently omit altogether the verification of a limit, which formed one of the most difficult processes of the method of Exhaustions.

The principles of this method were insensibly adopted by the greatest part of the geometers of that period, who considered the want of rigour in the demonstrations founded upon them, as fully compensated, by their superior facility: they verified their conclusions, by shewing that they were identical with those of the ancient geometers, when treating of the same propositions; an agreement

which was always appealed to, as the test of the truth of new principles, or new methods of investigation; but one principal advantage resulting from the introduction of this method, consisted in the increased facilities which it gave to the applications of Algebra to the theory of curves, and particularly, by its forming, in the hands of Leibnitz, the basis of his reasoning in the discovery and demonstration of the principles of the Differential and Integral Calculus.

Newton finally succeeded in embodying, in his method, of prime and ultimate ratios, the accurate reasoning of the method of Exhaustions, without its prolixity. He considers quantities and their ratios as *ultimately equal*, which constantly approximate to each other, and in any finite time (for his principal object was the application of this method to mechanical philosophy), approach nearer to equality, than any difference that can be assigned. He then proceeds to make use of this principle, in the demonstration of a series of propositions or lemmas, which are applicable generally to all curves of continuous curvature, and to the theory of variable motions, and which likewise serve as first principles in all succeeding investigations.

It is not our intention to enter into a detailed explanation of the nature of the reasoning employed in these demonstrations, which principally consists in exhibiting to the eye, as well as to the mind, the relation of quantities invisible and evanescent by means of others, which are visible and finite. For the benefit, however, of such of our readers as are not acquainted with the original work, we will give a slight sketch of the process pursued in one of the most important of the lemmas, in which it is proposed to prove, that in all curves of continuous curvature, the chord, the arc, and the tangent are ultimately equal. Assume two similar arcs (a) and (b), commencing from a point of contact of two similar curves (A) and (B); since the arcs are similar and similarly situated, and commence

from a common point, their chords must be in the same straight line: their tangents also, drawn from the point of contact, must coincide with each other; let them be determined by secants drawn from the centres of equal curvature, or from corresponding points in the perpendicular to the tangent at the point of contact, and passing through the extremities of the arcs, and let the tangent (t), corresponding to the arc (b), be assumed as fixed and unchangeable. Suppose the arc (a) to be perpetually diminished, either by successive bisection, or in any other manner: if the arc (b) of the same curve (B) be changed proportionally, its tangent will be also changed, which is contrary to the hypothesis we have made: but by supposing the dimensions of the curve (B) to be perpetually increased, we shall always be able to find some state of it, in which the arc (b) may be successively taken similar to the successive values of the arc (a), and have its tangent equal to the determinate tangent (t). But whilst the arc (a) diminishes, the angle between the chord and the tangent, diminishes also; and by a preceding lemma, in its ultimate state, it becomes evanescent: the same changes must likewise take place in the value of the angle between the chord and the tangent of the arc (b), which in its ultimate state must be likewise evanescent. In this state, therefore, the chord, the tangent (t), and the arc, which is always intermediate to them, must be coincident and equal, and the same must be the case with the chord, arc, and tangent of the arc (a), which constantly bear to each other the same proportion as the corresponding lines in the arc (b). We must refer to the other lemmas, and particularly to the 9th, for further examples of the application of this refined and beautiful artifice.

In this method we speak of the *prime*, as well as of the *ultimate* ratios of variable quantities, according as we consider them as receding from, or approaching to, the fixed and determinate quantities which are their proper limits.

The excellence of Newton's method consisted in the strict demonstration of a number of first principles, which afterwards entirely superseded the tedious verification of a limit which embarrassed the proof of every new proposition, by the method of Exhaustions; and though in the brevity, and even elegance of its processes, it must sometimes yield to the methods of infinitesimals and indivisibles, yet its evidence is never weakened by the introduction of assumptions which are not axiomatic, nor vitiated by hypotheses which are contrary to the first principles of Geometry.*

We have been thus particular in our account of limits, and of the different systems in which their theory has been embodied, not only on account of the importance of the subject itself, but more particularly with reference to different systems of the Differential and Integral Calculus, of which they formed the bases. The student will find some previous knowledge of these different methods, almost essential to his fully understanding the nature and spirit of the reasoning employed in the analytical theories derived from them; and although the consideration of

---

* These are the reasons assigned by Newton, in the Scholium to the Lemmas, as having induced him to invent the method of prime and ultimate ratios: "Præmisi verò hæc lemmata, ut effugerem tædium deducendi longas demonstrationes, more veterum geometrarum, ad absurdum. Contractiores enim redduntur demonstrationes per methodum indivisibilium. Sed quoniam *durior* est indivisibilium hypothesis, et propterea minus geometrica censetur; malui demonstrationes rerum sequentium ad ultimas quantitatum evanescentium summas et rationes, primasque nascentium, id est, ad limites summarum et rationum deducere; et propterea limitum illorum demonstrationes quâ potui brevitate præmittere. His enim idem præstatur quod per methodum indivisibilium: et principiis demonstratis jam tutiùs utemur." The whole of this Scholium is well worth the diligent attention of the student.

limits or infinitesimals, in the establishment of this calculus, is calculated to mislead the mind from its true meaning and origin, the same excellencies and defects will be found to distinguish the reasonings employed in these different systems, which have already been remarked in the geometrical methods from which they severally originate.

We will now mention a few results deducible algebraically, from our definition of a limit, some of which may be found useful in succeeding investigations. Suppose it was required to find the limit of the ratio of the chord of a circular arc to its sine: if we make x = versed sine, we have

$$\frac{\text{chord}}{\text{sine}} = \frac{\sqrt{2x}}{\sqrt{2x-x^2}} = \frac{\sqrt{2}}{\sqrt{2-x}};$$

but the limit of $\sqrt{2-x}$ is $\sqrt{2}$, since it may be made to differ from it by a quantity less than any assignable; we may consequently conclude, that the limit of this ratio is $\frac{\sqrt{2}}{\sqrt{2}}$, or 1; or in other words, that the chord and the sine are ultimately equal. In all investigations therefore, in which the limits only of these quantities are considered, we may make use indifferently of one or the other.

In a similar manner, we find

$$\frac{\text{sine}}{\text{cosine}} = \frac{\sqrt{x}\,.\,\sqrt{2-x}}{1-x},$$

and

$$\frac{\text{sine}}{\text{versed sine}} = \frac{\sqrt{x}\,.\,\sqrt{2-x}}{x} = \frac{\sqrt{2-x}}{\sqrt{x}};$$

we may readily prove, that the limit of the first ratio is zero, and of the second infinity: we hence conclude, that in their ultimate state, the sine bears no assignable ratio to the cosine, nor the versed sine to the sine.

Again, we have

$$\frac{\text{tangent}}{\text{sine}} = \frac{1}{1-x},$$

$$\text{and } \frac{\text{tangent}}{\text{chord}} = \frac{\sqrt{2-x}}{(1-x)\sqrt{2}};$$

ratios which have respectively unity for their limit. When speaking of limits, therefore, the chord, the sine, and the tangent may be considered as equal to each other.

Let us next endeavour to ascertain the limit of the ratio of the arc and the sine; for this purpose, we shall assume as true the principle of Archimedes, by which it appears, that a circular arc is greater than its sine, and less than its tangent: from thence we conclude, that

$$\frac{\text{tangent}}{\text{sine}} > \frac{\text{arc}}{\text{sine}} > 1:$$

we have already proved, that the quotient $\left\{\frac{\text{tangent}}{\text{sine}}\right\}$ may be made to differ from unity by a quantity less than any that can be assigned; and since the quotient $\left\{\frac{\text{arc}}{\text{sine}}\right\}$ is less than $\left\{\frac{\text{tangent}}{\text{sine}}\right\}$ and greater than unity, it must differ less from unity than $\left\{\frac{\text{tangent}}{\text{sine}}\right\}$ itself, although it can never actually become equal to it: we may consequently conclude, that unity is its limit. The tangent, chord, and sine of an arc, and the arc itself, may therefore be taken as equivalent quantities, when considered in their ultimate state.

We will now take a very common example, which is connected, however, with some enquiries which will succeed it. Let it be required to find the sum, or rather the limit of the sum of the indefinite geometric series,

$$a + ar + ar^2 + ar^3 + \text{\&c.}$$

where r, which expresses the inverse ratio of any two consecutive terms, is supposed to be less than unity. The sum of n terms of this series, determined by the common method, is $\frac{a}{1-r} - \frac{a r^n}{1-r}$; but since it is a proper fraction, the second part $\frac{a r^n}{1-r}$ of this expression, approaches to zero, whilst n increases, and may be made to differ from it less than any assignable quantity; we may consequently conclude, that the limit of the sum of the series, or the sum itself, is equal to $\frac{a}{1-r}$.

It is frequently an enquiry of great importance in analysis, to ascertain whether an indefinite series of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c. \qquad (a)$$

can be made convergent, by assigning a determinate value of x; the inverse ratio of any two consecutive coefficients, being always supposed to be finite. Assuming r to represent the greatest value of this inverse ratio, if we construct the geometric series

$$a_0 + a_0 r x + a_0 r^2 x^2 + a_0 r^3 x^3 + \&c. \qquad (b)$$

we shall have the coefficients $a_0 r$, $a_0 r^2$, $a_0 r^3$, &c. severally greater than those corresponding to them in the original series. There are some cases, however, which form, in some degree, exceptions to this assertion, which are those in which $r = \frac{a_1}{a_0}$, and consequently $a_0 r = a_1$, and also those in which $r = \frac{a_1}{a}$, and any given number of the immediately succeeding ratios are respectively equal to the first, when we have $a_1 = a_0 r$, $a_2 = a_0 r^2$, &c. and so on, till we arrive at a ratio which is less than r.

Since the terms of the series (b) are severally equal to or greater than, those corresponding to them, in the series

(a), it is obvious that any value of x, which renders the first convergent, must likewise produce a similar effect in the second, and in a greater degree, at least from the point where its terms become less than those corresponding to them, in the limiting geometric series. It is hardly necessary to remark, that any value of x which makes rx a proper fraction, will render both the series convergent, and this will be the case with the original series, even if rx be made equal to unity, although the convergency will not, in all cases, commence immediately from its first term.

Let us take, as an example, the following series:

$$1+2 \cdot 3^3 \cdot x + 3 \cdot 3^6 \cdot x^2 + 4 \cdot 3^9 \cdot x^3 + \&c.$$

two of whose consecutive terms are expressible generally by

$$n \cdot 3^{3n-3} x^{n-1} + (n+1) 3^{3n} x^n;$$

the inverse ratio of two consecutive coefficients, is expressed in general by

$$\frac{n+1}{n} \cdot 3^3,$$

and if we make successively

$$n=1, \quad n=2, \quad n=3, \quad \&c.$$

the quantity $\frac{n+1}{n} \cdot 3^3$ will become

$$2 \cdot 3^3, \quad \frac{3}{2} \cdot 3^3, \quad \frac{4}{3} \cdot 3^3, \quad \&c.$$

and its greatest value, or r, is obviously $2 \cdot 3^3$, or 54. If we substitute therefore $\frac{1}{54}$ for x, in the given series, it will become convergent from its first term.

The problem, however, is impossible, when the ratios

$$\frac{a_1}{a_0}, \quad \frac{a_2}{a_1}, \quad \frac{a_3}{a_2}, \quad \&c.$$

perpetually increase in value; of this we have an instance in the series

$$1 + 1 . 2 . x + 1 . 2 . 3 . x^2 + 1 . 2 . 3 . 4 . x^3 + \&c.$$

The inverse ratio of whose terms, which is equal to $(n+2)x$, increases perpetually with the unlimited number n; whatever value therefore we assign to x, the product $(n+2)x$ will finally become equal to unity, and afterwards exceed it; and the series must from that point become divergent.

The sum of the series (b) is equal to $\frac{a_0}{1 - rx}$, which has been shewn to be always greater than that of the given series (a): if for x we put $\frac{1}{2r}$, the quantity $\frac{a_0}{1-rx}$ will be found to be equal to $2a_0$, or, in other words, the first term of the geometric series will be equal to the sum of all the rest: we may from this conclude, that the substitution of the same value of x in the series (a), will make the first term superior to the sum of all those which succeed it. If it was required to find some determinate value of x, which would make the sum of the given series (a), less than $a_0 + \delta$, we should be able to solve the problem, by assuming $\frac{a_0}{1-rx} = a_0 + \delta$, from which we obtain $x = \frac{\delta}{r(a_0+\delta)}$, and substituting this value in the series (a), its sum would be necessarily less than $a_0 + \delta$, the sum of the geometric series (b) corresponding to the same value of x. We thus see, that by making $x = \frac{\delta}{r(a_0+\delta)}$, the sum of the series (a) will differ less from its first term, than the quantity δ, however small this quantity may be assumed.

It is hardly necessary to observe, that the preceding observations are applicable *à fortiori* to the series (a), when any number whatever of its coefficients become negative or evanescent.

We will now proceed to the demonstration of a theorem

of great importance, particularly in the applications of the Differential Calculus, when established on its true principles, to the theory of curves.

Let the three expressions

$$a_0+a_1x+a_2x^2+a_3x^3 + \&c. \qquad (1)$$

$$a_0'+a_1'x+a_2'x^2+a_3'x^3 + \&c. \qquad (2)$$

$$a_0''+a_1''x+a_2''x^2+a_3''x^3 + \&c. \qquad (3)$$

be so related to each other, that the values of the second may be always less than those of the first, and greater than those of the third: if $a_0=a_0''$, *then also must* $a_0'=a_0$.

We have already shewn, that it is possible to determine a value of x, which will make the value of the series (1) less than $a_0+\delta$, however small δ may be assumed: let its true value, resulting from this substitution be a_0+d, and let $a_0'+d'$, $a_0''+d''$, be the values of the series (2) and (3), under the same circumstances: since the series themselves are arranged in the order of their magnitudes, the magnitudes of a_0+d, $a_0'+d'$, $a_0''+d''$, must also follow the same order, and the differences $(a_0+d)-(a_0'+d')$ and $(a_0'+d')-(a_0''+d'')$, or $a_0-a_0'+d-d'$, and $a_0'-a_0''+d'-d''$, must be positive quantities. Assume $a_0''=a_0$, and $a_0'=a_0+\Delta$; the differences will thus be reduced to $-\Delta+d-d'$, and $\Delta+d'-d''$; but since it is possible to assign a value to x, which will make both d and d' severally less than any given quantity, we must have $d-d'$, under these circumstances, less than Δ, and consequently the difference $-\Delta+d-d'$ must be negative, a result which is contrary to the condition above specified: it is obvious that this conclusion will not be affected, by supposing d' negative, since the sum of the quantities d and d' may be also made less than Δ. Again, assume $a_0'=a_0-\Delta$; the differences upon this hypothesis become $\Delta+d-d'$, and $-\Delta+d'-d''$: we may prove as before, that $-\Delta+d'-d''$ may become a negative quantity, a conclusion which is likewise inconsistent with

what has been already established. We are thus reduced to the only remaining hypothesis, which is that of $a_0' = a_0$. We will give a single instance of the application of this theorem: in the developement of $\sin x$ in terms of x and its powers, we have direct means of shewing, that it must be of the following form:

$$\sin x = a_0 x - \frac{a_0^3}{1.2.3} x^3 + \frac{a_0^5}{1.2.3.4.5} . x^5 - \&c.$$

it yet remains to determine the constant quantity a_0.

We have already seen, that $x < \tan x$, and $> \sin x$: but

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sin x}{\sqrt{(1-\sin^2 x)}};$$

and consequently

$$\frac{\sin x}{\sqrt{(1-\sin^2 x)}} > x;$$

we hence get

$$\sin x > x\sqrt{(1-\sin^2 x)},$$

$$\sin^2 x > x^2 \quad (1-\sin^2 x).$$

By transposition, we have

$$\sin^2 x\,(1+x^2) > x^2;$$

therefore

$$\sin^2 x > \frac{x^2}{1+x^2},$$

and

$$\sin x > \frac{x}{\sqrt{(1+x^2)}}:$$

We consequently have

$$\sin x < x$$

$$\sin x = a_0 x - \frac{a_0^3}{1.2.3} x^3 + \frac{a_0^5}{1.2.3.4.5} . x^5 - \&c.$$

$$\sin x > \frac{x}{\sqrt{(1-x^2)}} > x\left(1 - \frac{1}{2}x^2 + \frac{1.3}{2.4} . x^4 - \&c.\right)$$

and therefore

$$\frac{\sin x}{x} < 1$$

$$\frac{\sin x}{x} = a_0 - \frac{a_0^3}{1.2.3} \cdot x^2 + \frac{a_0^5}{1.2.3.4.5} \cdot x^4 - \&c.$$

$$\frac{\sin x}{x} > 1 - \frac{1}{2} x^2 + \frac{1.3}{2.4} \cdot x^4 - \&c.$$

We thus obtain three expressions, which answer the conditions expressed in the enunciation of the preceding theorem; and since the first terms of the greatest and least are identical, and equal to unity, we may conclude that a_0, the first term of the second, is equal to unity also.

The series for $\sin x$ will thus become

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$$

NOTE (B).

The method which is made use of by our author, in the exposition and demonstration of the principles of the Differential Calculus, was first given by D'Alembert, in the Encyclopédie.* We shall not at present stop to discuss its merits, but shall proceed directly to shew in what manner this calculus may be established upon principles which are entirely independent of infinitesimals or limits. We shall afterwards endeavour to explain the reason why the same conclusions have invariably been deduced from apparently different first principles.

It is hardly necessary to inform the reader, that we are indebted for the principal part of the contents of this note, to the Calcul des Fonctions of Lagrange, and the large trea-

* Art. Differentiel.

tise by our author, on the Differential and Integral Calculus.

Let $f(x)$ represent a function of x, which becomes $f(x+h)$ when x, its base, is changed into $x+h$: the increment of the function corresponding to the increment h of its base, or as it is commonly called, the *difference* of the function, will be $f(x+h)-f(x)$, which we shall consider as being always capable of developement, in a series according to powers of h. Our next enquiry must be directed to the form and properties of this developement.

Suppose

$$f(x+h)=u+A_1 h^{a_1}+A_2 h^{a_2}+A_3 h^{a_3}+\ \&c. \qquad (a)$$

where A_1, A_2, A_3, &c. are functions of x, which are severally the coefficients of the different powers of h.

In the first place $u=f(x)$; for by supposing $h=0$, $f(x+h)$ becomes $f(x)$, and the series (a) is reduced to its first term. Again, the indices a_1, a_2, a_3, &c. must all be positive; for otherwise all those terms which involve negative powers of h, must become infinite by its evanescence; a supposition which would entirely destroy the essential generality of the function $f(x)$.

Since $u=f(x)$, we will assume, for greater brevity, $u'=f(x+h)$, and $u''=f(x+h+k)$. We shall also consider A_1', A_2', A_3', &c. as the representatives of the new values of the functions A_1, A_2, A_3, &c. which result from the substitution of $x+h$, in the place of x.

The function u'' may be developed under two different forms, according as we consider it as representing $f(x+(h+k))$, or $f((x+h)+k)$. The first hypothesis will give,

$$u''=u+A_1(h+k)^{a_1}+A_2(h+k)^{a_2}+A_3(h+k)^{a_3}+\&c. \qquad (a')$$

and the second,

$$u''=u'+A_1'k^{a_1}+A_2'k^{a_2}+A_3'k^{a_3}+\ \&c. \qquad (a'')$$

But we have already seen (a), that

$$u' = u + A_1 h^{\overset{a}{1}} + A_2 h^{\overset{a}{2}} + A_3 h^{\overset{a}{3}} + \&c.$$

and since A_1', A_2', A_3', &c. are severally functions of $x+h$, they will all admit of developements in series, according to powers of h: consequently,

$$A_1' = A_1 + B_1 h^{\overset{b}{1}} + B_2 h^{\overset{b}{2}} + B_3 h^{\overset{b}{3}} + \&c.$$

$$A_2' = A_2 + C_1 h^{\overset{c}{1}} + C_2 h^{\overset{c}{2}} + C_3 h^{\overset{c}{3}} + \&c.$$

$$A_3' = A_3 + D_1 h^{\overset{d}{1}} + D_2 h^{\overset{d}{2}} + D_3 h^{\overset{d}{3}} + \&c.$$

$$\&c. \quad \&c.$$

The series (a'') may therefore be exhibited under the following form:

$$u'' = \left\{ \begin{array}{l} u + A_1 h^{\overset{a}{1}} + A_2 h^{\overset{a}{2}} + A_3 h^{\overset{a}{3}} + \&c. \\ + A_1 k^{\overset{a}{1}} + B_1 h^{\overset{b}{1}} k^{\overset{a}{1}} + B_2 h^{\overset{b}{2}} k^{\overset{a}{1}} + B_3 h^{\overset{b}{3}} k^{\overset{a}{1}} + \&c. \\ + A_2 k^{\overset{a}{2}} + C_1 h^{\overset{c}{1}} k^{\overset{a}{2}} + C_2 h^{\overset{c}{2}} k^{\overset{a}{2}} + C_3 h^{\overset{c}{3}} k^{\overset{a}{2}} + \&c. \\ + A_3 k^{\overset{a}{3}} + D_1 h^{\overset{d}{1}} k^{\overset{a}{3}} + D_2 h^{\overset{d}{2}} k^{\overset{a}{3}} + D_3 h^{\overset{d}{3}} k^{\overset{a}{3}} + \&c. \\ \qquad + \&c. \end{array} \right.$$

The series (a') and (b) are true, under all circumstances, whatever be the relative values of h and k: we may assume, therefore, $h=k$; and since the series are identical, being developements of the same quantity u'', the coefficients of the powers of h in the resulting form of (a'), or

$$u + A_1 . 2^{\overset{a}{1}} h^{\overset{a}{1}} + A_2 . 2^{\overset{a}{2}} h^{\overset{a}{2}} + A_3 . 2^{\overset{a}{3}} h^{\overset{a}{3}},$$

must be identical with those corresponding to them, in the new form of (b), which also arises from this hypothesis: we shall find, by actual comparison,

$$2 A_1 = A_1 . 2^{\overset{a}{1}}, \text{ or } 2 = 2^{\overset{a}{1}},$$

and therefore a_1 must be equal to unity.

We have, therefore, in all cases whatever,

$$f(x+h)=f(x)+A_1 h+A_2 h^2+A_3 h^3+\&c.$$

the second term, involving the first power of h only. We may conclude from this also, that b_1, c_1, d_1, &c. which are the indices of h in the second terms of the several developements of A_1', A_2', A_3', &c. are respectively equal to unity.

We will arrest, for a short time, our enquiries concerning the other terms of this developement, in order to consider more particularly the second term, the determination of which, in different functions, constitutes an essential part of the Differential Calculus.

We have just shewn, that the *difference* of the function $f(x)$ is

$$A_1 h.+A_2 h^2+A_3 h^3+\&c.$$

Analysts have agreed to call its first term $A_1 h$, the *differential* of the function, and A_1, which is the function of x by which h is multiplied, is denominated the *differential coefficient*. The student should be very careful not to attach to these terms any meaning, which is not distinctly implied in the definitions we have just given.

It is usual to denote the difference of a function by the characteristic Δ, and its differential by the characteristic d: thus $\Delta u=u'-u=\Delta f(x)=f(x+h)-f(x)$, and $du=df(x)=A_1 h$. This notation is perfectly arbitrary, and is different with different authors: we shall state hereafter the reasons which have induced us to make choice of the above.

The process of finding the differential of a function is called *differentiation*: we shall proceed to apply it to a few simple examples.

1. Let $u=x$; in this case $u'=x+h$, and consequently $\Delta x=h=dx$; the difference and differential of x and its

increment h are therefore identical quantities, and may be interchanged at pleasure; we shall follow the common custom of replacing h by dx: we shall thus have generally $du = A_1 dx$, and A_1, or the differential coefficient $= \frac{du}{dx}$.

2. Let $u = ax$: therefore $u' = a(x+h) = ax + ah$; from which we get $\Delta u = ah = adx = du$: in this example also, the difference and differential are identical.

3. Let $u = x^2$; we have $u' = (x+h)^2 = x^2 + 2xh + h^2$, and $\Delta u = 2xh + h^2$; we hence get $du = 2xh = 2xdx$.

4. Let $u = ax^2 + bx + c$; in this case $u' = a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + 2xh + bh + h^2$: we hence obtain $\Delta u = 2xh + bh + h^2$, and $du = 2xdx + bdx$. We ought to observe, that the constant quantity c, which was connected by addition with the given function, disappears in the difference and differential: the same must be the case with all constant quantities whatever, which are not connected with the variable parts of functions, by multiplication or division.

5. Let $u = x^n$, where n is any integral or fractional number; in this case $u' = (x+h)^n = x^n + A_1 h + A_2 h^{\alpha_2} +$ &c. and our first enquiry must be directed to the determination of A_1; to effect this, we must put this developement under a somewhat different form: since $(x+h)^n = x^n\left(1 + \frac{h}{x}\right)^n$, we have $(x+h)^n = x^n\left\{1 + \alpha_1 \frac{h}{x} + \alpha_2 \left(\frac{h}{x}\right)^{a_2} + \text{\&c.}\right\} = x^n + \alpha_1 x^{n-1} h + \alpha_2 x^{n-a_2} h^{a_2} +$ &c. expressions which are obviously identical. The coefficients α_1, α_2, &c. being independent of x, must be functions of n, the only quantity upon which their value can be dependent. We may assume, therefore, $\alpha_1 = f(n)$. If n be supposed to become $n+1$, we shall have

$$(x+h)^{n+1}=x^{n+1}+f(n+1)\,.\,x^n h+\ \&c. \qquad (a)$$

and multiplying the developement of $(x+h)^n$ by $x+h$, we shall have another expression for $(x+h)^{n+1}$, under a different form, namely,

$$\left.\begin{array}{r}(x+h)^{n+1}=x^{n+1}+f(n)\,x^n h+\ \&c.\\ x^n h+\ \&c.\end{array}\right\}\ (b)$$

a comparison of those terms in the identical developements (a) and (b), which involve the first powers of h, will give us the following equation:

$$f(n+1)=f(n)+1.$$

By supposing $n=0$, the second term of $(x+h)^n$ can have no existence, and consequently $f(n)=0$: it follows from this, that n must be a factor of $f(n)$ in all cases. Again, the first power of n alone can enter into this function; for otherwise the difference $f(n+1)-f(n)$ would be a function of n also, as may be easily shewn. We must have, therefore $f(n)=a\,n=n$; for a must be equal to unity, in virtue of the equation $f(n+1)-f(n)=a(n+1)-a\,n=a=1$.

We thus obtain

$$(x+h)^n=x^n+n\,x^{n-1}+\ \&c.$$

$$=n\,x^{n-1}\,h=n\,x^{n-1}\,d\,x$$

$$\text{and}\quad \frac{d\,u}{d\,x}=n\,x^{n-1}.$$

6. Let $u=x^{-n}$, where n is an integral or fractional number: by a process similar to the above, we shall be able to prove, that

$$(x+h)^{-n}=x^{-n}-n\,x^{-n-1}h+\ \&c.$$

we consequently obtain $d\,u=-\,n\,x^{-n-1}\,d\,x$, and $\dfrac{d\,u}{d\,x}=-n\,x^{-n-1}$.

We will now resume the consideration of the form and

properties of the remaining terms of the series (a), in the determination of which, the knowledge of the second term of the expansion of $(x+h)^n$ will be found of essential importance.

We are thus enabled to exhibit the series (a') under the following form:

$$u'' = \left\{ \begin{array}{l} u \\ + A_1 h + A_1 k. \\ + A_2 h^{a_2} + A_2 a_2 h^{a_2 - 1} k + \&c. \\ + A_3 h^{a_3} + A_3 a_3 h^{a_3 - 1} k + \&c. \\ \quad \&c. \quad \&c. \end{array} \right. \qquad (a''')$$

By comparing the terms of this series with those of the identical series (b), (the indices a_1, b_1, c_1, d_1, &c. being severally replaced by unity), we shall obtain the following equations:

$$a_2 A_2 h^{a_2 - 1} k = B_1 h k.$$

$$a_3 A_3 h^{a_3 - 1} k = B_2 h^{b_3} k.$$

$$a_4 A_4 h^{a_4 - 1} k = B_3 h^{b_3} k.$$

$$\&c. \qquad \&c.$$

In order that the powers of h in each equation may be identical, the following equations must obtain:

$$a_2 - 1 = 1, \quad a_3 - 1 = b_2, \quad a_4 - 1 = b_3, \ \&c.$$

From the first of these we get $a_2 = 2$; and this general conclusion being applicable to the developements of A_1', A_2', A_3', &c. will give $b_2 = 2$, $c_2 = 2$, $d_2 = 2$, &c.

From the second equation we obtain $a_3 = b_2 + 1 = 3$; and in a similar manner we may conclude, that $b_3 = 3$, $c_3 = 3$, &c.

From the third equation we obtain $a_4 = b_3 + 1 = 4$, and by a continuation of the same process and reasoning, we shall

get $a_5=5$, $a_6=6$, and generally $a_n=n$: $b_4=4$; $b_5=5$, and $b_n=n$: $c_4=4$, $c_5=5$, and $c_n=n$; and so on, for the series of the indices of h, in each successive developement.

The general form of the series (a) will, therefore, be the following:

$$f(x+h)=f(x)+A_1 h+A_2 h^2+A_3 h^3+ \&c.$$

the indices of h, in the successive terms, forming the series of natural numbers.

Again, by referring to the equations obtained above, we shall find

$$A_2=\tfrac{1}{2} . B_1, \quad A_3=\tfrac{1}{3} . B_2, \quad A_4=\tfrac{1}{4} . B_3, \ \&c.$$

Now, since $A_1=\dfrac{du}{dx}$, and $B_1=\dfrac{dA_1}{dx}$, we shall have

$$A_2=\frac{1}{2} . \frac{dA_1}{dx}=\frac{1}{2} . \frac{d\left(\frac{du}{dx}\right)}{dx}=\frac{1}{2} . \frac{d(du)}{dx^2};$$

but $d(du)$, which is called the second differential of u, is usually written d^2u, where the index 2 is symbolical of the number of operations: we thus obtain $A_2=\dfrac{1}{1.2} . \dfrac{d^2u}{dx^2}$: a conclusion, which being extended to the other developements, gives us $B_2=\dfrac{1}{1.2} . \dfrac{d^2A_1}{dx^2}$, $C_2=\dfrac{1}{1.2} . \dfrac{d^2A_2}{dx^2}$, &c.

Again,

$$A_3=\frac{1}{3} . B_2=\frac{1}{1.2.3} . \frac{d^2A_1}{dx^2}=\frac{1}{1.2.3} . \frac{d^2\left(\frac{du}{dx}\right)}{dx^2}$$

$$=\frac{1}{1.2.3} . \frac{d^2.(du)}{dx^3}=\frac{1}{1.2.3} . \frac{d^3u}{dx^3},$$ by a further extension of the principle of notation mentioned above. We from hence conclude, that

$$B_3 = \frac{1}{1.2.3}\frac{d^3 A_1}{d x^3},\ C_3 = \frac{1}{1.2.3}\frac{d^3 A_2}{d x^2},$$

and so on, for D_3, &c.

Again,

$$A_4 = \frac{1}{4} . B_3 = \frac{1}{1.2.3.4} . \frac{d^3 A_1}{d x^3} = \frac{1}{1.2.3.4} . \frac{d^3\left(\frac{du}{dx}\right)}{d x^3}$$

$$= \frac{1}{1.2.3.4} . \frac{d^4 u}{d x^4}.$$

The law of the derivation and formation of the other coefficients is now sufficiently manifest.

We thus arrive at another form of the series (a),

$$u' = u + \frac{du}{dx} . \frac{h}{1} + \frac{d^2 u}{dx^2} . \frac{h^2}{1.2} + \frac{d^3 u}{dx^3} . \frac{h^3}{1.2.3} + \frac{d^4 u}{dx^4} . \frac{h^4}{1.2.3.4} + \&c.$$

known by the name of Taylor's Theorem, and which exhibits the whole theory of the Differential Calculus.

This theorem might have been deduced by a process somewhat more simple, by a slight change in the form of series (a); for by interchanging the quantities h and k, we should have

$$u'' = u + A_1(k+h) + A_2(k+h)^{\overset{a}{2}} + \&c.$$

and by expanding the several binomials, we get

$$u'' = \begin{cases} u + A_1 k + A_1 h \\ + A_2 k^{\overset{a}{2}} + A_2 a_2 k^{\overset{a}{2}-1} h + \&c. \\ + A_3 k^{\overset{a}{3}} + A_3 a_3 k^{\overset{a}{3}-1} h + \&c. \\ + \&c. \end{cases}$$

A comparison of this with the series (b), will give us the following equations:

$A_2 a_2 k^{\overset{a}{2}-1} h = B_1 k h$, and therefore $a_2 - 1 = 1$, or $a_2 = 2$.

$A_3 a_3 k^{\overset{a}{3}-1} k = B_1 k^{\overset{a}{2}} k$, and therefore $a_3 - 1 = a_2$, or $a^3 = 3$.

$A_4 a_4 k^{\overset{a}{4}-1} k = D_1 k^{\overset{a}{3}} h$, and therefore $a_4 - 1 = a_3$, or $a_4 = 4$.

&c. &c. &c. &c.

We thus immediately obtain the values of a_2, a_3, a_4, &c. from which we find

$$A_2 = \frac{1}{2} \cdot B_1 = \frac{1}{2} \cdot \frac{d A_1}{d x}, \text{ since } B_1 = \frac{d A_1}{d x}.$$

$$A_3 = \frac{1}{3} \cdot C_1 = \frac{1}{3} \cdot \frac{d A_2}{d x}, \text{ since } C_1 = \frac{d A_2}{d x}.$$

$$A_4 = \frac{1}{4} \cdot D_1 = \frac{1}{4} \cdot \frac{d A_3}{d x}, \text{ since } D_1 = \frac{d A_3}{d x}.$$

&c. &c.

We from hence get

$$A_2 = \frac{1}{2} \cdot \frac{d\left(\frac{d u}{d x}\right)}{d x} \left(\text{since } A_1 = \frac{d u}{d x}\right) = \frac{1}{2} \cdot \frac{d (d u)}{d x^2} = \frac{1}{2} \cdot \frac{d^2 u}{d x^2}$$

$$A_3 = \frac{1}{3} \cdot \frac{d\left(\frac{d^2 u}{2 . d x^2}\right)}{d x} \left(\text{since } A_2 = \frac{d^2 u}{2 . d x^2}\right) = \frac{1}{2 . 3} \cdot \frac{d (d^2 u)}{d x^3}$$

$$= \frac{1}{2 . 3} \cdot \frac{d^3 u}{d x^3}$$

$$A_4 = \frac{1}{4} \cdot \frac{d\left(\frac{d^3 u}{2 . 3 . d x^3}\right)}{d x} \left(\text{since } A_3 = \frac{d^3 u}{2 . 3 . d x^3}\right)$$

$$= \frac{1}{2 . 3 . 4} \frac{d (d^3 u)}{d x^4} = \frac{1}{2 . 3 . 4} \cdot \frac{d^4 u}{d x^4}$$

&c. &c.

We thus obtain the series of Taylor in the same form as before.

We make no apology for giving two different methods of deducing this theorem, which is unquestionably the most important in the whole range of Analysis: we shall

at present proceed to apply it to the demonstration of the binomial theorem of Newton.

Let $u=x^n$: then

$$u'=(x+h)^n=u+\frac{du}{dx}\cdot\frac{h}{1}+\frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2}$$

$$+\frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3}+\&c.$$

Now $\frac{du}{dx}=nx^{n-1}$, as has been shewn above;

$$\frac{d^2u}{dx^2}=n.\frac{d.x^{n-1}}{dx}=\frac{n}{dx}.(n-1)x^{n-2}dx=n(n-1)x^{n-2}$$

$$\frac{d^3u}{dx^3}=n(n-1)(n-2)x^{n-3}$$

$$\frac{d^4u}{dx^4}=n(n-1)(n-2)(n-3)x^{n-4}$$

&c.

consequently

$$(x+h)^n=x^n+nx^{n-1}h+n(n-1)x^{n-2}\frac{h^2}{1.2}$$

$$+n(n-1)(n-2)x^{n-3}\frac{h^3}{1.2.3}+\&c.$$

whatever be the value of n.*

---

* The same method which was used in the determination of the first coefficient of this series, may be very readily applied to determine the other coefficients also.

By what has been already demonstrated, we may assume

$$(x+h)^n=x^n+A_1h+A_2h^2+\&c.$$

$$=x^n\left(1+\frac{h}{x}\right)^n=x^n\left(1+f_1(n).\frac{h}{x}+f_2.(n).\frac{h^2}{x^2}+\&c.\right)$$

$$=x^n+f_1(n)x^{n-1}h+f_2(n).x^{n-2}h^2+f_3(n).x^{n-3}h^3+\&c.$$

By the same process as that pursued in the determination of $f_1(n)$, we have

$(x+h)^{n+1}$

It is evident, that the only difficulty we can encounter in the developement of $f(x+h)$ in all other cases, must arise from the determination of the series of derivative

---

$$(x+h)^{n+1}=x^{n+1}+f_1(n+1)\,.\,x^n h+f_2(n+1)\,x^{n-1}h^2$$
$$+f_3(n+1)\,x^{n-2}h^3+\text{\&c.}$$
$$=x^{n+1}+\{f_1(n)+1\}\,.\,x^n h+\{f_2(n)+f_1(n)\}\,x^{n-1}h^2$$
$$(f_3(n)+f_2(n))\,x^{n-2}h^3+\text{\&c.}$$

But we have already proved $f_1(n)=n$: we have therefore $f_2(n+1)=f_2(n)+n$. and $f_2(n+1)-f_2(n)=n$: but in general, if $f(n)=A\,n^m+B\,n^{m-1}+$ &c.: then

$$f(n+1)-f(n)=A\,((n+1)^m-n^m)+B\,((n+1)^{m-1}-n^{m-1})+\text{\&c.}$$
$$=A\,n^{m-1}+P\,n^{m-2}+\text{\&c.}$$

the highest power of n being less by unity than in the original function. We may from hence conclude, that the highest power of n in $f_2(n)$ is the second. Also n and $n-1$ are factors of $f_2(n)$, since $f_2(n)=0$, if $n=0$ or $n=1$; consequently

$$f_2(n)=a\,n\,(n-1),\text{ and therefore } f_2(n+1)-f_2(n)$$
$$=a\,n\,(n+1)-a\,n\,(n-1)=2\,a\,n=n;$$

or $a=\frac{1}{2}$, and $f_2(n)=\dfrac{n(n-1)}{2}$.

Again,

$$f_3(n+1)-f_3(n)=f_2(n)=\frac{n(n-1)}{2};$$

hence $f_3(n)$ is of the third dimension.

Also n, $n-1$, and $n-2$, are factors of $f_3(n)$; since this function vanishes when $n=0$, $n=1$, $n=2$: we have therefore $f_3(n)=a\,n\,(n-1)(n-2)$; and

$$f_3(n+1)-f_3(n)=a\,n\,(n-1)\,\{n+1-n+2\}=3\,a\,.\,n\,(n-1)$$
$$=\frac{n(n-1)}{2},\text{ and therefore } a=\frac{1}{2\,.\,3},$$

and $f_3(n)=\dfrac{n(n-1)(n-2)}{2\,.\,3}$, and so on, for the other terms.

functions of x, which form the coefficients of h, $\frac{h^2}{1.2}$, $\frac{h^3}{1.2.3}$, &c. in the general series. We have already determined the law of derivation in the most common functions: we shall now endeavour to exhibit it in all other algebraical functions whatever.

1. Let $u = f(x \times f_1(x) = v . v_1$; we, in this case, have

$$u' = f(x+h) \times f_1(x+h) = \left\{v + \frac{dv}{dx} . h + \frac{d^2v}{dx^2} . \frac{h^2}{1.2} + \&c.\right\} \times$$

$$\left\{v_1 + \frac{dv_1}{dx} . \frac{h}{1} + \frac{d^2v_1}{dx^2} . \frac{h^2}{1.2} + \&c.\right\}$$

$$= vv_1 + v . \frac{dv_1}{dx} . \frac{h}{1} + v \frac{d^2v_1}{dx^2} \frac{h^2}{1.2} + \&c.$$

$$+ v_1 . \frac{dv}{dx} . \frac{h}{1} + \frac{dv_1}{dx} . \frac{dv}{dx} . \frac{h^2}{1.1} + \&c.$$

$$+ v_1 . \frac{d^2v}{dx^2} . \frac{h^2}{1.2} + \&c.$$

and therefore, by the definition of a differential, we have $du = v . \frac{dv_1}{dx} h + v_1 . \frac{dv}{dx} h = v\,dv_1 + v_1\,dv$, putting dx for h.

2. Let $u = f(x) \times f_1(x) \times f_2(x) = v . v_1 . v_2$; assume $z = v_1 . v_2$: we thus have $u = vz$,

$$\text{and therefore } du = v\,dz + z\,dv:$$

$$\text{but } dz = d . v_1 v_2 = v_1 dv_2 + v_2 dv_1;$$

consequently

$$du = v\{v_1 dv_2 + v_2 dv_1\} + v_1 v_2 dv$$

$$= vv_1 dv_2 + vv_2 dv_1 + v_1 v_2 dv.$$

It is obvious, that the same principle may be applied to

the differentiation of a function, consisting of the products of any number of functions whatever.

3. Let $u = \{f(x)\}^n = v^n$; we have here

$$du = n v^{n-1} dv;$$

or thus,

$$u' = \{f(x+h)\}^n = \left(v + \frac{dv}{dx} . h + \frac{d^2 v}{dx^2} . \frac{h^2}{1.2} + \&c.\right)^n$$

$$= \left\{v + h\left(\frac{dv}{dx} + \frac{d^2 v}{dx^2} . \frac{h}{1.2} + \&c.\right)\right\}^n$$

$$= v^n + n v^{n-1} h \left\{\frac{dv}{dx} + \frac{d^2 v}{dx^2} . \frac{h}{1.2} + \&c.\right\}$$

$$+ \frac{n(n-1)}{1.2} . v^{n-2} h^2 \left(\frac{dv}{dx} + \frac{d^2 v}{dx^2} . \frac{h}{1.2}\right)^2 + \&c.$$

$$= v^n + n v^{n-1} \frac{dv}{dx} h + \left(n v^{n-1} \frac{d^2 v}{dx^2}\right.$$

$$\left. + n . \frac{(n-1)}{1.2} . v^{n-2} \frac{dv^2}{dx^2}\right) h^2 + \&c.$$

and consequently

$$du = n v^{n-1} \frac{dv}{dx} h = n v^{n-1} dv.$$

4. Let $u = \frac{f(x)}{f_1(x)} = \frac{v}{v_1}$; since $u = v . v_1^{-1}$, we may differentiate this function as a product of two others, and

$$du = v_1^{-1} . dv + v . dv_1^{-1}$$

$$= v_1^{-1} . dv + v(-v_1^{-2} dv_1)$$

$$= \frac{dv}{v_1} - \frac{v\,dv_1}{v_1^2} = \frac{v_1 dv - v dv_1}{v_1^2};$$

or thus,

$$u' = \frac{f(x+h)}{f_1(x+h)} = \frac{v + \frac{dv}{dx} . h + \frac{d^2 v}{dx^2} . \frac{h^2}{1.2} + \&c.}{v_1 + \frac{dv_1}{dx} h + \frac{d^2 v_1}{dx^2} \frac{h^2}{1.2} + \&c.}$$

$$=\frac{v+ph}{v_1+p_1h}=(v+ph)\left(\frac{1}{v_1}-\frac{p_1h}{v_1^2}+\frac{p_1^2h^2}{v_1^3}-\&c.\right)$$

$$=\left\{\begin{array}{l}\frac{v}{v_1}-\frac{vp_1h}{v_1^2}+\frac{vp_1^2h^2}{v_1^3}-\&c.\\ \quad+\frac{ph}{v_1}-\frac{pp_1h^2}{v_1^2}+\&c.\end{array}\right.$$

$$=\left\{\begin{array}{l}\frac{v}{v_1}-v\cdot\frac{\frac{dv_1}{dx}h}{v_1^2}+qh^2\\ \quad+\frac{\frac{dv}{dx}h}{v_1}-q_1h^2;\end{array}\right.$$

and therefore

$$du=\frac{\frac{dv}{dx}h}{v_1}-\frac{v\cdot\frac{dv_1}{dx}h}{v_1^2}=\frac{dv}{v_1}-\frac{v\,dv}{v_1^2}$$

$$=\frac{v_1\,dv-v\,dv_1}{v_1^2}.$$

For the rules of differentiation deducible from these conclusions, and for their application to particular examples, the reader is referred to Nos. 11, 12, 13, 14, 15 and 16 of this work; our only object here being to demonstrate them, in conformity with our definition of a differential.

The investigation of the differentials of logarithmic and circular functions will be found in Note (D).

Enough has been already said, to give the reader an insight into the nature and objects of the Differential Calculus. We are required to determine a series of derivative functions, whose forms and values are all dependent upon a primitive function, and which are also connected with it, and with each other by a determinate law. We readily discover its *direct* uses in the developement of functions, and shall have hereafter occasion to remark them in nearly

every other part of analysis, and in almost every department of the theory of curves and mechanical philosophy; its *indirect* uses will be found to consist in furnishing expressions in terms of the derivative functions of some primitive one, which is required to be determined, either simply by themselves, or combined with each other, and with other variables, in the form of equations; being a state preparatory to the application of the Integral Calculus, by which the primitive function is elicited from the differential expressions which are thus presented to its operation.

The same general rules of differentiation, and the same series for the developement of functions, have been deduced by our author, by the method of limits which he has adopted. The reason of this agreement will be best explained by a statement of the general principle of that method.

Assume

$$f(x+h)=f(x)+A_1 h+A_2 h^2+A_3 h^3 + \&c.$$

which gives

$$\frac{f(x+h)-f(x)}{h} = \frac{\Delta f(x)}{h} = A_1+A_2 h+A_3 h^2 + \&c.$$

and since A_1, A_2, A_3, &c. which are functions of x, must be of finite, though indeterminate value, the sum of all the terms of this series, after the first, may become, by the diminution of h, less than any assigned quantity whatever. (See note A.) Its true limit, therefore, which is likewise that of $\frac{\Delta f(x)}{h}$, is A_1; and this, by convention, is represented by $\frac{d f(x)}{h}$, or $\frac{d f(x)}{d x}$.

We thus see, that the term *differential*, in its representation at least, is identical in the two systems; and since the limit of the ratio of the respective differences of

the function and its base, is independent of the absolute value of h or dx, we may consider it as finite in every differential expression in which it appears. We shall find also, that the general principle being once established, the theory of limits will sometimes enable us to determine the coefficient A_1 more readily than by the actual developement of $f(x+h)$, an advantage which is still more observable in the applications of this Calculus to the theory of curves.

These reasons would, in a great measure, determine the superiority of this system, if it was true, as is sometimes maintained, that it is of little importance, when the rules of Analysis are once established, whether we clearly comprehend or not, the principle of their derivation: but it will be found, that in most cases, a process of reasoning similar to that employed in deducing these rules, must likewise direct their application, and that whatever difficulty or obscurity attends their investigation, will be again traced in every theory which they are employed to demonstrate.

Our notion, indeed, of a ratio, whose terms are evanescent, is necessarily obscure, however rigorously its existence and magnitude may be demonstrated; and its introduction into all our reasonings in the establishment of this Calculus, is calculated to throw a mystery over all its operations, which can only be removed by our knowledge of its more simple and natural origin. We will not attempt, however, to enumerate all the objections which may be made to this system, and shall only mention that which we consider as insuperable, its tendency to separate the principles and departments of the Differential Calculus from those of common Algebra.

We have, in the preceding Note, alluded to the method of Infinitisimals, which formed the foundation of the system of Leibnitz: its principles may be briefly exhibited as follows:

Assume

$$f(x+h)-f(x)=\Delta f(x)=A_1 h+A_2 h^2+A_3 h^3+\text{\&c.}$$

He considers h as an infinitesimal quantity; since $\frac{1}{h}=\frac{h}{h^2}=\frac{h^2}{h^3}=$ &c. and 1 is assumed to be infinitely greater than h, we have h also infinitely greater than h^2, h^2 infinitely greater than h^3, and so on: thus h, h^2, h^3, h^4, &c. and consequently $A_1 h$, $A_2 h^2$, $A_3 h^3$, &c. form a series of infinitesimal quantities, each of which is infinitely greater than the sum of all those which succeed it.

He therefore assumes

$$\Delta f(x)=d f(x)=A_1 h=A\, d\, x.$$

Upon the same principle we have $d A_1=q\, d x$, $dq=r\, d x$, $d r=s\, d x$, and consequently $d^2 f(x)=q\, d x^2$, $d^3 f(x)=r dx^3$, $d^4 f(x)=s\, d x^4$, and so on; the different orders of differentials constituting so many orders of infinitesimals.

The differential being considered as a particular state of the difference, the rules of differentiation are deducible with great readiness and simplicity. Thus, let $u=vv_1$, then $u+\Delta u=(v+\Delta v)(v_1+\Delta v_1)$,

or

$$u+d u=(v+d v)(v_1+dv_1)=vv_1+vdv_1+v_1dv+dv\,.\,dv_1$$

and $du=vdv_1+v_1dv+dv\,.\,dv_1$

$=vdv_1+v_1dv$, since $dv\,.\,dv_1$ may be neglected as an infinitesimal of an order superior to that of dv or dv_1.

The differentials of products involving any number of functions as factors, of powers of functions, both fractional and integral, and of functions under fractional forms, may, without difficulty, be deduced from the result above given.

Again, let $u=\sin x$; then $u+\Delta u=\sin(x+\Delta x)$, or $u+du=\sin(x+d x)=\sin x\,.\,\cos d x+\cos x\,.\,\sin d x$; but in the infinitesimal system $\cos d x=1$, and $\sin dx=d x$: we have therefore

$$u+du = \sin x+\cos x\,dx,$$

$$\text{and } du=\cos x\,dx.$$

In a similar manner it may be applied, to find the differentials of the other trigonometrical lines.

The rules of differentiation are more easily deduced by this method, than by any other; and it also admits of a very ready application to the more common questions concerning curve lines: but it is otherwise liable to the same objections as the system of D'Alembert, and to many others from which that system is free; for its first principles hardly admit of demonstration, and by considering the successive differentials as infinitesimals of successive orders, we are unable to form any notion whatever of their connection with each other, and with the function from which they are derived.

We will now endeavour to give some account of the method of fluxions, which was for a long time the rival of the Differential Calculus of Leibnitz, and which is adopted even at this day almost universally by English mathematicians.

If we consider a curve as generated by the uniform motion of a point, we may decompose the velocity of this point into two others, one parallel to the axis of the abscissæ, and the other parallel to that of the ordinates. These velocities are severally the *fluxions* of the arc, the abscissa, and the ordinate. It is also evident, that unless the moving point describe a straight line, the fluxions of the abscissa and the ordinate must be variable quantities, and their ratio at each instant of this motion, must depend upon the nature of the curve, or the relation of the coordinates. We may also suppose the motion in the direction of one of the co-ordinates to be uniform, and consequently the velocities in the direction of the curve and of the other co-ordinate, must be considered as variable: we may also, as is generally the case, confine our attention to

the motions in the directions of the two co-ordinates only, one of them being always supposed to be uniform and constant.

Fluxions, being considered as velocities, must admit of estimation in the same manner as these velocities themselves, and must be finite or not, according to the nature of the quantities of which they are the representatives. They will not therefore be equal to the quotients, which arise from the division of the actual increments of the curve or co-ordinates, by the time in which the change is effected; but to those which would result from supposing the increments such as would be generated by the continuance of that velocity with which the change commenced uniformly throughout the whole time. The time itself is assumed as a unit; and the fluxions may therefore be severally represented by the whole increment of that part which is generated by a uniform motion and by the *potential* increments of the other two.

The fluxions, which are not constant, may be taken as the co-ordinates of another curve, which will likewise have their fluxions, of finite and estimable magnitude. These may again be taken as the co-ordinates of a third curve, and so on, so that there will never enter into consideration any quantities which are not finite.

The method of fluxions will thus consist in finding the relation that subsists between the fluxions, when we know the relation between the co-ordinates; and reciprocally, since the relation between the co-ordinates must depend upon that of their fluxions at each instant of time, it will be another problem to find the relation of the co-ordinates, when we know that of their fluxions, either simply, or combined with the co-ordinates themselves. This is the object of the inverse method of fluxions, which is also called the method of fluents.

The method of fluxions is not confined to the lines

generated by the motion of a point; it may be extended also, by analogy, to the areas of curves generated by the uniform motion of ordinates of variable length: for if we assume two co-ordinates, one to represent the area generated, and the other some area described upon the abscissa of the given curve, by a given and invariable line moving with the same velocity as the variable ordinate itself, their changes will be analogous to those of the areas. Suppose now, that the increment of the area of the curve is so modified, as to bear the same ratio to the increment of the other area, that the *fluxions* of the co-ordinates bear to each other: and since the increment of the rectangular area varies as the fluxion of the co-ordinate by which it is represented, since they both increase in the same ratio with the units of time, the other modified increment will also vary as the fluxion of its representative co-ordinate, and may consequently, by analogy, be called the *fluxion* of the area of the curve. We may also, upon a similar principle, speak of the fluxions of the volumes and curve surfaces of solid bodies; of forces to communicate motion, and even of functions themselves: for we may always assume two co-ordinates, one to represent the volume, curve surface, force, function, or other variable quantity, which is the subject of consideration, and the other some quantity of a nature and dimension similar to the former, the increase of the second co-ordinate being always supposed proportional to the time. We consequently must have the cotemporaneous changes of the quantities proportional to those of the co-ordinates; and their fluxions also must be proportional, according to the hypothesis we have already made.

We thus see, that if the term fluxion be applied to the actual increment of the auxiliary quantity, the fluxion of the variable quantity itself must be such a modified state of its increment also, as will make it a fourth term in the proportion alluded to. The denomination is thus transferred to

a *potential* increment of the variable quantity, from the actual fluxion of the co-ordinate, which is assumed as its representative.

The method of fluxions is naturally deduced from the method of Prime and Ultimate Ratios; for a fluxion is the *ultimate* value of the quotient, which arises from the division of the space which represents the increment of the co-ordinate, by the whole time of the motion; for this quotient, by the diminution of the time, may be made to differ from the fluxion by a quantity less than any that can be assigned. A fluxion, therefore, in its analogical sense, may be defined to be the *limit* of the increment of the variable quantity upon which it is dependent, and may be consequently considered as identical, both in its representation and properties, with the term *differential*, in the system of D'Alembert.

If $$\Delta f(x) = A_1 h + A_2 h^2 + A_3 h^3 + \&c.$$

represent the increment of one of the co-ordinates corresponding to the increment h of the other; then $A_1 h$ and h will represent their fluxions; for h is proportional to the time and velocity jointly, and is consequently the fluxion of the second co-ordinate, when the time is assumed as a unit; and $A_1 h$ is the only part of $\Delta f(x)$, whose magnitude increases in the same proportion with h, or with the fluxion of the other co-ordinate; and since the proportion of the fluxions is independent of h, we evidently see, that $A_1 h$ is the only part of the difference which will answer that condition.

We may readily extend this principle to determine the fluxion of any function whatever, by considering the changes of the function and its base, as analogous to the changes of the co-ordinates; and the first term of the increment will still be found to correspond to the fluxion.

We thus find, that all the different systems which we have examined, terminate in the same general principle or

conclusion, that the *differential* or *fluxion* of a *function is the first term of the developement of its difference or increment.*

The consideration of motion, which is essential to the method of fluxions, is foreign to the spirit of pure Analysis; and the analogy by which the name and properties of a fluxion are transferred to a modification of the difference of a function, is strained and unnatural. The different orders of fluxions also are involved in considerable obscurity, and we are utterly unable to comprehend the connection which they respectively bear to their primitive function.

In the brevity of its demonstrations, and in the facility of its applications, it is unquestionably inferior to all the other methods; and the mixture of mechanical and geometrical considerations upon which it is founded, are little calculated to assist us in investigating the properties of functions which are always algebraical in their form, and generally in their nature also.

But the most important distinction between this system and the Differential Calculus, consists in a different notation. The student will best judge of its merits by comparing it with the Differential notation, in a few examples.

1. We denote dx or h by $\dot{x}$: in this case, they may be considered as equally simple.

2. Again, du, d^2u, d^3u, are severally denoted by $\dot{u}$, $\ddot{u}$, $\dddot{u}$, $\overset{n}{\dot{u}}$; this notation becomes complicated, when the number of dots is considerable.

3. Take Laplace's series,

$$u = u_0 + z\frac{du}{dy}\cdot\frac{x}{1} + \frac{d\,.\,z^2\frac{du}{dy}}{dy}\cdot\frac{x^2}{1\,.\,2} + \frac{d^2\,.\,z^3\frac{du}{dy}}{dy^2}\cdot\frac{x^3}{1\,.\,2\,.\,3}$$

$$+\ \&c. + \frac{d^{n-1}\,.\,z^n\frac{du}{dy}}{dy^{n-1}}\cdot\frac{x^n}{1\,.\,2\,.\,3\ \ldots\ n} + \&c.$$

In the fluxional notation it will become

$$u = U + z\frac{\dot{u}}{\dot{y}} \cdot \frac{x}{1} + \frac{\left(z^2\frac{\dot{u}}{\dot{y}}\right)^{\cdot}}{\dot{y}} \cdot \frac{x^2}{1.2} + \frac{\left(z^3\frac{\dot{u}}{\dot{y}}\right)^{\cdot\cdot}}{\dot{y}^2} \cdot \frac{x^3}{1.2.3}$$

$$+ \&c. + \frac{\left(z^n\frac{\dot{u}}{\dot{y}}\right)^{(n-1)}}{\dot{y}^{n-1}} \cdot \frac{x^n}{1.2.3\ldots n} + \&c.$$

The notation for cases of this kind is deficient, both in symmetry and simplicity.

4. The difficulty of denoting the operations of finding the different orders of fluxions is very great, when for u we put the function itself, which it represents.

Thus,

1. $d.f(x)$ is denoted by $(f(x))^{\cdot}$ or $\dot{\overline{f(x)}}$
2. $d.\sqrt{1-x^2}$ —— by $(\sqrt{1-x^2})^{\cdot}$ or $\dot{\overline{\sqrt{1-x^2}}}$
3. $d^3\frac{1}{(1-x^2)^{\frac{3}{2}}}$ —— by $\left(\frac{1}{(1-x^2)^{\frac{3}{2}}}\right)^{\cdot\cdot\cdot}$ or $\overset{\cdot\cdot\cdot}{\overline{\frac{1}{(1-x^2)^{\frac{3}{2}}}}}$
4. $d^n.(1-x^2)^m$ —— by $((1-x^2)^m)^{(n)}$ or $\overline{(1-x^2)^m}^{(n)}$
5. $d.\sin x$ —— by $(\sin x)^{\cdot}$ or $\dot{\overline{\sin x}}$
6. $d.\log x$ —— by $(\log x)^{\cdot}$ or $\dot{\overline{\log x}}$
7. $d.a^x$ —— by $(a^x)^{\cdot}$ or $\dot{\overline{a^x}}$.

We have taken examples of the most simple and common kind, in order that we may not be accused of misrepresenting the real merits of the question: by taking functions of a more complicated nature, the aukwardness of the fluxional notation will become more and more manifest, particularly when the order of the fluxion is considerable. But the best argument of its utter insufficiency in

most cases of this nature, is derived from the practices of fluxionists themselves, who usually denote the operation by a verbal statement, or by prefixing some abbreviation of the word fluxion.

5. The beautiful theorem of Lagrange, so important in the theory of Finite Differences,

$$\Delta^n u_x = \left(\epsilon^{\frac{d}{dx}} - 1\right)^n u_x, \qquad \text{(App. Art. 387.)}$$

and many others connected with it, are incapable of representation by the fluxional notation.

But this note has already exceeded its proper bounds, and we must come to a conclusion: the differential notation is equally convenient for representing both operation and quantity; its symbols are distinct, and never ambiguous; it is symmetrical in all cases, and it continues equally simple, whatever be the order of the differential, or the nature of the function to which it is applied; whilst that of fluxions is deficient in nearly all the essential particulars which we have just enumerated, and in the representation of many important theorems it absolutely fails.

Note (C).

The series of Maclaurin,

$$u = U_0 + U_1 \frac{x}{1} + U_2 \frac{x^2}{1.2} + U_3 \cdot \frac{x^3}{1.2.3} + \&c.$$

which is made use of by our author in the investigation of the more general series of Taylor, may more naturally be derived from the latter.

For, since

$$u' = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

or,

$$f(x+h) = f(x) + f_1(x)\frac{h}{1} + f_2(x)\frac{h^2}{1.2} + f_3(x)\frac{h^3}{1.2.3} + \&c.$$

$f_1(x)$, $f_2(x)$, $f_3(x)$, &c. severally representing

$$\frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}, \text{ \&c.}$$

Let $x=0$; an hypothesis which gives

$$f(h)=f(0)+f_1(0)\frac{h}{1}+f_2(0)\frac{h^2}{1.2}+f_3(0)\frac{h^3}{1.2.3}+\text{ \&c.}$$

where $f(0)$, $f_1(0)$, $f_2(0)$, &c. severally represent the values of $f(x)$, $f_1(x)$, $f_2(x)$, &c. when x becomes equal to nothing. Changing h into x, we get

$$f(x)=f(0)+f_1(0)\frac{x}{1}+f_2(0)\frac{x^2}{1.2}+_3(0)\frac{x^3}{1.2.3}+\text{ \&c.}$$

or replacing $f(0)$, $f_1(0)$, $f_2(0)$, &c. by U_0, U_1, U_2, &c. we have

$$f(x)=u=U_0+U_1\frac{x}{1}+U_2\frac{x^2}{1.2}+U_3\frac{x^3}{1.2.3}+\text{ \&c.}$$

It is evident that this series will furnish the means of developing any function of x whatever, in terms of ascending integer powers of x, and constant coefficients, if the function itself be capable of such a form; but if $f(x)$ be of such a nature, as to involve in its developement negative or fractional powers of x, it will then be found, that this theorem will fail in effecting it.

The supposition of x having a particular value, which is necessary to deduce this theorem from the more general one of Taylor, deprives it of that generality which is essential to the latter: it is on this account that its application is not general. We will give some instances of its failure, when we come to note (H).

NOTE (D).

The differential of a^x is $A a^x dx$, where A is equal to the series

$$\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\&c.;$$

for this is shewn by our author to be the first term of the difference $a^x(a^h-1)$.

The differential of a logarithm is deducible from that of a^x. (See No. 26.)

The general series of Taylor furnishes us with a means of developing a^x, even without a previous knowledge of its differential. The constant A, however, which it involves, will remain indeterminate.

Thus, let $a^x=f(x)$ and $a^y=f(y)=u$: then $a^x\times a^y=a^{x+y}=f(x+y)$: we hence obtain

$$f(y+x)=f(y).f(x)=f(y)+\frac{du}{dy}.\frac{x}{1}+\frac{d^2u}{dy^2}.\frac{x^2}{1.2}$$
$$+\frac{d^3u}{dy^3}.\frac{x^3}{1.2.3}+\&c.$$

Consequently, by dividing by $f(y)$ or u, we have

$$f(x)=1+\frac{du}{dy}.\frac{x}{u.1}+\frac{d^2u}{dy^2}.\frac{x^2}{u.1.2}$$
$$+\frac{d^3u}{dy^3}.\frac{x^3}{u.1.2.3}+\&c.$$

and since y cannot enter into a developement of $f(x)$, the several coefficients of $\frac{x}{1}$, $\frac{x^2}{1.2}$, $\frac{x^3}{1.2.3}$, &c. must be constant quantities: we must have, therefore, $\frac{du}{dy.u}=A$, a constant quantity, and $\frac{du}{dy}=Aa^y$; also

$$\frac{d^2 u}{dy^2} = A \,.\, \frac{du}{dy} = A \,.\, A \,.\, a^y = A^2 a^2;$$

$$\frac{d^3 u}{dy^3} = A^2 \,.\, \frac{du}{dy} = A^2 \,.\, A a = A^3 a^y.$$

$$\frac{d^4 u}{dy^4} = A^4 a \text{, and so on:}$$

hence

$$\frac{d^2 u}{dy^2 . u} = A^2, \quad \frac{d^3 u}{dy^3 . u} = A^3, \text{ and } \frac{d^4 u}{dy^4 . u} = A^4, \text{ \&c.}$$

and therefore

$$a^x = 1 + A x + \frac{A^2 x^2}{1 . 2} + \frac{A^3 x^3}{1 . 2 . 3} + \text{\&c.}$$

If $x = \frac{1}{A}$, we shall have

$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1 . 2} + \frac{1}{1 . 2 . 3} + \&c. = e$, where e is that value of a, which makes $A = 1$. We thus get $e^A = a$; and consequently $A = \log a$ to base e, or it is equal to the Napierian logarithm of a. Since $e^x = a^x = N$, we have log N to base $a = Ax = A \times \log N$ to base e. Thus logarithms of the same number, corresponding to the different bases, e and a, are connected with each other by the constant quantity A.

We thus see, that it is unnecessary to embarrass our calculations with logarithms calculated to different bases, since they admit of so ready a change from one system to another. We shall find it convenient to choose e for our base; and whenever the contrary is not specified, log u or l u may be considered as calculated to that system.

The reader will find no difficulty in demonstrating the following logarithmic series:

$$1. \ \log u = n \left\{ (\sqrt[n]{u} - 1) - \frac{(\sqrt[n]{u} - 1)^2}{2} + \frac{(\sqrt[n]{u} - 1)^3}{3} - \frac{(\sqrt[n]{n} - 1)^4}{4} + \text{\&c.} \right\}$$

$$2.\ \log u = n\left\{\left(1-\frac{1}{\sqrt[n]{u}}\right)+\frac{\left(1-\frac{1}{\sqrt[n]{u}}\right)^2}{2}+\frac{\left(1-\frac{1}{\sqrt[n]{u}}\right)^3}{3}\right.$$

$$\left.+\frac{\left(1-\frac{1}{\sqrt[n]{u}}\right)^4}{4}+\&c.\right\}$$

$$3.\ \log u = (u-u^{-1})-\frac{(u^2-u^{-2})}{2}+\frac{(u^3-u^{-3})}{3}-\&c.$$

The uses of the first two series may be seen in Lagrange *Calcul des Fonctions*, sec. 4. The third is remarkable for the elegance of its form, and will be found useful in the course of this Note, in the demonstration of a very curious theorem.

The developements of $\cos x$ and $\sin x$ may also be effected, by means of Taylor's theorem, without a previous knowledge of their differentials. The constants also which are severally found in them, will require a separate determination.

Let $\cos x = f(x)$, and therefore $\cos y$, $\cos(y+x)$ and $\cos(y-x)$ may be represented respectively by $f(y)$, $f(y+x)$, $f(y-x)$: a very common trigonometrical formula gives us the equation

$$2f(x)\times f(y)=f(y+x)+f(y-x).$$

Also, representing $f(y)$ by u, we have

$$f(y+x)=u+\frac{du}{dy}\cdot\frac{x}{1}+\frac{d^2u}{dy^2}\cdot\frac{x^2}{1.2}+\frac{d^3u}{dy^3}\cdot\frac{x^3}{1.2.3}+\&c.$$

and

$$f(y-x)=u-\frac{du}{dy}\cdot\frac{x}{1}+\frac{d^2u}{dy^2}\cdot\frac{x^2}{1.2}-\frac{d^3u}{dy^3}\cdot\frac{x^3}{1.2.3}+\&c.$$

By adding these two series together, and dividing their sum by $2f(y)$, we get

$$f(x)=1+\frac{d^2u}{u.dy^2}\cdot\frac{x^2}{1.2}+\frac{d^4u}{u.dy^4}\cdot\frac{x^4}{1.2.3.4}+\&c.$$

a series into which no function of y can enter: the coefficients $\frac{d^2 u}{u \cdot dy^2}$, $\frac{d^4 u}{u \cdot dy^4}$, &c. must therefore be constant quantities. Since $f(x)$ or $\cos x$, must be less than unity, we may assume $\frac{d^2 u}{u \cdot dy^2} = -a^2$, and therefore $\frac{d^2 u}{dy^2} = -a^2 u$ $= -a^2 \cos y$. We hence deduce

$\frac{d^4 u}{dy^4} = -a^2 \cdot \frac{d^2 u}{dy^2} = -a^2 \cdot -a^2 u = a^4 u$, and therefore $\frac{d^4 u}{u \cdot dy^4} = a^4$.

In the same manner, we shall find

$\frac{d^6 u}{u \cdot dy^6} = -a^6$, $\frac{d^8 u}{u \cdot dy^8} = a^8$, and so on; consequently

$$\cos x = 1 - \frac{a^2 x^2}{1 \cdot 2} + \frac{a^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{a^6 x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \&c.$$

in which a alone remains to be determined.

Assuming $\sin x = \varphi(x)$, and therefore $\sin(y+x) = \varphi(y+x)$, and $\sin(y-x) = \varphi(y-x)$: we readily deduce the following equation,

$$2\varphi(x) \times f(y) = \varphi(y+x) - \varphi(y-x).$$

Developing $\varphi(y+x)$ and $\varphi(y-x)$, and dividing the difference of the resulting series by $2f(y)$, we shall get

$$\varphi(x) = \frac{dv}{u \cdot dy} \cdot \frac{x}{1} + \frac{d^3 v}{u \cdot d^3 y} \cdot \frac{x^3}{1 \cdot 2 \cdot 3}$$
$$+ \frac{d^5 v}{u \cdot dy^5} \cdot \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c.$$

where v represents $\varphi(y)$. We may conclude, for the same reason as before, that $\frac{dv}{u \cdot dy}$ $\frac{d^3 v}{u \cdot dy^3}$, &c. are severally constant quantities.

Assume $\frac{dv}{u.dy}=b$, and consequently $\frac{dv}{dy}=bu=b\cos y$; hence also $\frac{d^3v}{dy^3}=b\,.\frac{d^2u}{dy^2}=-a^2b\,.\cos y$; and therefore $\frac{d^3v}{u.dy^3}=-a^2b$: in the same manner we shall find $\frac{d^5v}{u.dy^5}$ $=+a^4b$, and so on, for the other coefficients.

We consequently obtain

$$\sin x=bx-a^2b\,.\frac{x^3}{1.2.3}+a^4b\,.\frac{x^5}{1.2.3.4.5}-\&c.$$

We have already shewn in what manner b may be determined to be equal to unity (See note A). We may also prove, that $a^2=1$, in virtue of the equation

$$\cos x^2+\sin x^2=1\ ;$$

or more readily thus;

since $\frac{d\cdot\sin v}{dx}=b\cos x=\cos x$, we have $d\sin x=\cos x\,dx$; also, $\cos x=\sin\left(\frac{\pi}{2}-x\right)$, and therefore $d\,.\cos x$

$$=d\,.\sin\left(\frac{\pi}{2}-x\right)=\cos\left(\frac{\pi}{2}-x\right)d\,.\left(\frac{\pi}{2}-x\right)$$

$=\sin x\,.-dx=-\sin x\,dx$; consequently $\frac{d^2\cos x}{dx^2}=-\frac{d\,.\sin x}{dx}=-\cos x=-a^2.\cos x$; and therefore $a^2=1$.

We thus get

$$\sin x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\&c.$$

$$\cos x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\&c.$$

The reader must already have remarked, that the differential coefficients of e^x are severally equal to each other

and to the original function; a property by which the developement of e^x is very easily effected. Thus, assuming

$$e^x = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.$$

we have

$$\frac{d e^x}{d x} = e^x = a_1 + 2 a_2 x + 3 a_3 x^2 + 4 a_4 x^3 + \&c.$$

and equating the corresponding terms of these identical series, we find

$$a_1 = 1,\ a_2 = \frac{1}{1.2},\ a_3 = \frac{1}{1.2.3},\ \&c.\ \&c.$$

therefore,

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

The differential coefficients of $\sin x$ and $\cos x$, which are of an even order, also reproduce their original functions, though with signs alternately negative and positive: thus

$$\frac{d^2 \sin x}{d x^2} = -\sin x,\ \frac{d^4 \sin x}{d x^4} = \sin x,\ \frac{d^6 \sin x}{d x^6} = -\sin x,\ \&c.$$

and

$$\frac{d^2 \cos x}{d x^2} = -\cos x,\ \frac{d^4 \cos x}{d x^4} = \cos x,\ \frac{d^6 \cos x}{d x^6} = -\cos x,\ \&c.$$

This property likewise furnishes a very ready means of developing $\sin x$ and $\cos x$.

For, assuming

$$\sin x = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \&c.$$

we have

$$\frac{d^2 \sin x}{d x^2} = -\sin x = 1.2 \,.\, a_2 + 2.3\, a_3 x + 3.4 \,.\, a_4 x^2$$
$$+ 4.5 \,.\, a_5 x^3 + \&c.$$
$$= -a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 - \&c.$$

we thus get

$$a_2 = 0,\ a_3 = \frac{a_1}{1.2.3}\ a_4 = 0,\ a_5 = \frac{a_1}{1.2.3.4.5},\ \&c.$$

and consequently, if $a_1 = 1$,

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$$

In the same manner it may be applied to the developement of cos x.

Also, if
$$u, \frac{d^n u}{d x^n}, \frac{d^{2n} u}{d x^{2n}}, \&c.$$

were severally equal to each other, the developement of u might easily be effected: we should find

$$u = a_1 + \frac{a_1 x^n}{1.2.3.n} + \frac{a_1 x^{2n}}{1.2 \ldots 2n} + \&c.$$

or

$$u = a_1 - \frac{a_1 x^n}{1.2.3.n} + \frac{a_1 x^{2n}}{1.2.3 \ldots 2n} - \&c.$$

if

$$u, \frac{d^n u}{d x^n}, \frac{d^{2n} u}{d x^{2n}}, \&c.$$

were alternately negative and positive.

We have here assumed in some degree the form of the developement, particularly with respect to the first term: if we had supposed the first term to be $a_m x^m$, we should have had

$$u = a_m x^m + \frac{a_m x^{m+n}}{m.(m+1)\ldots(m+n)} + \frac{a_m x^{m+2n}}{m.(m+1)\ldots(m+2n)} + \&c.$$

and

$$u = a_m x^m - \frac{a_m x^{m+n}}{m(m+1)\ldots(m+n)} + \frac{a_m x^{m+2n}}{m(m+1)\ldots(m+2n)} - \&c.$$

for the second hypothesis.

We shall find no difficulty in deducing the following expressions for sin x and cos x, which furnish the means of demonstrating many curious and important theorems.

$$\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c. = \cos x.$$

$$\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\&c.=\sin x.$$

We hence find

$$e^{x\sqrt{-1}}=\cos x+\sqrt{-1}\sin x, \text{ and } e^{-x\sqrt{-1}}=\cos x-\sqrt{-1}\sin x.$$

Also, by putting $n x$ for x, we obtain

$$e^{nx\sqrt{-1}}=\cos n x+\sqrt{-1}\,.\sin n x,$$

and

$$e^{-nx\sqrt{-1}}=\cos n x-\sqrt{-1}\,.\sin n x.$$

Since $e^{nx\sqrt{-1}}=(e^{x\sqrt{-1}})^n$, and $e^{-nx\sqrt{-1}}=(e^{-x\sqrt{-1}})^n$, we get the following remarkable equations:

$$(\cos x+\sqrt{-1}\,.\sin x)^n=\cos n x+\sqrt{-1}\,.\sin n x$$

$$(\cos x-\sqrt{-1}\,.\sin x)^n=\cos n x-\sqrt{-1}\sin n x,$$

which were first discovered by Demoivre.

Again, since

$$e^{x\sqrt{-1}}=\cos x+\sqrt{-1}.\sin x=\cos x(1+\sqrt{-1}\,.\tan x),$$

we have

$$\log e^{x\sqrt{-1}}=x\sqrt{-1}=\log\cos x+\log(1+\sqrt{-1}\,.\tan x)$$

$$=\log\cos x+\sqrt{-1}\,.\tan x+\frac{\tan^2 x}{2}-\sqrt{-1}\frac{\tan^3 x}{3}-\frac{\tan^4 x}{4}+\&c.$$

The following equation must be true:

$$x\sqrt{-1}=\sqrt{-1}\left(\tan x-\frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}+\&c.\right)$$

and therefore

$$x=\tan x-\frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}-\&c.$$

which is the expression for the arc in terms of the tangent.

Again, since

$$\log u=u-u^{-1}-\frac{(u^2-u^{-2})}{2}+\frac{(u^3-u^{-3})}{3}-\frac{(u^4-u^{-4})}{4};$$

if we put $e^{x\sqrt{-1}}$ in the place of u, and divide the whole by $2\sqrt{-1}$, we shall have

$$\frac{x}{2} = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}} - \frac{1}{2} \cdot \frac{(e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}})}{2\sqrt{-1}}$$

$$+ \frac{1}{3} \cdot \frac{(e^{3x\sqrt{-1}} - e^{-3x\sqrt{-1}})}{2\sqrt{-1}} - \&c.$$

$$= \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \&c.$$

an expression for x, in terms of the sines of x, and its multiples, which was first given by Euler.

By taking the successive differential coefficients of this equation, we shall obtain

$$\tfrac{1}{2} = \cos x - \cos 2x + \cos 3x - \cos 4x + \&c.$$

$$0 = -\sin x + 2 \sin 2x - 3 \sin 3x + 4 \sin 4x - \&c.$$

$-\cos 3x$ $$0 = -\cos x + 2^2 \cos 2x - 3^2 \cos 2x + 4^2 \cos 4x - \&c.$$

and generally

$$0 = \cos x - 2^{2n} \cos 2x + 3^{2n} \cos 3x - 4^{3n} \cos 4x + \&c.$$

$$0 = \sin x - 2^{2n+1} \sin 2x + 3^{2n+1} \sin 3x - 4^{2n+1} \sin 4x + \&c.$$

If in the first of these expressions, we make $x=0$, we get

$$0 = 1 - 2^{2n} + 3^{2n} - 4^{2n} + \&c.$$

If in the second, we put $x = \frac{\pi}{2}$, we have

$$0 = 1 - 3^{2n+1} + 5^{2n+1} - 7^{2n+1} + \&c.$$

But the narrow limits of this Note will not allow us to enumerate all the other curious and elegant results, which are deducible from these exponential formulæ for the sine and cosine. We shall proceed to consider the developement of the tangent in terms of the arc, which is alluded to by our author.

We may make use of the series of Maclaurin for this purpose. Thus,

$$u = \tan x$$

$$\frac{du}{dx} = \frac{1}{\cos x^2} = 1 + \tan^2 x = 1 + u^2,$$

$$\frac{d^2 u}{dx^2} = 2u \frac{du}{dx} = 2u + 2u^3,$$

$$\frac{d^3 u}{dx^3} = 2\frac{du}{dx}(1+3u^2) = 2 + 2.4.u^2 + 2.3.u^4,$$

$$\frac{d^4 u}{dx^4} = 2\frac{du}{dx}(2.4u + 3.4.u^3)$$

$$= 2.2.4u + 2.4.5u^3 + 2.3.4.u^5,$$

$$\frac{d^5 u}{dx^5} = \frac{du}{dx}(2.2.4 + 2.3.4.5u^2 + 2.3.4.5.u^4)$$

$$= 2.2.4 + 2.4.17.u^2 + 3.4^2.5u^4 + 2.3.4.5u^6,$$

&c.

consequently,

$$U_0 = 0,\ U_1 = 1,\ U_2 = 0,\ U_3 = 2,\ U_4 = 0,\ U_5 = 16, \text{ \&c.}$$

and therefore

$$\tan x = x + \frac{2x^3}{1.2.3} + \frac{16x^5}{1.2.3.4.5} + \text{\&c.}$$

There is nothing in this result which can enable us to determine the law of the formation of the other terms, and the process by which we deduced it, is tedious and embarrassing, particularly when applied to find terms beyond the third. The following method is much more simple, and will also afford us the means of assigning the law which connects each term with all those which precede it.

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \text{\&c.}}{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \text{\&c.}}$$

$$= a_1 x + a_3 x^3 + a_5 x^5 + \text{\&c. } a_{2n+1} x^{2n+1} + \text{\&c.}$$

for it is obvious, that the even powers of x cannot enter into this series; consequently

$$x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\&c.=(a_1x+a_3x^3+a_5x^5+\&c.)\times$$

$$\left(1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\&c.\right)=$$

$$\begin{aligned}&a_1x+a_3x^3+a_5x^5+\&c. \quad +a_{2n+1}x^{2n+1}+\&c.\\ &\quad-\frac{a_1x^3}{1.2}-\frac{a_3}{1.2}x^5-\&c. \quad -\frac{a_{2n-1}}{1.2}x^{2n+1}-\&c.\\ &\quad+\frac{a_1}{1.2.3.4}x^5+\&c. \quad +\frac{a_{2n-3}}{1.2.3.4}x^{2n+1}+\&c.\\ &\quad\quad-\&c. \quad -\frac{a_{2n-5}}{1.2.3.4.5.6}x^{2n+1}-\&c.\\ &\quad\quad+\&c.\end{aligned}$$

Comparing the coefficients of the same powers of x in these two identical series, we shall get

$$a_1=1$$

$$a_3=\frac{a_1}{1.2}-\frac{1}{1.2.3}$$

$$a_5=\frac{a_3}{1.2}-\frac{a_1}{1.2.3.4}+\frac{1}{1.2.3.4.5}$$

$$a_7=\frac{a_5}{1.2}-\frac{a_3}{1.2.3.4}+\frac{a_1}{1.2.3.4.5.6}-\frac{1}{1.2.3.4.5.6.7}$$

and generally

$$a_{2n+1}=\frac{a_{2n-1}}{1.2}-\frac{a_{2n-3}}{1.2.3.4}+\frac{a_{2n-5}}{1.2.3.4.5.6}-\frac{a_{2n-7}}{1.2.3.4.5.6.7.8}+\&c.$$

$$\pm\frac{a_1}{1.2\ldots 2n}\mp\frac{1}{1.2.3\ldots(2n+1)}$$

We hence get

$$a_1=1,\ a_3=\frac{2}{1.2.3},\ a_5=\frac{16}{1.2.3.4.5},\ a_7=\frac{272}{1.2\ldots7},\ \&c.$$

It is evident, that the same method may be applied to the developement of cot x, and the remaining trigonometrical lines.

Note (E).

We will here give the theorem of Lagrange, which is so important in the developement of functions, and in the reversion of series.

Let $u=f(y)$, and $y=z+x\,\varphi(y)$, where z is considered as independent of the variation of x. The theorem of Maclaurin will give us

$$u=U_0+U_1\,\frac{x}{1}+U_2\,\frac{x^2}{1.2}+U_3\,\frac{x^3}{1.2.3}+\&c.$$

our next object is the determination of the form of the coefficients U_0, U_1, U_2, &c.

We shall assume Y_0, Y_1, Y_2, &c. to represent the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. when $x=0$: we shall also put

$\varphi(y)=v$, $\varphi(z)=p$, $f(z)=Q$, and $\frac{df(z)}{dz}=q$.

In the first place,

$$y=z+x\,v,$$

$$\frac{dy}{dx}=v+\frac{x\,dv}{dx},$$

$$\frac{d^2y}{dx^2}=\frac{2dv}{dx}+\frac{x\,d^2v}{dx^2},$$

$$\frac{d^3y}{dx^3}=\frac{3\,d^2v}{dx^2}+\frac{x\,d^3v}{dx^3},$$

$$\frac{d^4y}{dx^4}=\frac{4\,d^3v}{dx^3}+\frac{x\,d^4v}{dx^4},$$

$$\&c. = \&c.$$

Again,

$$v = \varphi(y),$$

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx},$$

$$\frac{d^2v}{dx^2} = \frac{d^2v}{dy^2} \cdot \frac{dy^2}{dx^2} + \frac{dv}{dy} \cdot \frac{d^2y}{dx^2},$$

$$\frac{d^3v}{dx^3} = \frac{d^3v}{dy^3} \cdot \frac{dy^3}{dx^3} + \frac{3\,d^2v}{dy^2} \cdot \frac{dy\,d^2y}{dx^3} + \frac{dv}{dy} \cdot \frac{d^3y}{dx^3},$$

&c. = &c.

Consequently,

$$Y_0 = z,$$

$$Y_1 = \varphi(z) = p,$$

$$Y_2 = 2p \cdot \frac{dp}{dz} = \frac{d \cdot p^2}{dz},$$

$$Y_3 = 3p^2 \cdot \frac{d^2p}{dz^2} + \frac{3\,dp}{dz} \cdot \frac{d.p^2}{dz} = \frac{d^2.p^3}{dz^2},$$

$$Y_4 = 4p^3 \cdot \frac{d^3p}{dz^3} + \frac{4\,d.p^3}{dz} \cdot \frac{d^2p}{dz^2} + \frac{4\,dp}{dz} \cdot \frac{d^2.p^3}{dz^2} = \frac{d^3.p^4}{dz^3}$$

&c. = &c.

Also,

$$u = f(y)$$

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx},$$

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dy^2} \cdot \frac{dy^2}{dx^2} + \frac{du}{dy} \cdot \frac{d^2y}{dx^2},$$

$$\frac{d^3u}{dx^3} = \frac{d^3u}{dy^3} \cdot \frac{dy^3}{dx^3} + \frac{3\,d^2u}{d^2y} \cdot \frac{dy\,d^2y}{dx^3} + \frac{du}{dx} \cdot \frac{d^3y}{dx^3},$$

&c. = &c.

We hence deduce

$$U_0 = f(z) = Q,$$

$$U_1 = p \cdot \frac{dQ}{dz} = p\,q,$$

$$U_2 = p^2 \cdot \frac{dq}{dz} + q \cdot \frac{d \cdot p^2}{dz} = \frac{d \cdot p^2 q}{dz},$$

$$U_3 = p^3 \cdot \frac{d^2 q}{dz^2} + \frac{d \cdot p^3}{dz} \cdot \frac{dq}{dz} + \frac{d^2 \cdot p^3}{dz^2} \cdot q = \frac{d^2 \cdot p^3 q}{dz^2},$$

&c. = &c.

We consequently have

$$u = Q + p\,q \cdot \frac{x}{1} + \frac{d \cdot p^2 q}{dz} \cdot \frac{x^2}{1.2} + \frac{d^2 \cdot p^3 q}{dz^2} \cdot \frac{x^3}{1.2.3} + \text{\&c.}$$

or, replacing Q, q, and p by the quantities which they severally represent, we have

$$u = f(z) + \varphi(z) \frac{df(z)}{dz} \cdot \frac{x}{1} + d \cdot \frac{(\varphi(z))^2 \frac{df(z)}{dz}}{dz} \cdot \frac{x^2}{1.2} + \text{\&c.} \quad (1)$$

If we suppose $x = 1$, the series becomes

$$u = f(z) + \varphi(z) \frac{df(z)}{dz} + d \cdot \frac{(\varphi(z))^2 \frac{df(z)}{dz}}{dz} \cdot \frac{1}{1.2} + \text{\&c.} \quad (2)$$

which is the celebrated theorem, first given by Lagrange, in the *Mémoires de l'Académie de Berlin*, for the year 1768. He has considered at great length its different properties and applications, in his *Traité de la Résolution des Equations Numériques*, Note xi.

The proof which we have given of this theorem, though sufficiently simple in principle, will probably be a little embarrassing to a student to whom this species of reasoning cannot yet be supposed to be familiar. We shall, on this account, offer no apology for the introduction of a second.

$$u = f(y) = f\{z + x\varphi(y)\} = f(z) + \frac{df(z)}{dz} \cdot x\,\varphi(y)$$

$$+ \frac{d^2 f(z)}{dz^2} \cdot \frac{x^2\,\varphi(y)^2}{1.2} + \text{\&c.}$$

$$= q + q_1\, x\, v + q_2 \frac{x^2 v^2}{1.2} + q_3 \frac{x^3 v^3}{1.2.3} + \&c.$$

where q, q_1, q_2, &c. represent

$$f(z),\ \frac{d f(z)}{d z},\ \frac{d^2 f(z)}{d z^2},\ \&c.$$

and $v = \phi(y)$.

Again,

$$v = \phi\{z + x\,\phi(y)\} = \phi(z) + \frac{d\phi(z)}{dz} \cdot x\,v$$

$$+ \frac{d^2\phi(z)}{dz} \cdot \frac{x^2 v^2}{1.2} + \&c.$$

$$= p + p_1\, x\, v + p_2 \frac{x^2 v^2}{1.2} + \&c.$$

where p, p_1, p_2, &c. are the differential coefficients of $\phi(z)$.

Also,

$$v^2 = p' + p_1'\, x\, v + p_2' \frac{x^2 v^2}{1.2} + \&c.$$

$$v^3 = p'' + p_1''\, x\, v + p_2'' \frac{x^2 v^2}{1.2} + \&c.$$

$$\&c. = \&c.$$

where p', p'', p''', severally represent p^2, p^3, p^4, &c. p_1', p_1'', p_1''', &c. the first differential coefficients of these quantities, p_2', p_2'', p_2''', &c. the second differential coefficients, and so on.

Consequently,

$$\begin{aligned} u &= q \\ &+ x\,\{ q_1 p \} \\ &+ \frac{x^2}{1.2}\{ q_2 p' + 2\, q_1 p_1 v \} \\ &+ \frac{x^3}{1.2.3}\{ q_3 p'' + 3\, q_2 p_1' v + 3\, q_1 p_2 v^2 \} \end{aligned}$$

$$+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}\left\{q_4 p''' + 4\, q_3 p_1''\, v + 6\, q_2 p_2'\, v^2 + 4\, q_1 p_3 v^3\right\}$$

$$+ \text{ \&c.}$$

$$= U_0 + U_1 \frac{x}{1} + U_2 \cdot \frac{x^2}{1\cdot 2} + U_3 \cdot \frac{x^3}{1\cdot 2\cdot 3} + \text{ \&c.}$$

Supposing $x=0$, in the quantities v, v^2, v^3, &c. which enter into the several coefficients of the series, or in other words, considering v, v^2, v^3, &c. as severally equal to p, p^2, p^3, &c.

We hence get

$$U_0 = q = f(z),$$

$$U_1 = q_1 p = \frac{p\, d q}{d z} = \phi(z) \frac{d f(z)}{d z},$$

$$U_2 = q_2 p' + 2\, q_1 p_1 p = \frac{d\cdot \frac{p^2 d q}{d z}}{d z} = \frac{d\cdot \phi(z)^2 \frac{d f(z)}{d z}}{d z},$$

$$U_3 = q_3 p'' + 3\, q_2 p_1' p + 3\, q_1 p_2 p^2 = \frac{d^2 \cdot \frac{p^3 d q}{d z}}{d z^2} = \frac{d^2\cdot \phi(z)^3 \frac{d f(z)}{d z}}{d z^2}$$

$$U_4 = q_4 p''' + 4\, q_3 p_1'' p + 6\, q_2 p_2' p^2 + 4\, q_1 p_3 p^3 = \frac{d^3 \cdot \frac{p^4\, d q}{d z}}{d z^3}$$

$$= \frac{d^3 \cdot \phi(z)^4 \frac{d(z)}{d z}}{d z^3},$$

&c. = &c.

If we suppose $u = f(y)$, and $y = f(z + x\,\phi(y))$, the same series, *mutatis mutandis*, will be found to be true.

For, $u = f(y) = f\{f(z + x\,\phi(y))\} = \psi(z + x\,\phi(y))$

$$= \psi(z) + \frac{d\,\psi(z)}{d z}\, x\, \phi(y) + \frac{d^2\,\psi(z)}{d z^2}\, \frac{x^2\,\phi(y)^2}{1\cdot 2} + \text{ \&c.}$$

Also,

$$\phi(y)=\phi\{f(z+x\,\phi(y))=\phi_1(z+x\,\phi\,y))$$

$$=\phi_1(z)+\frac{d\,\phi_1(z)}{d\,z}\,x\,\phi(y)+\frac{d^2\phi_1(z)}{d\,z^2}\,\frac{x^2\,\phi(y)^2}{1.2}+\text{\&c.}$$

and it will easily be found, by a continuation of the same process as before, that

$$u=\psi(z)+\phi_1(z)\frac{d\,\psi(z)}{d\,z}\cdot\frac{x}{1}+d\,.\,\frac{\phi_1(z)^2\frac{d\,\psi(z)}{d\,z}}{d\,z}\cdot\frac{x^2}{1.2}+\text{\&c.}$$

which was given by Laplace, in the *Mémoires de l'Académie des Sciences*, for 1777.

The extensive uses of this important theorem will be readily discovered, by applying it to a few examples.

1. Suppose it was required to deduce an expression for y^n, when we have $\alpha-\beta\,y+\gamma\,y^m=0$; in this case $u=y^n$ $=f(y)$, and $y=\frac{\alpha}{\beta}+\frac{\gamma}{\beta}\,y^m=z+x\,\phi(y)$.

We hence obtain, in the series (1),

$$z=\frac{\alpha}{\beta},\quad x=\frac{\gamma}{\beta},\; f(z)=Q=z^n,\; q=n\,z^{n-1},\quad \phi(z)=p=z^m.$$

Also, $p\,q=n\,z^{n+m-1}$,

$$\frac{d\,.\,p^2\,q}{d\,z^2}=n\,.\,(n+2\,m-1)\,z^{n+2m-2},$$

$$\frac{d^2\,.\,p^3\,q}{d\,z}=n\,(n+3\,m-1)\,(n+3\,m-2)\,z^{n+3m-3},$$

&c.

Therefore,

$$u=y^n=z^n+n\,z^{n+m-1}\cdot\frac{x}{1}+n\,(n+2\,m-1)\,z^{n+2m-2}\cdot\frac{x^2}{1.2}$$

$$+n\,(n+3\,m-1)\,(n+3\,m-2)\,z^{n+3m-3}\cdot\frac{x^3}{1.2.3}+\text{\&c.}$$

and substituting the values of z and x, we get

$$y^n = \frac{\alpha^n}{\beta^n}\left\{1 + \frac{n\,\alpha^{m-1}\gamma}{\beta^m} + \frac{n(n+2m-1)}{1\cdot 2}\;\frac{\alpha^{2m-2}\gamma^2}{\beta^{2m}} + \&c.\right\}$$

If a_1, a_2, a_3, &c. be supposed to be the roots of the given equation $\alpha-\beta\,y+\gamma\,y^m=0$, we shall have m values of y^n, corresponding to the different roots. The developement just given, however, admits but of one value, which will be found in all cases to correspond to the least root of the equation. This property may be readily verified in the equation $\alpha-\beta\,y+\gamma\,y^2=0$.

For, in this case,

$$y = \frac{\beta \pm \sqrt{\beta^2 - 4\,\alpha\gamma}}{2\gamma} = \frac{\beta}{2\gamma}\left\{1 \pm \sqrt{1 - \frac{4\,\alpha\gamma}{\beta^2}}\right\}$$

$$= \frac{\beta}{2\gamma}\left\{1 \pm \left(1 - \frac{1}{2}\cdot\frac{4\,\alpha\gamma}{\beta^2} - \frac{1.1}{1.2.2^2}\cdot\frac{(4\,\alpha\gamma)^2}{\beta^4} - \frac{1.1.3}{1.2.3.2^3}\cdot\frac{(4\,\alpha\gamma)^3}{\beta^6} - \&c.\right)\right\}$$

$$= \frac{\beta}{2\gamma}\left\{1 \pm \left(1 - 2\,.\,\frac{\alpha\gamma}{\beta^2} - 1.2\,.\,\frac{\alpha^2\gamma^2}{\beta^4} - 1.1.4\,\frac{\alpha^3\gamma^3}{\beta^6} - \&c.\right)\right\}$$

and taking the inferior sign, we shall find

$$y = \frac{\alpha}{\beta} + \frac{\alpha^2\gamma}{\beta^3} + \frac{2\,\alpha^3\gamma^2}{\beta^5} + \&c.$$

$$= \frac{\alpha}{\beta}\left(1 + \frac{\alpha\gamma}{\beta} + \frac{2\,\alpha^2\gamma^2}{\beta^4} + \&c.\right);$$

and the same value of y will be found from the developement given above, by making $m=2$ and $n=1$. This property of the general formula is demonstrated by Lagrange, in the Memoir and Note referred to before. In the course of this investigation, he has also proved the following property of his theorem, equally remarkable for its elegance and extensive utility. If a_1, a_2, a_3, &c. be the roots of the equation $z-y+\phi(y)=0$, then the sum of the reciprocals of the nth powers of these roots will be found to be equal to the sum of all those terms of the developement of

y^{-n}, which involve the negative powers of z: the series itself for this case being

$$u = y^{-n} = \frac{1}{z^n} - \frac{n}{1} \cdot \frac{\phi(z)}{z^{n+1}} - \frac{n}{1.2 \cdot dz} d \cdot \frac{\phi(z)^2}{z^{n+1}} - \&c.$$

we shall give the demonstration of this important property, in the shortest and simplest form that it seems to admit of.

Since

$$z - y + \phi(y) = \gamma(a_1 - y)(a_2 - y) \ldots\ldots (a_n - y),$$

by taking the logarithms of each member of the equation, and differentiating, we shall get

$$\frac{1 - \frac{d\phi(y)}{dy}}{z - y + \phi(y)} = \frac{1}{a_1 - y} + \frac{1}{a_2 - y} + \&c. + \frac{1}{a_n - y};$$

representing $1 - \frac{d\phi(y)}{dy}$ by $f(y)$, and the sums of the reciprocals of the first, second, third, &c. powers of the roots by s_{-1}, s_{-2}, s_{-3}, &c. we shall find

$$\frac{f(y)}{z - y + \phi(y)} = \frac{f(y)}{z - y} - \frac{f(y)\phi(y)}{(z-y)^2} + \frac{f(y)\phi(y)^2}{(z-y)^3}$$

$$= s_{-1} + s_{-2}y + s_{-3}y^2 + \&c. \quad + s_{-n}y^{n-1} + s_{-n-1}y^n + \&c.$$

Assume

$$f(y) = A_0 + A_1 y + A_2 y^2 + \&c. \quad + A_n y^n + \&c.$$

$$f(y)\phi(y) = B_0 + B_1 y + B_2 y^2 + \&c. \quad + B_n y^n + \&c.$$

$$f(y)\phi(y)^2 = C_0 + C_1 y + C_2 y^2 + \&c. \quad + C_n y^n + \&c.$$

$$\&c. = \&c.$$

Also,

$$\frac{1}{z-y} = \frac{1}{z} + \frac{y}{z^2} + \frac{y^2}{z^3} + \&c. + \frac{y^n}{z^{n+1}} + \&c.$$

$$\frac{1}{(z-y)^2} = \frac{1}{z^2} + \frac{y2}{z^3} + \frac{3y^2}{z^4} + \&c. + \frac{(n+1)y^n}{z^{n+2}} + \&c.$$

$$\frac{1}{(z-y)^3} = \frac{1}{z^3} + \frac{y}{z^4} + \frac{3 \cdot 4}{1 \cdot 2}\frac{y^2}{z^5} + \&c. + \frac{(n+1)(n+2)}{1 \cdot 2}\frac{y^n}{z^{n+3}}$$

$$\&c. = \&c.$$

Collecting together those terms of the developement of $\frac{f(y)}{z-y}$, which involve y^n, and representing their sum by $t_n \frac{f(y)}{z-y}$, we shall find

$$t_n \cdot \frac{f(y)}{z-y} = \frac{A_0 y^n}{z^{n+1}} + \frac{A_1 y^n}{z^n} + \frac{A_2 y^n}{z^{n-1}} + \text{\&c.} + \frac{A_n y^n}{z}$$

$$= \frac{A_0 + A_1 z + A_2 z^2 + A_3 z^3 + \text{\&c.} + A_n z^n}{z^{n+1}} y^n$$

$= \frac{f(z)}{z^{n+1}} y^n$, if $\frac{f(z)}{z^{n+1}}$ be restricted to such terms as involve negative powers of z only.

In the same manner we shall find

$$t_n \frac{f(y)\phi(y)}{z-y} = \frac{B_0 + B_1 z + B_2 z^2 + \text{\&c.} + B_n z^n}{z^{n+1}} y^n$$

$= \frac{f(z)\phi(z)}{z^{n+1}} y^n$, under the same restrictions as above. The same process will give us

$$t_n \frac{f(y)\phi(y)}{(z-y)^2} = \frac{(n+1)B_0 + nB_1 z + (n-1)B_2 z^2 + \text{\&c.} + B_n z^n}{z^{n+2}} y^n$$

$= -\frac{1}{dz} d \cdot \frac{f(z)\phi(z)}{z^{n+1}} \cdot y^n$, a result which will be immediately verified by the actual differentiation of the series which is equal to $\frac{f(z)\phi(z)}{z^{n+1}}$.

Again,

$$t_n \frac{f(y)\phi(y)^2}{(z-y)^2} = \frac{(n+1)C_0 + nC_1 z + (n-1)C_2 z^2 + \text{\&c.} + C_n z^n}{z^{n+2}} y^n$$

$$= -\frac{1}{dz} \cdot d \cdot \frac{f(z)\phi(z)^2}{z^{n+1}} y^n ;$$

and consequently

$$t_n \frac{f(y)\phi(y)^2}{(z-y)^3} = \frac{\frac{(n+1)(n+2)}{1.2}C_0 + \frac{n(n+1)}{1.2}C_1 z + \&c\ldots C_n z^n}{z^{n+3}} y^n$$

$$= \frac{1}{1.2.dz^2} d^2 . \frac{f(z)\phi(z)^2}{z^{n+1}} . y^n,$$

a result likewise admitting of very easy verification.

A continuation of this process will give us

$$t_n . \frac{f(y)}{z-y} - t_n . \frac{f(y)\phi(y)}{(z-y)^2} + t_n . \frac{f(y)\phi(y)^2}{(z-y)^3} - \&c.$$

$$= \left(\frac{f(z)}{z^{n+1}} + \frac{1}{dz} d . \frac{f(z)\phi(z)}{z^{n+1}} + \frac{1}{1.2.dz^2} . d^2 . \frac{f(z)\phi(z)^2}{z^{n+1}} + \&c.\right) y^n$$

$$= s_{-n-1} y^n.$$

But $f(z) = 1 - \frac{d\phi(z)}{dz} = 1 - p_1$, $\phi(z) = p$, and $\frac{1}{z^{n+1}} = q$;

and therefore

$$s_{-n-1} = \left\{q - qp_1 + \frac{1}{dz}(d . qp - d . qpp_1) + \frac{1}{1.2.dz^2} \times \right.$$

$$\left. (d^2 . qp^2 - d^2 . qp^2 p_1) + \&c. \right\}$$

But $\frac{1}{dz} d . qp = qp_1 + pq_1$

$$\frac{1}{2.dz^2} d^2 . qp^2 = \frac{1}{dz} d\left(\frac{1}{2dz} d . qp^2\right) = \frac{1}{dz} \times$$

$$(d . qpp_1 + \tfrac{1}{2} d . p^2 q_1)$$

$$\frac{1}{3.dz^3} d^3 . qp^3 = \frac{1}{dz^2} d^2\left(\frac{1}{3dz} d . qp^3\right) = \frac{1}{dz^2} \times$$

$$(d^2 . qp^2 p_1 + \tfrac{1}{3} d^2 . p^3 q_1).$$

By substituting these values, and reducing, we finally get

$$s_{-n-1} = \left\{q + pq_1 + \frac{1}{1.2.dz} d . p^2 q_1 + \frac{1}{1.2.3\, dz^2} d^2 . p^3 q_1 + \&c.\right\}$$

But $q_1 = -\frac{(n+1)}{z^{n+2}}$, and consequently, replacing p by its value, we get s_{-n-1}

$$= \left\{\frac{1}{z^{n+1}} - \frac{(n+1)}{1}\frac{\phi(z)}{z^{n+2}} - \frac{(n+1)}{1.2.dz}d.\frac{\phi(z)^2}{z^{n+2}} - \&c.\right\};$$

putting n in the place of $n+1$, we have

$$s_{-n} = \left\{\frac{1}{z^n} - \frac{n}{1}\cdot\frac{\phi(z)}{z^{n+1}} - \frac{n}{1.2.dz}d.\frac{\phi(z)^2}{z^{n+1}}\right.$$

$$\left. - \frac{n}{1.2.3.dz^2}d^2.\frac{\phi(z)^3}{z^{n+1}} - \&c.\right\} \quad (4)$$

We will now apply this result to a few examples. In the equation $\alpha - \beta y + \gamma y^2 = 0$, we have already determined an expression for y^n; and by changing its sign, we shall have

$$y^{-n} = \frac{\beta^n}{\alpha^n}\left\{1 - \frac{n\alpha\gamma}{\beta^2} + \frac{n(n-3)}{1.2}\frac{\alpha^2\gamma^2}{\beta^4} - \frac{n(n-5)(n-6)}{1.2.3}\times\right.$$

$$\left.\frac{\alpha^3\gamma^3}{\beta^6} + \&c.\right\}$$

$= \frac{1}{a_1^n} + \frac{1}{a_2^n}$, by the theorem which we have just demonstrated.

If $\alpha = 1$, and $\gamma = 1$, we have

$$\frac{1}{a_1^n} + \frac{1}{a_2^n} = a_1^n + a_2^n \left(\text{since } a_1 = \frac{1}{a_2}\right)$$

$$= \beta^n - n\beta^{n-2} + \frac{n(n-3)}{1.2}\beta^{n-4} - \frac{n.(n-5)(n-6)}{1.2.3}\beta^{n-6} + \&c.$$

This last result is applicable to the developement of $\cos mA$ in terms of $\cos A$, and its powers: for if we assume $2\cos A = a_1 + \frac{1}{a_1}$, we must also have $2\cos mA = a_1^n + \frac{1}{a_1^n}$, (See Woodhouse's Trigonometry, page 41), where a_1 and

$\frac{1}{a_1}$ are the roots of the equation $x^2-2\cos A\,x+1=0$: we hence get, by substituting $2\cos A$ for β, in the expression just given,

$$a_1^n+\frac{1}{a_1^n}=2\cos mA\pm(2\cos A)^n-n(2\cos A)^{n-2}$$
$$+\frac{n(n-3)}{1.2}(2\cos A)^{n-4}-\&c.$$

the series being supposed to terminate, when the powers of $2\cos A$ become negative.

The reader may see this result verified by referring to the Appendix of Mr. Woodhouse's Trigonometry, page 216, where the same conclusion is deduced by the successive application of the general series to the developement of a_1^n and $\frac{1}{a_1^n}$.

But the series (4) is not merely applicable to the determination of s_{-n}, or the sum of the reciprocals of the nth powers of the roots of an equation: by substituting $\frac{1}{y}$ for y, and applying the series to the resulting transformed equation, we shall obtain an expression for s_n, or the sum of nth powers of the roots.

Thus the equation $\alpha-\beta y+\gamma y^2=0$, being transformed into $\alpha y^2-\beta y+\gamma=0$, we shall get, by the application of this theorem,

$$a_1^n+a_2^n=\frac{\beta^n}{\gamma^n}\left\{1-\frac{n\alpha\gamma}{\beta^2}+\frac{n(n-3)}{1.2}\frac{\alpha^2\gamma^2}{\beta^4}-\&c.\right\}$$

Again, suppose the general equation

$$y^n-p_1y^{n-1}+p_2y^{n-2}-p_3y^{n-3}+\&c. \qquad \pm p_n=0,$$

to be transposed into

$$1-p_1y+p_2y^2-p_3y^3+\&c. \qquad \pm p_ny^n=0,$$

the roots of which are the reciprocals of the roots of the former. Comparing this equation with $z-y+\phi(y)=0$, we get

$$z=\frac{1}{p_1}, \text{ and } \phi(y)=\frac{y^2}{p_1}(p_2-p_3y+\&c.)$$

and

$$y^{-n}=s_n=\left\{\frac{1}{z^n}-\frac{n\phi(z)}{z^{n+1}}-\frac{n}{1.2.dz}d.\frac{\phi(z)^2}{z^{n+1}}-\&c.\right\}$$

Now $\frac{1}{z^n}=p_1^n$

$$-\frac{n}{1}\cdot\frac{\phi(z)}{z^{n+1}}=-\frac{nz^2}{p_1z^{n+1}}(p_2-p_3z+p_4z^2-p_5z^3+p_6z^4-\&c.)$$

$$=-np_2p_1^{n-2}+np_3p_1^{n-3}-np_4p_1^{n-4}$$
$$+np_5p_1^{n-5}-np_6p_1^{n-6}+\&c.$$

$$\frac{n}{1.2.dz}d.\frac{\phi(z)^2}{z^{n+1}}=-\frac{n}{1.2.dz.p_1^2}d\left(p_2^2z^{-n+3}-2p_2p_3z^{-n+4}\right.$$
$$\left.+\left.\begin{matrix}+p_3^2\\+2p_2p_4\end{matrix}\right\}z^{-n+5}-\&c.\right)$$

$$=\frac{n}{1.2.p_1^2}\left((n-3)p_2^2z^{-n+2}-2(n-4)p_2p_3z^{-n+3}\right.$$
$$\left.+\left.\begin{matrix}+(n-5)p_3^2\\+2(n-5)p_2p_4\end{matrix}\right\}z^{-n+4}-\&c.\right)$$

$$=\frac{n(n-3)}{1.2}p_2^2p^{n-4}-n(n-4)p_2p_3p_1^{n-5}$$
$$+\left.\begin{matrix}+\frac{n.(n-5)}{1.2}p_3^2\\+n(n-5)p_2p_4\end{matrix}\right\}p_1^{n-6}-\&c.$$

$$\frac{n}{2.3dz^2}d^2.\frac{\phi(z)^3}{z^{n+1}}=-\frac{n}{1.2.3.dz^2p_1^3}d^2\left(p_2^3z^{-n+5}-\&c.\right)$$
$$=-\frac{n(n-4)(n-5)}{1.2.3}p_2^3p_1^{n-6}+\&c.$$

Consequently,

$$s_n = p_1^n - n p_2 p_1^{n-2} + n p_3 p_1^{n-3} - \left.\begin{matrix} n p_4 \\ + \dfrac{n.(n-3)}{1.2} p_2^2 \end{matrix}\right\} p_1^{n-4} + \left.\begin{matrix} n p_5 \\ - n.(n-4) p_2 p_3 \end{matrix}\right\} \times$$

$$\left.\begin{matrix} -n p_6 \\ + n(n-5) p_2 p_4 \\ + \dfrac{n.(n-5)}{1.2} p_3^2 \\ - \dfrac{n.(n-4)(n-5)}{1.2.3} p_2^3 \end{matrix}\right\} p_1^{n-6} + \&c.$$

which is Waring's theorem, given in his Meditationes Algebraicæ, Cap. 1.

If, in the general series (2), we suppose $f(z) = z$, and therefore $\frac{d\,f(z)}{dz} = 1$, we shall have

$$u = y = z + \phi(z) + \frac{d.\phi(z)^2}{1.2.dz} + \frac{d^2.\phi(z)^3}{1.2.3.dz^2} + \&c. \quad (5)$$

a formula of great use in the reversion of series.

Thus, if

$$\alpha + \beta y + \gamma y^2 + \delta y^3 + \&c. = 0,$$

we have $z = -\frac{\alpha}{\beta}$, and $\phi(y) = -\frac{y^2}{\beta}(\gamma + \delta y + \&c.)$

and therefore

$$y = z - \frac{z^2}{\beta}(\gamma + \delta z + \&c.) + \frac{1}{1.2\,dz} d\left\{\frac{z^4}{\beta^2}(\gamma + \delta y + \&c.)^2\right\} - \&c.;$$

by performing the operations indicated, we shall get

$$\begin{aligned} y = z - \frac{\gamma z^2}{\beta} - \frac{\delta z^3}{\beta} - \frac{\epsilon z^4}{\beta} - \&c. \\ + \frac{2\gamma^2 z^3}{\beta^2} + \frac{5\gamma\delta z^4}{\beta^2} + \&c. \\ - \frac{5\gamma^3 z^4}{\beta^4} - \&c. \\ + \&c. \end{aligned}$$

It would extend this Note to too great a length to go through the demonstration of the general property of the series mentioned above, by which it appears that the value of y, which is given by this developement, is that of the least root of the given equation. Our principal object in this Note was to shew the great use of this theorem in the developement of functions; but the curious student will not fail to keep in mind that it is capable of many other applications, besides those we have mentioned; one of the most important of which consists in its furnishing the means of approximating to the least roots of equations.

NOTE (F).

Our author has given an instance, in which, by assigning a particular value of x, a fractional power of dx or h appears in the expression for the difference of a function. In cases of this kind, the series of Taylor is said to be defective: we will endeavour to give the theory of this apparent failure.

In the first place, suppose $f(x)=\frac{F(x)}{(x-a)^m}$, where m is a whole number, and where $F(x)$ neither becomes infinite nor nothing, by supposing $x=a$. If x becomes $x+h$, we have $f(x+h)=\frac{F(x+h)}{(x-a+h)^m}$, or supposing $x=a$, $f(a+h)=\frac{F(a+h)}{h^m}$. We thus see, that the developement of $f(x+h)$, for this particular value of x, will involve negative powers of h; consequently, whenever the assignation of a particular value of x makes a function infinite, we may conclude that the developement of $f(a+h)$, for this case, will involve negative powers of h, and conversely. Again, let $f(x)$ be supposed to involve a radical, such as $(x-a)^{\frac{m}{n}}$, which dis-

appears by supposing $x=a$. Then $f(x+h)$ must involve $(x-a+h)^{\frac{m}{n}}$, which becomes $h^{\frac{m}{n}}$ when $x=a$: this is a case in which Taylor's series fails to give the developement of the function $f(x+h)$ corresponding to a particular value of x. But there will be no failure, when a radical disappears under these circumstances, merely in consequence of the evanescence of a quantity by which it is multiplied. Thus, suppose a given function of x should involve the product of a radical, and some such quantity as $(x-a)^m$, in which the radical does not vanish by making $x=a$. It is obvious, that this radical will be multiplied by $(x-a+h)^m$ in the function $f(x+h)$, or by h^m, when $f(x+h)$ becomes $f(a+h)$. We consequently see, that this hypothesis will make the radical disappear in all the differential coefficients before the mth: but it will be again found in all the others, and the series of Taylor will give the true developement of $f(x+h)$, corresponding to this particular value of x.

The following statement will serve as an illustration of the preceding observations: suppose

$$f(x+h)=A_1 h^{a_1}+A_2 h^{a_2}+A_3 h^{a_3}+\text{\&c.}$$

to be the developement corresponding to the hypothesis of $x=a$; in which the indices a_1, a_2, a_3, &c. are arranged in the order of their magnitudes. It is evident, that if a_1 be negative, the term $A_1 h^{a_1}$ will become infinite, when $h=0$, and therefore also $f(x)$ itself must be infinite, corresponding to this particular value of x. Again, differentiating $f(x+h)$ as a functionof h, we shall have

$$\frac{df(x+h)}{dh}=a_1 A_1 h^{a_1-1}+a_2 A_2 h^{a_2-1}+a_3 A_3 h^{a_3-1}+\text{\&c.}$$

$$\frac{d^2 f(x+h)}{dh^2}=a_1(a_1-1)A_1 h^{a_1-2}+a_2(a_2-1)A_2 h^{a_2-2}+\text{\&c.}$$

$$\frac{d^3 f(x+h)}{dh^3}=a_1(a_1-1)(a_1-2)A_1 h^{a_1-3}+\text{\&c.}$$

&c.

Now it is evident, that if a_1, a_2, a_3, &c. be integer and positive numbers, the coefficients A_1, A_2, A_3, &c. will admit of accurate determination, by making $h=o$ and $x=a$, until we meet with a fractional index. Suppose a_m to be a proper fraction, and let n be the least whole number which is greater than m; it is obvious, that

$$\frac{d^n f(x+h)}{d h^n} = a_m (a_m - 1) \dots (a_m - n + 1) A_m h^{a_m - n} + \&c.;$$

the index $a_m - n$ is therefore negative, and consequently $\frac{d^n f(x+h)}{d h^n}$ is infinite, when $h=o$ and $x=a$.

We thus are enabled to discover, by the order of the differential coefficient of $f(x)$, which becomes infinite, between what whole numbers the fractional index is situated.

To determine the true form of the developement, when the series of Taylor fails, we must expand the function $f(x+h)$ by the common algebraical methods, putting a in the place of x.

An example or two will form the best commentary upon the preceding statement.

Let $f(x) = 2ax - x^2 + a\sqrt{(x^2 - a^2)}$: the developement is required, when $x=a$.

In this case

$$\frac{df(x)}{dx} = \frac{df(x)}{dh} \text{ (Lac. No. 21) } = 2(a-x) + \frac{ax}{\sqrt{(x^2 - a^2)}}.$$

We consequently have, making $x=a$,

$$f(x) = a^2, \quad \text{and} \quad \frac{df(x)}{dx} = \frac{1}{0}:$$

the developement of $f(a+h)$ must involve a term $A_1 h^{a_1}$, where a_1 is between o and 1.

We may readily verify this remark, by observing that

$$f(a+h) = a^2 - h^2 + a h^{\frac{1}{2}} \sqrt{(2a+h)}$$

$$= a^2 + \sqrt{2}\, a^{\frac{3}{2}} h^{\frac{1}{2}} + \frac{1}{2\sqrt{2}} a^{\frac{1}{2}} h^{\frac{3}{2}} - h^2 + \&c.$$

Again, let

$$f(x) = \sqrt{x} + (x-a)^2 \log (x-a),$$

$$\frac{d f(x)}{d x} = \frac{1}{2\sqrt{x}} + 2(x-a) \log (x-a) + x - a,$$

$$\frac{d^2 f(x)}{d x^2} = -\frac{1}{4x\sqrt{x}} + 2 \log (x-a) + 3,$$

and making $x=a$, we have

$$f(x) = a^{\frac{1}{2}}$$

$$\frac{d f(x)}{d x} = \frac{1}{2a^{\frac{1}{2}}}$$

$$\frac{d^2 f(x)}{d x^2} = \frac{1}{0}.$$

The developement of $f(a+h)$ must therefore involve a fractional power of h, between 1 and 2.

We will add a few remarks concerning implicit functions. Suppose $y=f(x)$, in which a radical disappears by making $x=a$, but which is found again in the differential coefficients; consequently $\frac{dy}{dx}$ must have more values than y or $f(x)$ itself, and therefore will not be expressible by a simple function of x and y, which does not involve explicitly this radical. Again, if in the equation $y=f(x)$ we exterminate this radical, the resulting equation may be represented by

$$f(x, y) = 0 = u,$$

the differential of which will give us

$$\frac{du}{dx} dx + \frac{du}{dy} dy = 0; \text{ or } \frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}}.$$

Now this expression must be defective, when $x=a$, unless $\frac{du}{dy}$ and $\frac{du}{dx}$ vanish simultaneously; for it is evident, that

$\frac{dy}{dx}$ can have no more values from this equation, than y itself, except by supposing $\frac{dy}{dx}=\frac{0}{0}$; when it will be necessary to resort to the second differential of $u=0$, in order to discover its true values. By differentiating, therefore, for a second time, we get

$$\frac{d^2 u}{dx^2}+\frac{2\,d^2 u}{dx\,dy}\cdot\frac{dy}{dx}+\frac{d^2 u}{dy^2}\cdot\frac{dy^2}{dx^2}+\frac{du}{dy}\cdot\frac{d^2 y}{dx^2}=0,$$

which becomes, since $\frac{du}{dx}=0$,

$$\frac{d^2 u}{dx^2}+\frac{2\,d^2 u}{dx\,dy}\cdot\frac{dy}{dx}+\frac{d^2 u}{dy^2}\cdot\frac{dy^2}{dx^2}=0,$$

an equation from which two values of $\frac{dy}{dx}$ may be deduced.

Thus, let

$$f(x)=x+(x-a)\sqrt{x-b}.$$

$$\frac{df(x)}{dx}=1+\sqrt{x-b}+\frac{x-a}{2\sqrt{x-b}}.$$

By making $x=a$, we get

$f(x)=a$, and $\frac{df(x)}{dx}=1+\sqrt{a-b}$, the radical re-appearing in the differential coefficient $\frac{df(x)}{dx}$, and thereby restoring the number of values which is essential to the consecutive state of the function $f(x)$.

But if we make $f(x)=y$, and exterminate the radical, we shall get the implicit function

$$(y-x)^2-(x-a)^2(x-b)=0,$$

from which, by differentiating, we deduce

$$\frac{dy}{dx}=1+\frac{2(x-a)(x-b)+(x-a)^2}{2(y-x)}=\frac{0}{0}, \text{ when } x=a,$$

and therefore $y=x$.

The second differential of the equation gives us

$$2(y-x)\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}-1\right)^2=4(x-a)+2(x-b);$$

and if $x=a=y$, we shall get

$$\frac{dy}{dx}=1+\sqrt{(a-b)},$$

the same result as before.

If the same value of x, which destroys the numerator and denominator of the expression for $\frac{dy}{dx}$, deduced from the first differential of u, likewise destroy these terms in the expression deduced from the second; we must then proceed to the third differential, which, in consequence of the evanescence of the terms which involve $\frac{d^2u}{dy^2}$ and $\frac{d^3u}{dy^3}$, will furnish us with a cubic equation for the determination of $\frac{dy}{dx}$. We must proceed, in the same manner, when the third and higher differentials fail in giving us the requisite values. This evidently depends upon the nature of the radical which disappears, which must be replaced by the degree of the equation upon which the determination of this differential coefficient depends.

But the same value of x which destroys the radical in y, may make it disappear in $\frac{dy}{dx}$, though not in $\frac{d^2y}{dx^2}$; in this case the number of values of y and $\frac{dy}{dx}$ will be the same, but less than those of $\frac{d^2y}{dx^2}$. If therefore we exterminate the radical in the equation $y=f(x)$, the value of $\frac{d^2y}{dx^2}$

which is deduced from it, will be found $=\frac{0}{0}$, and it will be necessary to proceed to the differential equations of a higher order, in order to determine its different values.

Thus, if

$$y=x+(x-a)^2\sqrt{(x-b)}$$

$$\frac{dy}{dx}=1+2(x-a)\sqrt{(x-b)}+\frac{(x-a)^2}{2\sqrt{(x-b)}},$$

$$\frac{d^2y}{dx^2}=2\sqrt{(x-b)}+\frac{2(x-a)}{\sqrt{(x-b)}}-\frac{(x-a)^2}{4(x-b)^{\frac{3}{2}}}.$$

Making $x=a$, we have

$$y=a,\ \frac{dy}{dx}=1,\ \text{and}\ \frac{d^2y}{dx^2}=2\sqrt{(a-b)}.$$

But if we reduce the equation to a rational form

$$(y-x)^2=(x-a)^4(x-b):$$

we deduce, by differentiating,

$$2(y-x)\left(\frac{dy}{dx}-1\right)=4(x-a)^3(x-b)+(x-a)^4,$$

from which we get $\frac{dy}{dx}=\frac{0}{0}$, when $x=a=y$.

Again, the second differential gives us

$$(y-x)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}-1\right)^2=6(x-a)^2(x-b)+4(x-a)^3.$$

This gives, when $x=a$,

$$\left(\frac{dy}{dx}-1\right)^2=0,\ \text{and}\ \frac{dy}{dx}=1.$$

In order to determine the value of $\frac{d^2y}{dx^2}$, we must proceed to the fourth differential, which gives

$$(y-x)\frac{d^4y}{dx^4}+3\left(\frac{dy}{dx}-1\right)\frac{d^3y}{dx^3}+\frac{3d^2y^2}{dx^4}=48(x-a)$$
$$+12(x-b).$$

Making $x=a$, $y=a$, and $\frac{dy}{dx}=1$, we get

$$\frac{3\,d^2y^2}{dx^4}=12\,(a-b),$$

and therefore

$$\frac{d^2y}{dx^2}=2\sqrt{(a-b)},$$

the same result as before.

The reader will meet with many examples to illustrate this theory, when he comes to the investigation of the *singular points* of curve lines, the determination of which presupposes a failure, if so it may be termed, in the general series of Taylor.

We are indebted for nearly the whole of this Note to Leçon 8. of the *Calcul des Fonctions* of Lagrange.

NOTE (G).

Our object in this Note, is the application of the Differential Calculus to the theory of curves, without introducing the consideration of limits.

In the curves represented in Fig. 1, we suppose $AP=x$, and $PM=y=f(x)$. Assume $AP_{-1}=x-h$, and $AP_1=x+h$, and let P_1M_1, $P_{-1}M_{-1}$ be the ordinates corresponding to the points P_1 and P_{-1}: draw $M_{-1}Q$ and MQ_1 parallel to AP: join M_1 and M, M and M_{-1}, and let them be produced to meet the axis of the abscissæ in the points S_1 and S_{-1}; and suppose MT to be the tangent to the curve, at the point M. The reader will find no difficulty in supplying the several lines which are not drawn in the figures referred to.

We shall adopt the common definition of a tangent, which considers it as a line which meets the curve at the point M, but does not cut it when produced on either side of this point, at least within assignable limits. Its position at each

point must depend upon the nature of the curve, and consequently its sub-tangent PT, by which its position is determined, must be expressible equally with the ordinate y, by some function of the other co-ordinate x. We shall now proceed to shew in what manner this function is deducible in all cases, from the given equation of the curve. We readily discover that

$$P_1M_1 = y + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \&c.$$

$$P_{-1}M_{-1} = y - \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} - \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \&c.$$

$$P_1M_1 - PM = M_1Q_1 = \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \&c.$$

$$PM - P_{-1}M_{-1} = MQ = \frac{dy}{dx}\cdot\frac{h}{1} - \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} - \&c.$$

We hence obtain

$$PS_1 = PM\cdot\frac{MQ_1}{M_1Q_1} = \frac{yh}{\frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2}} + \&c.$$

$$= \frac{y\,dx}{dy} + p_1h + p_2h^2 + \&c.$$

$$PT = \phi(x)$$

$$PS_{-1} = PM\cdot\frac{M_{-1}Q}{MQ} = \frac{yh}{\frac{dy}{dx}\cdot\frac{h}{1} - \frac{d^2y}{dx^3}\cdot\frac{h^2}{1.2}} + \&c.$$

$$= \frac{y\,dx}{dy} + q_1h + q_2h^2 + \&c.$$

But it necessarily follows, from our definition of a tangent, that it is intermediate in position to the secants M_1MS_1 and $MM_{-1}S_{-1}$; its subtangent PT must consequently be less than one of the subsecants, and greater than the other; and the three expressions for PS_1, PT, and PS_{-1} are thus arranged in the order of their magni-

tudes: and since this order is not affected by any variation of the value of h, and the first terms of the first and last of these expressions are identical, we may conclude, from the general principle demonstrated in Note (A), that

$$\phi(x)=PT=\frac{y\,dx}{dy}.$$

The reader will have no difficulty in proving that $\tan\theta=\frac{dy}{dx}$, if θ be assumed to represent the angle of inclination of the tangent at M to the axis of the abscissæ. Our author has shewn that the equation of the tangent to the curve at M is

$$_1y-y=\frac{dy}{dx}(_1x-x),$$

where $_1y$ and $_1x$ are the co-ordinates of any point of the tangent to the curve at M. If $_1x-x$ be taken equal to h, we have

$$_1y=y+\frac{dy}{dx}h;$$

and consequently $\frac{dy}{dx}h=Q_1N_1$, if N_1 be that point of the tangent which is determined by the production of P_1M_1.

Our definition of a tangent in some degree implies, that it is the only one which can be drawn to the curve at the point M. We shall find no difficulty, however, in proving this to be the case; for if y_1 and x_1 be supposed to be the co-ordinates to a point of the second tangent, corresponding to the point M of the curve, we shall find

$$y'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\cdot\frac{h^2}{1\cdot 2}+\&c.$$

$$y_1=y+ah$$

$$_1y=y+\frac{dy}{dx}h.$$

Now y_1 is, by hypothesis, intermediate in value to y' and y; the quantities $y'-y$, y_1-y, ${}_1y-y$ are therefore arranged in the order of their magnitudes, and by the general principle made use of above, $a\,h=\frac{dy}{dx}.h$, and therefore $a=\frac{dy}{dx}$. This second tangent must therefore coincide with the other.

We thus see, that if two curves have a common point, they will likewise have a common tangent, when $\frac{dy}{dx}=\frac{dy_1}{dx_1}$, y_1 and x_1 being the co-ordinates of the second curve. We shall also find no difficulty in shewing, that no curve whatever can be drawn through the point of contact, between a curve and its tangent, unless this condition holds good; for supposing it possible, we should have

$$y'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.$$

$$y_1'=y_1+\frac{dy_1}{dx_1}h+\frac{d^2y_1}{dx_1^2}\frac{h^2}{1.2}+\&c.$$

$${}_1y=y+\frac{dy}{dx}.h;$$

and since $y_1=y$ and $y'-y$, $y_1'-y$, ${}_1y-y$, are in the order of magnitude, we may conclude, as before, that $\frac{dy}{dx}=\frac{dy_1}{dx_1}$.

We will proceed a little further: suppose that in two curves at a common point, $\frac{dy}{dx}=\frac{dy_1}{dx_1}$ and $\frac{d^2y}{dx^2}=\frac{d^2y_1}{dx_1^2}$: they have therefore a common tangent, and the second condition will shew, that no curve whatever can be drawn *between* them through this point of contact, the second differential coefficient of whose ordinate is not identical with the same differential coefficient in the other curves; for since

$$y' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

$$_1y' = {}_1y + \frac{d\,_1y}{d\,_1x}h + \frac{d^2\,_1y}{d\,_1x^2}\frac{h^2}{1.2} + \frac{d^3\,_1y}{d\,_1x^3}\frac{h^3}{1.2.3} + \&c.$$

$$y_1' = y_1 + \frac{dy_1}{dx_1}h + \frac{d^2y_1}{dx_1^2}\frac{h^2}{1.2} + \frac{d^3y_1}{dx_1^3}\frac{h^3}{1.2.3} + \&c.$$

and since, upon this hypothesis, y', $_1y'$, y_1', are arranged in the order of magnitude, and $y = {}_1y = y_1$, we readily infer not only that $\frac{d\,_1y}{d\,_1x} = \frac{dy}{dx}$; but also that $\frac{d^2\,_1y}{d\,_1x^2} = \frac{d^2y}{dx^2}$.

It is evident, that the same process may be applied to prove the following general proposition. If two curves have a common point, and any number of the differential coefficients of their ordinates y and y_1 be supposed to be respectively equal to each other, then no curve can be drawn *between* them, through their point of contact, unless the same number of the differential coefficients of its ordinate be severally equal to those of the ordinate of either of the other curves. (See Nos. 94 and 97.)

The osculating circle, or circle of curvature, may be defined to be a circle which has a common point with the curve, and in which $\frac{dy_1}{dx_1} = \frac{dy}{dx}$ and $\frac{d^2y_1}{dx_1^2} = \frac{d^2y}{dx^2}$; the general equation of the circle being represented by

$$(y_1 - \beta)^2 + (x_1 - \alpha)^2 = \gamma^2. \qquad \text{(No. 94.)}$$

Consequently,

$$y_1 = \beta + \sqrt{(\gamma^2 - (x_1 - \alpha)^2)}$$

$$\frac{dy_1}{dx_1} = \frac{dy}{dx} = -\frac{(x_1 - \alpha)}{(y_1 - \beta)},$$

$$\frac{d^2y_1}{dx_1^2} = \frac{d^2y}{dx^2} = -\frac{\left(1 + \frac{dy_1^2}{dx_1^2}\right)}{y_1 - \beta} = -\frac{\left(1 + \frac{dy^2}{dx^2}\right)}{y_1 - \beta}.$$

We shall find no difficulty in determining expressions for γ, $y_1-\beta$, $x_1-\alpha$, in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, by means of these three equations. For the equation and properties of the evolute, the reader is referred to our author, Nos. 95, and 96.

The other properties of the circle of curvature, which are implied in the different definitions which are given of it, form very easy corollaries to the theory of contacts, which we have just been considering.

We may consider the arcs and areas of curves as functions of one of the co-ordinates, equally with the subtangent, radius of the circle of curvature, and other lines, the expressions for which we have just determined; and although the Differential Calculus does not enable us to assign the functions to which they are severally equal, yet it furnishes us with their differentials, from which they can, in most cases, be determined by the processes taught in the Integral Calculus.

Thus, if $s=f(x)$, where s is the length of the curve CM, Fig. 5, we shall have $\Delta s = MO'M$. From the principle of Archimedes we may conclude, that

$$MO'M < MN+NM' \text{ and } > MM';$$

but

$$MN+NM' = \sqrt{\left(h^2+\frac{dy^2}{dx^2}h^2\right)} - \frac{d^2y}{dx^2}\frac{h^2}{1.2} - \&c.$$

$$= h\sqrt{\left(1+\frac{dy^2}{dx^2}\right)} + p_2 h^2 + p_3 h^3 + \&c.$$

$$MOM' = \Delta s = \frac{ds}{dx}h + \frac{d^2s}{dx^2}\frac{h^2}{1.2} + \&c.$$

$$MM' = \sqrt{\left(h^2+\left(\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.\right)^2\right)}$$

$$= h\sqrt{\left(1+\frac{dy^2}{dx^2}\right)} + q_2 h^2 + q_3 h^3 + \&c.$$

The general principle so frequently made use of, gives us

$$\frac{ds}{dx}h = h\sqrt{\left(1+\frac{dy^2}{dx^2}\right)};$$

and therefore

$$ds = dx\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}.$$

Again, let $u=f(x)$, where u represents the area of the curve, Fig. 6. In this case Δu, or $PMM'P' > PMQP'$, and $< PNM'P'$; but

$$PNM'P' = PP' \times P'M'$$
$$= yh + \frac{dy}{dx}h^2 + \&c.$$

$$PMM'P' = \Delta u = \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c.$$

$$PMQP' = PP' \times PM = yh.$$

We consequently have

$$\frac{du}{dx}h = yh, \text{ and therefore } du = y\,dx.$$

Note (H).

The different properties of the cycloid may be deduced with great simplicity, by considering its co ordinates as severally functions of the arc of the generating circle, which has been in contact with the base.

Thus if θ be the arc MQ, which is equal to AQ, Fig. 29, we shall have $x=\theta-\sin\theta$, and $y=1-\cos\theta$: or if we suppose the abscissæ to commence from the vertex K of the cycloid, and to be reckoned upon the axis KI, we shall also get $x_1 = KI - PM = 2 - y = 1+\cos\theta$, and $y_1 = PI = AI - AP = \pi - x = \pi - \theta + \sin\theta$. Assuming $\phi = \pi - \theta = \text{arc } QM$, these equations will become $x_1 = 1-\cos\phi$, and $y_1 = \phi + \sin\phi$.

We have supposed the radius of the generating circle to be equal to unity. If it be equal to a, we shall have $x=a(\theta-\sin\theta)$, and $y=a(1-\cos\theta)$, or $x_1=a(1-\cos\phi)$ and $y_1=a(\phi+\sin\phi)$. The second equation to the cycloid is sometimes more convenient than the first, though they both admit of a very ready application to the solution of the different problems which have been considered in a different way by our author.

Thus,

$$\frac{dy}{dx}=\frac{d(1-\cos\theta)}{d(\theta-\sin\theta)}=\frac{\sin\theta}{1-\cos\theta}=\frac{MN}{NQ}=\frac{PM}{PT}.$$

We hence deduce immediately the value of the subtangent PT: this conclusion likewise suggests the geometrical construction by which it is determined. We need not explain the method of applying these equations to the determination of the normal and subnormal, in which the reader can experience no difficulty.

The ordinary formula will require no modification, as long as the first differentials only of x and y are involved; but when x is considered as the function of some third variable, the expressions which involve the second and higher differential coefficients, must be altered in a manner which is explained by our author, in No. 116. Thus the expression for the length of the radius of curvature, will become

$$\gamma=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dy\,d^2x-dx\,d^2y},$$

where y and x are considered as functions of some third quantity, such as θ, in the case before us. We shall be easily able to determine γ for the point M of the cycloid, by substituting for dx, dy, d^2x, and d^2y, the values deduced from the differentiation of the trigonometrical values of x and y. It will be found to be equal to twice the normal MQ, a property which furnishes us with a very ready

and simple method of determining the equation of the evolute. Referring to Fig. 29, we find $AE = AP + 2PQ = \theta + \sin\theta$; and also $EM' = NQ = 1 - \cos\theta$. If the point M' of the evolute be referred to a line drawn from the point A, perpendicular to AI, and if x_1 and y_1 be considered as the co-ordinates of that point, we shall have $x_1 = EM' = 1 - \cos\theta$, and $y_1 = AE = \theta + \sin\theta$, an equation to a cycloid, whose vertex is A, and whose generating circle is equal to that of the original cycloid.

We may conceive a cycloid, in which $x_1 = 1 - \cos\theta$, and $y_1 = m\theta + \sin\theta$, considering the axis of the curve as the axis of the abscissæ. This cycloid will be generated by the combined motion of a point in a circle, and a uniform motion of the circle itself, parallel to the base. The properties of this curve were first considered by Wallis, who denominated it the *protracted* or *contracted* cycloid, according as m was greater or less than unity.

The trigonometrical values of x and y, will furnish expressions for the differentials of the arc, area, &c. of a cycloid, from which the Integral Calculus will enable us to deduce, if not more readily, at least more elegantly, the several primitive functions to which they correspond.

We will add a word or two concerning the *Quadratrix*, a curve formerly celebrated for its apparent connection with the quadrature of the circle, and which likewise admits of trigonometrical expressions for its co-ordinates, whose mutual relation is not assignable algebraically. Let BC be the quadrant of a circle, whose center is A; and suppose the radius BA to revolve uniformly through the whole quadrant, in the same time in which a line perpendicular to AB moves uniformly from B to A. The curve traced out by the intersections of the perpendicular line, and revolving radius, is called the *Quadratrix*. If x and y be the co-ordinates to any point M of this curve, x being reckoned from the point B, and if θ be the angle described by the

radius at that point, we shall easily be able to shew that $x = \frac{2\theta}{\pi}$, and $y = \left(1 - \frac{2\theta}{\pi}\right) \tan \theta$.

We shall not investigate the different properties of this curve, in which the student will experience no particular difficulty. We will merely call his attention to the value of y, when $\theta = \frac{\pi}{2}$: and remind him, that since $d^2x = 0$, the common formulæ are immediately applicable, without any previous modification.

If u be assumed to represent the *radius vector* AM, we readily get $u = (1 - x) \sec \theta = \left(1 - \frac{2\theta}{\pi}\right) \sec \theta$. The curve is thus transferred into the class of spirals, in which the distance from some fixed point, called the pole or focus, is equal to some function of the angle described by the radius vector from a given position.

We shall be able to deduce, though not so readily as in the case just given, the polar equations to the different conic sections from the given equations subsisting between their co-ordinates. Thus, in the parabola

$$u = \frac{a}{\sin^2 \frac{\theta}{2}},$$

where a is the distance between the focus and vertex, and θ the angle between the radius vector and the axis.

In the ellipse,

$$u = \frac{a(1 - e^2)}{1 - e \cos \theta},$$

and in the hyperbola,

$$u = \frac{a(e^2 - 1)}{1 - e \cos \theta},$$

in both of which equations a represents the semi-axis major,

ae the eccentricity, and θ the angle between the radius vector and that part of the axis which is formed by the production of the line joining the vertex and the focus. These polar equations to the Conic Sections, will be found of most extensive use in almost every department of the Mathematical Sciences, and particularly in Physical Astronomy; and the student will be amply recompensed for any labour he may devote, in order to render himself familiar with their different applications.

The nature and generation of those spirals, in which $u=f(\theta)=a\,.\,\theta^n$, n being any positive or negative number whatever, have been explained by our author, who has investigated their properties by means of trigonometrical values of rectangular co-ordinates referred to an assumed axis. This method is both general and simple: for the greater satisfaction, however, of the reader, we will here shew in what manner the common differential formulæ may be directly deduced from their polar equations.

In a spiral curve, such as is represented in Fig. 31, let AM_1, AM, AM_{-1}, be three distances, making equal angles h or $\Delta\,\theta$ with each other. Draw MQ_1 and $M_{-1}Q$ perpendiculars to AM_1 and AM; and let MT be the tangent at M, and AT the subtangent: join M_1 and M, M and M_{-1}, and produce them to S_1 and S_{-1}. Then, since $u=f(\theta)$, we readily get the following expressions:

$$AM_1=u+\frac{du}{d\theta}h+\frac{d^2u}{d\theta^2}\cdot\frac{h^2}{1\,.\,2}+\&c.$$

$$AM_{-1}=u-\frac{du}{d\theta}h+\frac{d^2u}{d\theta^2}\cdot\frac{h^2}{1\,.\,2}-\&c.$$

$$M_1Q_1=AM_1-AQ_1=u'-u\,.\cos h.=u'-u\left(1-\frac{h^2}{1\,.\,2}+\&c.\right)$$

$$=\frac{du}{d\theta}h+p_2h^2+\&c.$$

$$MQ = AM - AQ = u - \left(u - \frac{du}{d\theta}h + \&c.\right)\cos h.$$

$$= \frac{du}{d\theta}h + q_2 h^2 + q_3 h^3 + \&c.$$

From similar triangles, we also get

$$AS_1 = AM \times \frac{MQ_1}{M_1 Q_1} = \frac{u \times u \,.\, \sin h}{\frac{du}{d\theta}h + \&c.} = \frac{u^2 h - \frac{u^2 h^3}{1.2.3} + \&c.}{\frac{du}{d\theta}h + p_2 h^2 + \&c.}$$

$$= \frac{u^2 d\theta}{du} + r_1 h + r_2 h^2 + \&c.$$

$$AT = \phi(\theta)$$

$$AS_{-1} = AM \times \frac{M_{-1}Q}{MQ} = AM \times \frac{AM_{-1}\sin h}{MQ}$$

$$= \frac{u\left(u - \frac{du}{d\theta}h + \&c.\right)\left(h - \frac{h^3}{1.2.3} + \&c.\right)}{\frac{du}{d\theta}h + q_2 h^2 + \&c.}$$

$$= \frac{u^2 d\theta}{du} + s_1 h + s_2 h^2 + \&c.$$

By the general principle, of which we have made such extensive use in the preceding Note, we may conclude, that

$$AT = \phi(\theta) = \frac{u^2 d\theta}{du}.$$

The trigonometrical tangent of the angle AMT is evidently $= \frac{AT}{AM} = \frac{u\,d\theta}{du}$.

The same principle may be very easily applied to the determination of the differentials of those functions of θ, which are severally equal to the length of the spiral arc and its area, both being estimated from the given position

of the radius vector. If s and v be taken to represent them, we shall find

$$ds = \sqrt{(u^2\, d\theta^2 + d\,u^2)}, \text{ and } d\,v = \frac{u^2\, d\theta}{2}.$$

If we suppose another curve to pass through the point M, and to be referred to the same focus A, we shall have

$$u' = u + \Delta u = u + \frac{d\,u}{d\,\theta}\, h + \frac{d^2 u}{d\,\theta^2} \cdot \frac{h^2}{1.2} + \&c.$$

and

$$u_1 = u_1 + \Delta u_1 = u_1 + \frac{d\,u_1}{d\,\theta}\, h + \frac{d^2 u_1}{d\,\theta^2} \cdot \frac{h^2}{1.2} + \&c.$$

where $u_1 = \psi(\theta)$, is the polar equation to the second curve.

Now, let $u = v_1$: they must have, therefore, a common point M.

Again, let

$$\frac{d\,u}{d\,\theta} = \frac{d\,u_1}{d\,\theta}, \text{ and therefore } \frac{d\,u}{u\,d\,\theta} = \frac{d\,u_1}{u_1\,d\,\theta}:$$

they must consequently have a common tangent at M.

If, likewise, $\frac{d^2 u}{d\,\theta^2} = \frac{d^2 u_1}{d\,\theta^2}$, we may easily prove, that no spiral curve can be drawn through M between them, in which this condition does not hold.

The same theory of contents which has been demonstrated of curves, referred to rectangular co-ordinates in the preceding Note, is likewise applicable to spiral curves, in which any number of the differential coefficients of those functions of θ, which are equal to the polar distances in each, are respectively equal to each other.

Let the curve of contact be a circle, and suppose

$$u = u_1, \frac{d\,u}{d\,\theta} = \frac{d\,u_1}{d\,\theta} \text{ and } \frac{d^2 u}{d\,\theta^2} = \frac{d^2 u_1}{d\,\theta^2}:$$

let F be the center of the circle, and suppose $M\,F = r$,

$AF = a$, and the angle $AMF = \phi$. The triangle AFM furnishes the following equation:

$$a^2 = u_1^2 + \gamma^2 - 2\gamma u_1 . \cos\phi. \qquad (1)$$

By the hypothesis, $\frac{du}{d\theta} = \frac{du_1}{d\theta}$, and since $u_1 = u$, we have $\frac{du}{u\,d\theta} = \frac{du_1}{u_1\,d\theta}$; the tangent at M is therefore common to the circle and the curve: but

$$\cot\phi = \tan . AMT = \frac{u\,d\theta}{du_1}:$$

we hence get

$$\cos\phi = \frac{u_1\,d\theta}{\sqrt{\{u_1^2 d\theta^2 + du_1^2\}}}.$$

Differentiating the equation (1), we get

$$o = u_1\,du_1 - \gamma\,\{u_1 d . \cos\phi + \cos\phi . du_1\} \quad (2);$$

and therefore $\gamma = \frac{u_1\,du_1}{u_1 d . \cos\phi + \cos\phi . du_1}$

$$= \frac{(du_1^2 + u_1^2 d\theta^2)^{\frac{3}{2}}}{2\,du_1^2\,d\theta - u_1\,d\theta\,d^2 u_1 + u_1^2\,d\theta^3}.$$

But $u_1 = u$, $\frac{du_1}{d\theta} = \frac{du}{d\theta}$, and $\frac{d^2 u_1}{d\theta^2} = \frac{d^2 u}{d\theta^2}$.

By substituting these values, we get

$$\gamma = \frac{(du^2 + u^2\,d\theta^2)^{\frac{3}{2}}}{2\,du^2\,d\theta - u\,d\theta\,d^2 u + u^2\,d\theta^3}.$$

If FE be drawn perpendicular to AM, we shall find

$$EM = \gamma\cos\phi = \frac{u\,(du^2 + u^2\,d\theta^2)}{2\,du^2 - u\,d^2u + u^2 d\theta^2},$$

which is the expression for half the chord of the circle of curvature, which passes through the pole of the spiral.

English mathematicians very often define spiral curves, by an equation between the polar distance and the per-

pendicular upon the tangent. Assuming p to represent the perpendicular, we shall be able to assign the relation of p and u, by eliminating θ between the two equations,

$$u = f(\theta)$$

$$p = \frac{u^2 d\theta}{\sqrt{(u^2 d\theta^2 + d u^2)}} .$$

The reader will find no difficulty in verifying the following expressions:

$$\gamma = \frac{u\, d u}{d p} \text{ and } M E = \frac{p\, d u}{d p} .$$

Note (I). Page 230.

Our author has explained the method of integrating all differential functions of one variable, which are comprehended in the formula $\frac{P\, d x}{\sqrt{(\alpha + \beta x + \gamma x^2)}}$, where P is any rational function of x; the integrals in all cases being either algebraic functions, or involving with an algebraical part, either logarithmic or circular transcendents.

The term *transcendent* is applied to all differential expressions which do not admit of complete integration; but in the case of the formula just mentioned, this difficulty is nearly overcome, since approximate values of the transcendents to which its integral is reducible, may be obtained o any degree of accuracy from the logarithmic and trigonometrical tables. It is upon this principle, that we consider a differential function as suscept.ble of integration, when it can be made to depend upon transcendents whoes values are tabulated, or are otherwise determinable by approximation.

The integral of the very comprehensive formula $\frac{P\,d\,x}{R}$, where $R = \sqrt{(\alpha + \beta\, x + \gamma\, x^2 + \delta\, x^3 + \iota\, x^4)}$, is reducible to an algebraical part, and to transcendents of the forms

$$\int \frac{x^2\,d\,x}{R_1}, \int \frac{d\,x}{R_1}, \text{ and } \int \frac{d\,x}{(x^2+a)R_1},$$

where $R_1 = \sqrt{\alpha + \beta\, x^2 + \gamma\, x^4}$: our principal attention ought, therefore, to be paid to the consideration of the properties of these transcendental functions, and to the investigation of methods of approximating to their values and of comparing them with each other.

Legendre, in a Memoir presented to the *Académie des Sciences*, in 1792, which he subsequently embodied and expanded in his *Exercices de Calcul Intégral*, has shewn that this integral may be always reduced to an algebraical part, and to transcendents of the form $\int (A + B \sin^2\phi) \frac{d\,\phi}{\Delta}$, and $\int \frac{d\,\phi}{(1 + n \sin^2 \phi)\,\Delta}$, where $\Delta = \sqrt{(1 - c^2 \sin^2\phi)}$, c being less than unity. The first of these is always expressible by means of elliptic arcs, and the second bears such analogy to the former, that it may be considered as a transcendent of the same order and species. It is on this account that he has given them the name of Elliptic Transcendents.

We shall not attempt to give an abstract of the admirable work of this illustrious Geometer, which would involve discussions in some degree inconsistent with the nature of an elementary Treatise like that of our author's. The English reader will find a translation of the Memoir, in the New Series of the *Mathematical Repository*.

Little is known concerning the integration of differential formulæ, more complicated than the preceding. Lacroix, in his great work, has given some few instances

which are reducible to it; but they are neither very general nor very important. Particular formulæ have been integrated by ingenious transformations, or reduced to others which are capable of integration. We shall give two or three of these from the same work, as examples of the artifices employed to effect these reductions.

The differential function $P\,dx\,(x+\sqrt{(1+x^2)})^{\frac{p}{q}}$, in which P is a function of x and $\sqrt{1+x^2}$, may be rationalized. For this purpose, make

$$x+\sqrt{(1+x^2)}=u^q;$$

this gives

$$x=\frac{u^{2q}-1}{2u^q},\quad dx=\frac{q\,u^{2q}\,du+q\,du}{2u^{q+1}},$$

$$\sqrt{1+x^2}=u^q-x=\frac{u^{2q}+1}{2u^q}.$$

Again, the formulæ

$$\frac{dx}{(1-x^m)\sqrt[2m]{2x^m-1}},\ \text{and}\ \frac{x^{m-1}\,dx}{(1-x^m)\sqrt[m]{2x^m-1}},$$

may be rationalized, by making in the first,

$$u=\pm\frac{\sqrt[2m]{2x^m-1}}{x},\ \text{and in the second,}\ u=\sqrt[m]{2x^m-1}:$$

the one will become $\dfrac{u^{2m-2}\,du}{1-u^{2m}}$ and the other $\dfrac{2u^{2m-2}\,du}{1-u^{2m}}$.

Mr. Bromhead, in a very original and ingenious paper, published in the *Philosophical Transactions*, for 1816, on the *Fluents of Irrational Functions*, has given general methods of rationalizing differential functions of this nature, which comprehend an immense variety of forms, which have hitherto been considered as incapable of reduction. We shall not attempt to enumerate the very curious results

which he has obtained, nor to explain the principle upon which he has deduced them, since the original paper itself is accessible to all our readers.

Note (K).

Our author has shewn the method of integrating the general formula $dz \sin z^m \cos z^n$, where m and n are any whole numbers whatever, whether positive or negative. We will mention a few other circular functions, some of which are of considerable importance, whose integrals are assignable either completely, or at least by means of converging series.

1. Let the differential function be $\frac{dx}{a+b\cos x}$: if we make $\cos x = \frac{1-u^2}{1+u^2}$, the differential will be transformed into

$\frac{2\,du}{(a+b)+(a-b)u^2}$: we consequently have, when $a < b$,

$$\int \frac{2\,du}{(a+b)-(b-a)u^2} = \frac{1}{\sqrt{(b^2-a^2)}} \log \left\{ \frac{\sqrt{(b^2-a^2)}+(b-a)u}{\sqrt{(b^2-a^2)}-(b-a)u} \right\} + \text{const.}$$

$$= \frac{1}{\sqrt{(b^2-a^2)}} \log \left\{ \frac{\sqrt{(b+a)}\sqrt{(1+\cos x)}+\sqrt{(b-a)}\sqrt{(1-\cos x)}}{\sqrt{(b+a)}\sqrt{(1+\cos x)}-\sqrt{(b-a)}\sqrt{(1-\cos x)}} \right\} + \text{const.}$$

$$= \frac{1}{\sqrt{(b^2-a^2)}} \log \left\{ \frac{a\cos x+b+\sin x\sqrt{(b^2-a^2)}}{a+b\cos x} \right\} + \text{const.}$$

If $a > b$,

$$\int \frac{2\,du}{(a+b)+(a-b)u^2} = \frac{2}{\sqrt{(a^2-b^2)}} \text{ arc } \left(\tan = \frac{\sqrt{(a-b)}}{\sqrt{(a+b)}} u \right) + \text{const.}$$

$$= \frac{2}{\sqrt{(a^2-b^2)}} \text{ arc } \left(\tan = \frac{\sqrt{(a-b)}\sqrt{(1-\cos x)}}{\sqrt{(a+b)}\sqrt{(1+\cos x)}} \right) + \text{const.}$$

$$= \frac{1}{\sqrt{(a^2-b^2)}} \text{ arc } \left(\tan = \frac{\sin x\sqrt{(a^2-b^2)}}{b+a\cos x} \right) + \text{const.},$$

since $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$: expressing the arc by its cosine, we shall get

$$\int \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{(a^2 - b^2)}} \text{ arc} \left(\cos = \frac{b + a \cos x}{a + b \cos x}\right) + \text{const.}$$

2. The differential $\frac{dx \sin x}{a + b \cos x}$ may be immediately transformed into $\frac{du}{a + bu}$, by making $u = \cos x$; and the differential $\frac{dx \cos x}{a + b \cos x}$ may be put under the form

$$\frac{dx}{b} - \frac{a\,dx}{b(a + b \cos x)}.$$

The integrals of both of these forms are very easily assigned.

3. Let us take the very general form $\frac{dx(a_1 + b_1 \cos x)}{(a + b \cos x)^n}$: assume, as in No. 154,

$$\int \frac{dx(a_1 + b_1 \cos x)}{(a + b \cos x)^n} = \frac{A \sin x}{(a + b \cos x)^{n-1}} + \int \frac{dx(B + C \cos x)}{(a + b \cos x)^{n-1}},$$

where A, B, C are constant and indeterminate coefficients. By differentiating, we shall get the equation

$$a_1 + b_1 \cos x = A \cos x (a + b \cos x) + (n - 1) Ab \sin^2 x$$
$$+ (B + C \cos x)(a + b \cos x);$$

from which we shall be able to determine

$$A = \frac{ab_1 - ba_1}{(n - 1)(a^2 - b^2)}, \quad B = \frac{aa_1 - bb_1}{a^2 - b^2}, \quad C = \frac{(n - 2)(ab_1 - ba_1)}{(n - 1)(a^2 - b^2)}.$$

A continuation of this process will conduct us, when n is a whole number, to an ultimate integral of the form

$\int \frac{dx(p+q\cos x)}{a+b\cos x}$, the method of assigning whose value has been explained in the preceding article.

If we suppose $a_1=1$, and $b_1=0$, we shall have

$$\int \frac{dx}{(a+b\cos x)^n} = -\frac{b\sin x}{(n-1)(a^2-b^2)(a+b\cos x)^{n-1}}$$

$$+\frac{1}{(n-1)(a^2-b^2)}\int \frac{\{(n-1)a-(n-2)b\}dx}{(a+b\cos x)^{n-1}}.$$

This method, however, fails in effecting the integration of this formula, when n is a fractional or negative number, in which case we must have recourse to series. The following method, which depends upon the developement of $(a+b\cos x)^n$, will enable us to integrate the differential expression $dx(a+b\cos x)^n$, for all values of n whatever.

The function $(a+b\cos x)^n$ may, in the first place, be put under the form $a^n(1+e\cos x)^n$, where

$e=\frac{b}{a}$. Assume $\psi=1+e\cos x$, and suppose

$$\psi^n=(1+e\cos x)^n=a_0+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\&c.$$

Taking the logarithms of each member of this equation, and then differentiating, we shall get

$$\frac{ne\cos x}{1+e\cos x}=\frac{a_1\sin x+2a_2\sin 2x+3a_3\sin 3x+\&c.}{a_0+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\&c.}$$

Exterminating the denominators, transposing all the terms to one side of the equation, and substituting for all products of the form $\sin x\cos mx$, and $\cos x\sin mx$, the equivalent expressions $\frac{1}{2}\sin(m+1)x-\frac{1}{2}\sin(m-1)x$, and $\frac{1}{2}\sin(m+1)x+\frac{1}{2}\sin(m-1)x$, we shall get

$$
\begin{array}{l|l|l|l|l}
o = a_1 & \sin x + 2\,a_2 & \sin 2\,x + 3\,a_3 & \sin 3\,x + 4\,a_4 & \sin 4\,x + \&c. \\
 & +\frac{1}{2}\,a_1 e & +\frac{2}{2}\,a_2 e & +\frac{3}{2}\,a_3 e & \\
+\frac{2}{2}\,a_2 e & +\frac{3}{2}\,a_3 e & +\frac{4}{2}\,a_4 e & +\frac{5}{2}\,a_5 e & \\
-n\,a_0 e & -\frac{n}{2}\,a_1 e & -\frac{n}{2}\,a_2 e & -\frac{n}{2}\,a_3 e & \\
+\frac{n}{2}\,a_2 e & +\frac{n}{2}\,a_3 e & +\frac{n}{2}\,a_4 e & +\frac{n}{2}\,a_5 e &
\end{array}
$$

We hence deduce

$$a_2 = \frac{2\,n\,a_0 e - 2\,a_1}{(n+2)\,e},\ a_3 = \frac{(n-1)\,a_1 e - 4\,a_2}{(n+3)\,e},\ a_4 = \frac{(n-2)\,a_2 e - 6\,a_3}{(n+4)\,e},\ \&c.$$

The determination of the coefficients of this series is consequently dependent upon that of the two first of them, a_0 and a_1, to which our attention must now be directed.

Since

$$\psi^n = 1 + n\,e\cos x + \frac{n\,(n-1)}{1\,.\,2}\,e^2\cos^2 x + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,e^3\cos^3 x + \&c.$$

If we develope $\cos^2 x$, $\cos^3 x$, $\cos^4 x$, &c. into series, involving the cosines of the multiples of x, by means of the formulæ given in No. 199, we shall readily discover that

$$a_0 = 1 + \frac{n\,(n-1)}{2.2}\,e^2 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{2\,.\,2\ \ 4\,.\,4}\,e^4 + \&c.$$

$$a_1 = 2\,e\left\{\frac{n}{2} + \frac{n\,(n-1)\,(n-2)}{2\,.\,2\,.\,4}\,e^2 + \frac{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)}{2\,.\,2\,.\,4\,.\,4\,.\,6\,.\,6}\,e^4 + \&c.\right\}$$

If a_0 and a_1 can be determined from these series, the developement of ψ^n will be very easily effected, as well as the integration of the formula $\psi^n\,d\,x$.

Euler, in his *Institutiones Calculi Integralis*, Vol. I, Cap. 6, has given several methods of approximating to the value of a_0 for different values of n and e; and also of determining from it the other coefficients of the series. He has also given a method of deducing, by means of the Differential Calculus, the coefficients of ψ^{-n-1} from those of ψ^{-n}, a proposition of the greatest importance, since it allows us to chuse that value of n which furnishes the readiest determination of a_0, at least amongst those values of it which differ from the one which is the immediate subject of consideration by whole numbers. We feel less regret at being compelled to omit these investigations, since it gives us an opportunity of referring our readers to a work which first reduced the principles and results of the Integral Calculus to system and order, and which is invaluable to a student, not less from the profundity and variety of its researches, than from the singular simplicity and elegance with which the most difficult theories are illustrated and explained.

The form in which $(a+b\cos x)^n$ generally presents itself in Physical Astronomy, is that of

$$\frac{1}{(r^2+r_1^2-2\,rr_1\cos x)^{\frac{3}{2}}} \text{ or of } \frac{1}{(r^2+r_1^2-2\,rr_1\cos x)^{\frac{5}{2}}},$$

where r and r_1 denote the distances of two planets from the sun, and x the angle between them; at least this is the case in which e is little different from unity, and is consequently the only one which presents any considerable difficulty.

Lagrange, in the *Memoires de l'Académie de Berlin*, for 1781, by substituting $\dfrac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}$ for $\cos x$, and by multiplying together the developements of

$$(r-r_1e^{x\sqrt{-1}})^n \text{ and } (r-r_1e^{-x\sqrt{-1}})^n,$$

or the factors of

$(r^2 - rr_1(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}) + r_1^2)^n$, has deduced series for the determination of a_0 and a_1, which are rapidly convergent when $n = -\frac{1}{2}$; and the theorem of Euler, which we have mentioned above, will enable us easily to determine the values of these coefficients in the common case, in which $n = -\frac{3}{2}$.

Mr. Ivory, in a very ingenious paper, in the *Edinburgh Transactions* for 1798, has improved upon the process of Lagrange, and has deduced series for a_0 and a_1, when $n = -\frac{1}{2}$, which converge with very great rapidity, even in the most unfavourable case that can be supposed.

Mr. Wallace, in a paper in the same Transactions for 1805, has also considered this subject, and has given a method of determining a_0 and a_1, when $n = -\frac{3}{2}$, which depends upon the rectification of the ellipse. We particularly recommend both these papers to the attention of our readers, which are remarkable for their elegance, and not less so for the discovery of very important series for assigning the lengths of the perimeters of ellipses of all eccentricities.

4. The following method of integrating $dx \log(1 + e\cos x)$ is taken from the same work of Euler, which we have mentioned above: since

$$\log \psi = e\cos x - \frac{e^2 \cos^2 x}{2} + \frac{e^3 \cos^3 x}{3} - \&c.$$

we may also assume

$$\log \psi = -a_0 + a_1 \cos x - a_2 \cos 2x + a_3 \cos 3x - \&c.$$

we readily see, that

$$a_0 = \frac{1}{2}\cdot\frac{e^2}{2} + \frac{1.3}{2.4}\cdot\frac{e^4}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{e^6}{6} + \&c.$$

considering e as a variable quantity, and differentiating, we shall get

$$\frac{e\,d a_0}{d e} = \frac{1}{2}\cdot e^2 + \frac{1.3}{2.4}\cdot e^4 + \frac{1.3.5}{2.4.6} e^6 + \&c.$$

$$= \frac{1}{\sqrt{(1-e^2)}} - 1;$$

consequently $d a_0 = \dfrac{d e}{e\sqrt{(1-e^2)}} - \dfrac{d e}{e}$, and integrating,

$$a_0 = \log\left(\frac{1-\sqrt{(1-e^2)}}{e}\right) - \log e + \text{const.}$$

$$= \log\left(\frac{2-2\sqrt{(1-e^2)}}{e^2}\right),$$

the constant being $= \log \frac{1}{2}$, as will be readily seen, if we make $e=0$, and therefore $a_0=0$.

Again, since

$$\frac{a_1}{2} = \frac{1}{2} e + \frac{1.3}{2.4}\cdot\frac{e^3}{3} + \frac{1.3.5}{2.4.6}\cdot\frac{e^5}{5} + \&c.$$

and therefore

$$\frac{e^2\,d a_1}{2\,d e} = \frac{1}{2} e^2 + \frac{1.3}{2.4}\cdot e^4 + \frac{1.3.5}{2.4.6}\cdot e^6 + \&c.$$

$$= \frac{1}{\sqrt{(1-e^2)}} - 1,$$ we get

$d a_1 = \dfrac{2\,d e}{e^2\sqrt{(1-e^2)}} - \dfrac{2\,d e}{e^2}$, and by integrating, we obtain

$$a_1 = -\frac{2\sqrt{(1-e^2)}}{e} + \frac{2}{e} + \text{const.} = 2\left(\frac{1-\sqrt{(1-e^2)}}{e}\right),$$

the constant disappearing, when $e=0$.

The other coefficients are determined in the same man-

ner as those of $(1+e\cos x)^n$, which we have considered above: for, by differentiating $\log\psi$, we shall get

$$\frac{-e\sin x}{1+e\cos x}=-a_1\sin x+2\,a_2\sin 2\,x-3\,a_3\sin 3\,x+\&c.$$

from which we shall easily discover, that

$$a_2=\frac{a_1-e}{e}\,,\ a_3=\frac{4\,a_2-a_1\,e}{3\,e}\,,\ a_5=\frac{6\,a_3-2\,a_2\,e}{4\,e}\,,\ \&c.$$

and substituting for a_1, &c. its value, we find

$$a_2=\frac{2}{2}\left(\frac{1-\sqrt{(1-e^2)}}{e}\right)^2,\ a_3=\frac{2}{3}\left(\frac{1-\sqrt{(1-e^2)}}{e}\right)^3,\ \&c.$$

making, for greater brevity, $\frac{1-\sqrt{(1-e^2)}}{e}=m$, we shall have

$$\log(1+e\cos x)=-\log\frac{2\,m}{e}+\frac{2}{1}m\cos x-\frac{2}{2}m^2\cos 2\,x$$
$$+\frac{2}{3}m^3\cos 3\,x-\frac{2}{4}m^4\cos 4\,x+\&c.$$

and consequently

$$\int dx\log(1+e\cos x)=-x\log\frac{2\,m}{e}$$
$$+\frac{2}{1}m\sin x-\frac{2}{4}m^2\sin 2\,x+\frac{2}{9}m^3\sin 3\,x-\frac{2}{16}m^4\sin 4\,x+\&c.$$

If $e=1$, and therefore $m=1$, we have

$$\log(1+\cos x)=-\log 2+\frac{2}{1}\cos x-\frac{2}{2}\cos 2\,x+\frac{2}{3}\cos 3\,x-\&c.$$
$$=\log\left(2\cos^2\frac{x}{2}\right)=\log 2+2\log\cos\frac{x}{2}\,.$$

$$\log(1-\cos x)=-\log 2-\frac{2}{1}\cos x-\frac{2}{2}\cos 2\,x-\frac{2}{3}\cos 3\,x$$
$$=\log\left(2\sin^2\frac{x}{2}\right)=\log 2+2\log\sin\frac{x}{2}\,.$$

We hence get, by taking the difference of these series,

$$\log \tan \frac{x}{2} = -2 \cos x - \frac{2}{3} \cos 3x - \frac{2}{5} \cos 5x - \&c.$$

5. The integration of the differentials $e^{\alpha x} dx \sin x^n$, and $e^{\alpha x} dx \cos x^n$, may be obtained without the aid of the exponential formulæ for the sine and cosine, by a process similar to that made use of in Nos. 170 and 203: we thus obtain the following equations of reduction:

$$(1) \int e^{\alpha x} dx \sin x^n = \frac{e^{\alpha x} \sin x^{n-1} (\alpha \sin x - n \cos x)}{\alpha^2 + n^2}$$

$$+ \frac{n(n-1)}{\alpha^2+n^2} \int e^{\alpha x} dx \sin x^{n-2},$$

$$(2) \int e^{\alpha x} dx \cos x^n = \frac{e^{\alpha x} \cos x^{n-1} (\alpha \cos x + n \sin x)}{\alpha^2+n^2}$$

$$+ \frac{n(n-1)}{\alpha^2+n^2} \int e^{\alpha x} dx \cos x^{n-2}:$$

NOTE (L).

The same principle which was applied in Note (G), to the determination of the differentials of the arcs and areas of curves, may also be applied to find the differentials of the volumes and curve surfaces of solids of revolution.

Let $u=f(x)$ be the volume generated by the revolution AMP, Fig. 46, round the axis AP. If $h=PP'$, then Δu is the volume generated by the revolution of the area $PMM'P'$, which is intermediate in value to the volumes generated by the areas $PSM'P'$ and $PMRP'$. Representing one of these quantities by D, and the other by D_1, we shall have

$$D = \pi \times P'M'^2 \times PP' = \pi h \left(y + \frac{dy}{dx} h + \&c.\right)^2$$

$$= \pi y^2 h + p_2 h^2 + p_3 h^3 + \&c.$$

$$\Delta u = \frac{du}{dx} h + \frac{d^2 u}{dx^2} \cdot \frac{h^2}{1.2} + \&c.$$

$$D_1 = \pi \times PM^2 \times PP' = \pi y^2 h;$$

consequently, from the relation subsisting between the quantities D, Δu, and D_1, we may conclude that

$$\frac{du}{dx} \cdot h = \pi y^2 h, \text{ and therefore } du = \pi y^2 dx.$$

Again, let $u_1 = f_1(x)$ represent the curve surface of the solid body generated by the revolution of AMP: then Δu, the surface generated by the revolution of the arc MOM_1, will be greater than the surface generated by the revolution of the chord MM', and less than that generated by the revolution of the tangent MN, and the line NM'; the lines MN, and NM', which are not in the figure referred to, may be very easily supplied. Denoting these quantities by D and D_1, we shall have

$$D = \pi \cdot MM' \cdot (PM + P'M')$$

$$= \pi \sqrt{\left(h^2 + \frac{dy^2}{dx^2} h^2 + \&c.\right)} \left(y + y + \frac{dy}{dx} h^2\right)$$

$$= 2\pi y h \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} + p_2 h^2 + \&c.$$

$$\Delta u_1 = \frac{du_1}{dx} h + \frac{d^2 u_1}{dx^2} \frac{h^2}{1.2} + \&c.$$

$$D_1 = \pi \cdot MN \cdot (PM + P'N) + \pi \cdot (P'N^2 - P'M'^2)$$

$$= \pi \cdot h \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} \left(y + y + \frac{dy}{dx} h\right)$$

$$+ \pi \left\{ \left(y + \frac{dy}{dx} h\right)^2 - \left(y + \frac{dy}{dx} h + \frac{d^2 y}{dx^2} \frac{h^2}{1.2}\right)^2 \right\}$$

$$= 2\pi y h \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} + q_2 h^2 + q_3 h^3 + \&c.$$

consequently, as before,

$$\frac{d u_1}{d x} h = 2 \pi y h \sqrt{\left(1 + \frac{d y^2}{d x^2}\right)},$$

and therefore

$$d u_1 = 2 \pi y \, d x \sqrt{\left(1 + \frac{d y^2}{d x^2}\right)}.$$

NOTE (M).

The proposition which is here made use of by our author, being little known to English readers, we shall endeavour to explain the method by which it is demonstrated. If A be the area of any figure traced out upon a given plane, then the area of its orthographical projection upon any other plane, making with the former an angle θ, will be equal to $A \cos \theta$. This theorem, which in all cases admits of a very simple demonstration, can present no difficulty when the figure is a parallelogram, one of whose sides is coincident with the line of the common intersection of the planes.

Again, from A, the common point of intersection of three planes, each of which is perpendicular to the other two, draw $A M$ perpendicular to a plane which intersects them in the lines $Q Q_1$, $Q_1 Q_2$, $Q_2 Q$: from M draw $M N$, $M N_1$, $M N_2$, perpendiculars upon $Q Q_1$, $Q_1 Q_2$, $Q_2 Q$, and $M P$, $M P_1$, $M P_2$, perpendiculars upon the planes $A Q Q_1$, $A Q_1 Q_2$, $A Q_2 Q$, and join $A N$, $A N_1$, $A N_2$, which likewise pass through P, P_1, P_2: then $A P$, $A P_1$, $A P_2$, or their equals, $M P_2$, $M P_1$, $M P$, will form the adjacent edges of a rectangular parallelopiped, whose diagonal is $A M$, and therefore $A M^2 = M P^2 + M P_1^2 + M P_2^2$. Let θ, θ_1, θ_2, be the angles $A N M$, $A N_1 M$, $A N_2 M$, or the angles which the intersecting plane makes with the co-ordinate planes. The angles $A M P$, $A M P_1$, $A M P_2$, are respectively equal to θ, θ_1, θ_2, and therefore $M P =$

$AM \, . \cos \theta$, $MP_1 = AM \, . \cos \theta_1$, $MP_2 = AM \, . \cos \theta_2$; and consequently $MP^2 + MP_1^2 + MP_2^2 = AM^2 \, (\cos^2 \theta^2 + \cos^2 \theta_1^2 + \cos^2 \theta_2) = AM^2$; or $\cos^2 \theta + \cos^2 \theta_1 + \cos^2 \theta_2 = 1$.

Now if A be the area of the parallelogram $MXYZ$, fig. 49, and θ, θ_1, θ_2, be the angles, which it forms respectively with the three co-ordinate planes; then $A \cos \theta$, $A \cos \theta_1$, $A \cos \theta_2$ will be the areas of its respective projections upon them, and consequently $A = \sqrt{(A^2 \cos^2 \theta^2 + A \cos^2 \theta_1 + A \cos^2 \theta_2)}$, since $\cos^2 \theta + \cos^2 \theta_1 + \cos^2 \theta_2 = 1$.

It is hardly necessary to inform the reader, that when we speak of the square of an area as equal to the sum of the squares of its projections upon the three co-ordinate planes, we have reference, in all cases, to the relation subsisting between the squares of lines or numbers, by which these areas are supposed to be represented.

Note (N).

The classification of differential equations by their order, is founded on the most important of their properties, those relating to the arbitrary constants which complete their integrals. The number of these constants being equal to the exponent of the order (272), affords a natural and strongly-marked line of separation between equations of different orders. The subdivision of equations of the same order into classes, according to the degree to which the dependent variable, or its differential coefficients rise, possesses no such advantage; but, on the contrary, is productive of the utmost confusion. Our author has noticed this circumstance in the preface to his great work on the Differential and Integral Calculus; and takes, as an instance, the equation of Clairaut (270); $y - px = f(p)$, which is integrable by the same process, whatever be the form of the function f. The distribution of equations of

the same order into classes, is a point of very considerable importance, as well as obscurity, since it would furnish us with a general mode of classing transcendents, whose properties are almost universally expressible by means of such equations, in a simple manner (46). An example will illustrate our meaning. The transcendent

$$\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c.$$

which we will denote by $\phi\,(1+x)$, gives, by differentiation,

$$x \cdot \frac{d\phi\,(1+x)}{dx} = \frac{x}{1} - \frac{x^2}{2} + \&c. = \log\,(1+x).$$

whence

$$\phi\,(1+x) = \int \frac{dx}{x} \cdot \log\,(1+x).$$

This equation, therefore, includes all the properties of the above transcendent, and they may all be derived from it. Thus, if for x we write $\frac{1}{x}$, we have

$$\phi\left(1+\frac{1}{x}\right) = \int \frac{-dx}{x}\,(\log\,(1+x) - \log x)\,;$$

whence

$$\phi\,(1+x) + \phi\left(1+\frac{1}{x}\right) = \int \frac{dx}{x} \cdot \log x$$

$$= \tfrac{1}{2}\,(\log x)^2 + C.$$

Now, when $x=1$, this becomes

$$2\,\phi\,(2) = C = 2\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \&c.\right) = \frac{\pi^2}{6}\,;$$

so that

$$\phi\,(1+x) + \phi\left(1+\frac{1}{x}\right) = \tfrac{1}{2}\,(\log x)^2 + \frac{\pi^2}{6}\,,$$

which is one of the principal properties of the function in question. Again, if for x we write $x-1$, we get

$$\phi(x)=\int \frac{dx}{x-1}\log x,$$

and in this, for x writing $\frac{1}{x}$,

$$\phi\left(\frac{1}{x}\right)=\int \frac{dx}{x-x^2}\log x;$$

whence

$$\phi(x)+\phi\left(\frac{1}{x}\right)=\int \frac{dx}{x}\log x$$

$$=\tfrac{1}{2}(\log x)^2+C;$$

and making $x=1$, we find $C=o$; so that

$$\phi(x)+\phi\left(\frac{1}{x}\right)=\tfrac{1}{2}(\log x)^2.$$

Unfortunately, however, such is the difficulty of the subject, no system of classification has been proposed, which will enable us to include in one class all transcendents essentially of the same nature, or reducible to one another, to the exclusion of all such as are fundamentally different. Meanwhile the manner in which the arbitrary constants enter into the integral, affords a convenient method of arranging such equations as are integrable. The equation of the first degree, for instance,

$$\frac{d^n y}{dx^n}+P.\frac{d^{n-1}y}{dx^{n-1}}+\ldots+My+N=o,$$

where $P, Q, \ldots M, N$, are functions of x, has its integral always of the form (284),

$$y=C_1.X_1+C_2.X_2+\ldots C_n.X_n+X.$$

To confine ourselves, however, to the first order, we will first consider the manner in which the arbitrary constant is combined with the variable x, in the integral of any homogeneous equation. The general form of all equations of this class is

$$p=\frac{dy}{dx}=f\left(\frac{y}{x}\right).$$

Thus, the equation $x\,dy - y\,dx = dx\,.\,\sqrt{x^2+y^2}$, in the text (page 323), is the same with

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}.$$

If we take $\frac{y}{x} = u$, we have

$$\frac{dy}{dx} = x\frac{du}{dx} + u,$$

which, substituted in the proposed, gives

$$u + x\frac{du}{dx} = f(u);$$

whence

$$\frac{dx}{x} = \frac{du}{f(u) - u},$$

and

$$\log c\,x = \int \frac{du}{f(u) - u}.$$

Now this, when the integration is performed, being a certain function of u, if we find u from this equation, in terms of $\log c\,x$, we shall have

$$u = \frac{y}{x} = F(\log c\,x),$$

and instead of $\log c$, writing simply c, since the constant is arbitrary,

$$y = x\,.\,F(c + \log x),$$

which is the general form of the integral of every homogeneous equation.

This process extends of course to every conceivable form of the function f, although in the text no examples are given of other than algebraic ones. Suppose, for instance, we had

$$x\,dy - y\,dx = y\,(\log y - \log x)\,dx.$$

This, reduced to the above form, gives

$$\frac{dy}{dx} = \frac{y}{x}\left(1 + \log\frac{y}{x}\right);$$

and consequently

$$f(u) - u = u \,.\, \log u;$$

whence

$$c + \log x = \int \frac{du}{u \,.\, \log u} = \log \log u;$$

whence

$$x \,.\, e^c = \log u, \quad u = e^{x \,.\, e^c}, \quad y = x \,.\, e^{x \,.\, e^c}.$$

If the equation proposed were

$$x y\, dx - y^2\, dx = (x+y)^2\, dx \,.\, e^{-\frac{y}{x}},$$

we should have

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{x}{y} + 2 + \frac{y}{x}\right) e^{-\frac{y}{x}}$$

$$= u + \frac{(1+u)^2}{u} e^{-u};$$

consequently

$$c + \log x = \int \frac{e^u\, du}{(1+u)^2} = \frac{e^u}{1+u} \quad . \quad (191);$$

whence

$$(1+u)(c + \log x) = e^u,$$

or,

$$(x+y)(c + \log x) = x \,.\, e^{\frac{y}{x}}.$$

In this instance, the form of the function we have denoted by F is transcendental. It is the *inverse function* of

$$\frac{e^u}{1+u}, \quad (399).$$

If we eliminate c between the equation

$$y = c x + f(c),$$

and its differential, $\frac{dy}{dx}$, or $p = c$, we find the equation of (270),

$$y = x p + f(p).$$

Had we assumed, for the form of the integral,

$$y=Pc+Q\,.f(c)+R,$$

P, Q, and R being functions of x, we should have obtained the equation

$$o=\frac{(PR'-RP')+(yP'-pP)}{PQ'-QP'}+f\left(\frac{(QR'-RQ')+(yQ'-pQ)}{PQ'-QP'}\right),$$

P', Q', and R' denoting the differential coefficients of P, Q, R. The nature of the solution is not altered by using instead of c any function of c, as for instance, by supposing

$$y=(a+bc)\,x+f(c)$$

$$y=(c^m+c^n)\,x+f(c), \text{ \&c.}$$

for, since c is arbitrary, c^m+c^n may be conceived equivalent to one arbitrary constant c'; and hence, finding c in terms of c', and substituting in $f(c)$, it becomes a function of c', such as $F(c')$, and the form of the integral

$$y=c'x+F(c')$$

differs from the former only in the nature of the function represented by F, and therefore comes under the same class, which we suppose to include all possible forms of the function f.

The next general form of the integral in the order of its simplicity, is

$$y=P+Qc+Rc^2+S\,.f(c),$$

which, however, leads to a very complicated class of differential equations, unless certain relations subsist between P, Q, R, S. One of these is, when $S=1$, and

$$P+Qc+Rc^2=\frac{(x+c)^2}{4a};$$

for, in this case, we have

$$\frac{dy}{dx}=p=\frac{x+c}{2a},\quad c=2ap-x;$$

and consequently

$$y = a p^2 + f(2 a p - x).$$

Let the equation

$$o = 1 + \alpha (y \pm p^2) + \beta (x \pm 2 p) + \gamma (y \pm p^2)(x \pm 2 p)$$

be proposed, and we find

$$y = \mp p^2 - \frac{1 + \beta (x \pm 2 p)}{\alpha + \gamma (x \pm 2 p)},$$

which is of the above form, taking $a = \mp 1$, and assuming for the form of the function f

$$f(x) = - \frac{1 - \beta x}{\alpha - \gamma x}.$$

Its integral is therefore

$$y = \mp \left(\frac{x + c}{2}\right)^2 - \frac{1 - \beta c}{\alpha - \gamma c}.$$

The equation

$$y = P + (c + Q)^n + f(c),$$

treated in the same way, by the elimination of c, leads to the differential equation

$$y = P + \left(\frac{p - P'}{n Q'}\right)^{\frac{n}{n-1}} + f\left\{ \left(\frac{p - P'}{n Q'}\right)^{\frac{1}{n-1}} - Q \right\},$$

P' and Q' denoting the differential coefficients of P and Q. This is an equation of considerable generality. If $Q = \frac{1}{n}$ we have $n Q' = 1$, and taking $n = \frac{m+1}{m}$,

$$y = P + (p - P')^{m+1} + f\left((p - P')^m - \frac{m x}{m+1}\right),$$

the integral being

$$y = P + \left(\frac{m x}{m+1} + c\right)^{\frac{m+1}{m}} + f(c).$$

If, in this, we still farther suppose $P = o$, we shall get

$$y=p^{m+1}+f\left(p^m-\frac{mx}{m+1}\right),$$

an equation generally integrable.

The equation $y=px+f(p)$ is remarkable, as we have before seen (270), for the facility with which its particular solutions present themselves by differentiation: those of various other equations may be obtained in the same manner. Taking, for instance, the equation

$$y=ap^2+f(2ap-x).$$

If we make $2ap-x=u$, we find $y=ap^2+f(u)$, and differentiating,

$$\frac{dy}{dx}=p=2ap\cdot\frac{dp}{dx}+f'(u)\cdot\frac{du}{dx},$$

in which, if for $2a\frac{dp}{dx}$ we write $1+\frac{du}{dx}$, its equal, we get

$$p=p+p\frac{du}{dx}+f'(u)\cdot\frac{du}{dx},$$

or,

$$\frac{du}{dx}(p+f'(u))=0.$$

If we put $\frac{du}{dx}=0$, we get $u=c$, $p=\frac{c+x}{2a}$, and of course $y=\frac{(c+x)^2}{4a}+f(c)$, the complete integral; but if we make the other factor vanish, we have

$$p+f'(u)=0$$
$$y=ap^2+f(u)$$
$$u=2ap-x,$$

from which, eliminating p and u, an equation will result between y and x, differing essentially from the former, and involving no arbitrary constant, and therefore a particular solution. In fact, these equations are equivalent to

$$y=ap^2+f(2ap-x)$$
$$o=p+f'(2ap-x),$$

from which p is to be eliminated. Now, if we differentiate the complete integral, regarding c as variable, we get

$$p=\frac{c+x}{2a}+\frac{dc}{dx}\left(f'(c)+\frac{c+x}{2a}\right),$$

and making

$$f'(c)+\frac{c+x}{2a}=o,$$

we have also

$$p=\frac{c+x}{2a}.$$

Eliminating c from these, we get

$$o=p+f'(2ap-x),$$

which is the same with the second of the above equations, and thus it appears, that the particular solution obtained by differentiation, as above explained, is identical with that resulting from the general theory of such solutions delivered in the text.

Note (O).

In a subject so abstruse as the theory of partial differential equations, the beginner will find great advantage from the actual solution of particular cases, for which reason we subjoin the following:

Ex. 1. $$\frac{dz}{dx}=\frac{dz}{dy}, \text{ or } p=q.$$

Since $dz=pdx+qdy$, and $p=q$, we have

$$dz=pdx+pdy=p.d(x+y),$$

and consequently $p=\phi'(x+y)$, ϕ' denoting the differential coefficient of ϕ, or its *derived function*; whence

$$dz=\phi'(x+y).d(x+y)$$
$$=d\phi(x+y),$$

and

$$z=\phi(x+y).$$

Ex. 2. $$a.\frac{dz}{dx}+b.\frac{dz}{dy}=c, \text{ or } ap+bq=c,$$

If we eliminate q by means of the equation

$$dz=p\,dx+q\,dy,$$

we get

$$dz=\frac{c}{b}dy+p\left(dx-\frac{a}{b}dy\right),$$

or,

$$d\left(z-\frac{cy}{b}\right)=p.d\left(x-\frac{ay}{b}\right),$$

and we must therefore have

$$p=\phi'\left(x-\frac{ay}{b}\right),$$

and

$$d\left(z-\frac{cy}{b}\right)=d\phi\left(x-\frac{ay}{b}\right)$$

$$z=\frac{cy}{b}+\phi\left(x-\frac{ay}{b}\right);$$

which, since the form of the function denoted by ϕ is arbitrary, may also be written thus:

$$z=\frac{cy}{b}+\phi(bx-ay).$$

Ex. 3. $$px-q=x^2, \text{ or}$$

$$x\frac{dz}{dx}-\frac{dz}{dy}=x^2.$$

Since $p=x+\frac{q}{x}$, and also $p=\frac{dz-q\,dy}{dx}$,

we have

$$x\,dx+q\,.\,\frac{dx}{x}=dz-q\,dy;$$

whence

$$d\left(z-\frac{x^2}{2}\right)=q\,(d.\log x+dy);$$

hence $q=\phi'\,(y+\log x)$, and

$$d\left(z-\frac{x^2}{2}\right)=d\,.\,\phi(y+\log x).$$

Ex. 4.

$$\frac{dz}{dx}+\frac{x^2+y^2}{2xy}\,.\,\frac{dz}{dy}=0.$$

Eliminating, as before, p from the equations

$$p+\frac{x^2+y^2}{2xy}\,.\,q=0$$

$$p=\frac{dz-q\,dy}{dx},$$

we get

$$dz=q\,.\left(dy-\frac{x^2+y^2}{2xy}\,dx\right)$$

$$=\frac{x}{2y}\,.\,q\left(\frac{2y\,dy}{x}-\frac{x^2+y^2}{x^2}dx\right)$$

$$=\frac{x}{2y}\,q\,.\,d\left(\frac{y^2-x^2}{x}\right),$$

hence $\frac{x}{2y}\,q$ must be a function of $\frac{y^2-x^2}{x}$, or of the form $\phi'\left(\frac{y^2-x^2}{x}\right)$, and consequently

$$z=\phi\left(\frac{y^2-x^2}{x}\right).$$

Ex. 5.

$$y\frac{dz}{dy}+x\frac{dz}{dx}=\sqrt{x^2+y^2}\,.$$

If we eliminate $\frac{dz}{dy}$, or q, as before, by the equation

$$q=\frac{dz-p\,dx}{dy},$$

we have

$$dz=\frac{dy}{y}\sqrt{x^2+y^2}+p\cdot\frac{y\,dx-x\,dy}{y}.$$

Now if we take a new variable $S=\frac{x}{y}$, we have

$\frac{y\,dx-x\,dy}{y}=y\,dS$, and $x=yS$, which, being substituted, give

$$dz=dy\sqrt{1+S^2}+py\,dS.$$

Now z being a function of y and x, is also a function of y and S, and the second member of this equation must therefore be the complete differential of such a function; this function will therefore be by (261),

$$\int\sqrt{(1+S^2)}\,.\,dy+\phi\,(S)=y\sqrt{(1+S^2)}+\phi\,(S).$$

The value of S being substituted in this, gives, for the integral required,

$$z=\sqrt{(x^2+y^2)}+\phi\left(\frac{x}{y}\right).$$

This method of solution applies equally to all the foregoing examples, and in general, to the equation

$$0=\frac{dz}{dy}+\frac{dz}{dx}\cdot f(x,y)+F(x,y).$$

As another example, we may take

Ex. 6.

$$y^2\frac{dz}{dy}+x^2\frac{dz}{dx}=axy.$$

Eliminating $\frac{dz}{dy}$, or q, we get

$$dz=\frac{ax\,dy}{y}+p\left(dx-\frac{x^2}{y^2}\,dy\right),$$

$$dz=\frac{a x\,dy}{y}+p x^2\left(\frac{dx}{x^2}-\frac{dy}{y^2}\right).$$

If now we take $\frac{dx}{x^2}-\frac{dy}{y^2}=dS$, or $S=\frac{1}{y}-\frac{1}{x}$, we get $x=\frac{y}{1-yS}$, and substituting

$$dz=\frac{a\,dy}{1-Sy}+p\frac{y^2}{(1-Sy)^2}dS,$$

which must be a complete differential of some function of y and S. This function will therefore be

$$\int\frac{a\,dy}{1-Sy}+\phi(S)=-\frac{a}{S}\log(1-Sy)+\phi(S),$$

and

$$z=\frac{axy}{x-y}.\log\left(\frac{x}{y}\right)+\phi\left(\frac{x-y}{xy}\right).$$

Ex. 7.

$$x\frac{dz}{dy}+y\frac{dz}{dx}=az.$$

Assume $z=e^u$, e being the number whose hyperbolic or natural logarithm, is unity, and we have

$$\frac{dz}{dy}=e^u.\frac{du}{dy},\ \frac{dz}{dx}=e^u.\frac{du}{dx},$$

by substituting which, the whole becomes divisible by e^u, and we get, for determining u, the equation

$$x\frac{du}{dy}+y\frac{du}{dx}=a,$$

which is integrable by the method pursued in the last two examples; thus, eliminating $\frac{du}{dy}$, by means of the equation,

$$\frac{du}{dy}=\frac{du-\frac{du}{dx}dx}{dy},$$

we find

$$du = \frac{a\,dy}{x} + \frac{du}{dx} \cdot \frac{1}{x}\,(x\,dx - y\,dy).$$

If now we take $x\,dx - y\,dy = dS$, and $S = \frac{x^2 - y^2}{2}$, we have $x = \sqrt{(2S + y^2)}$, and

$$u = \int \frac{a\,dy}{\sqrt{(2S + y^2)}} + \phi(S) = a \,.\, \log(y + \sqrt{(2S + y^2)}) + \phi(S)$$

$$= a \,.\, \log(x + y) + \phi(x^2 - y^2),$$

writing $\phi(2S)$ instead of $\phi(S)$, since the form of ϕ is arbitrary. Thus, since $z = e^u$, we obtain

$$z = (x + y)^a \,.\, e^{\phi(x^2 - y^2)},$$

or changing again the form of $\phi(x^2 - y^2)$ to $\log \phi(x^2 - y^2)$, which is allowed, for the reason before noticed,

$$z = (x + y)^a \,.\, \phi(x^2 - y^2).$$

The reader may apply the same process to the equation

Ex. 8.

$$(\alpha x + \beta x)\frac{dz}{dy} + (\gamma x + \delta y)\frac{dz}{dx} = \iota z,$$

which leads to the following value of z:

$$z = V^{\epsilon} \cdot \phi\left(\frac{\alpha x^2 + (\beta - \gamma)\,xy - \delta y^2}{V^{\beta + \gamma}}\right),$$

where

$$\log V = \int \frac{d \,.\, \left(\frac{y}{x}\right)}{\alpha + (\beta - \gamma)\left(\frac{y}{x}\right) - \delta \,.\, \left(\frac{y}{x}\right)^2},$$

and the same method will suffice for the integration of the more general equation,

$$0 = \frac{dz}{dy} + \frac{dz}{dx} \cdot f(x, y) + z \,.\, F(x, y).$$

Ex. 9.

$$x^m z^n = \frac{dz}{dx} + \frac{y}{x} \cdot \frac{dz}{dy}$$

Let z be regarded as a function of x, and a new variable u, u being a function of x and y, to be found, and let $\left(\frac{dz}{dx}\right)$ represent the differential coefficient of z, relative to x, on this supposition. (This will not be the same with $\frac{dz}{dx}$, for, u being a function of both x and y, the composition of z, when expressed in terms of u and x, will differ from that of the same function expressed in terms of y and x, and x will be involved in a different manner in the two expressions). We have then

$$\frac{dz}{dy} = \frac{dz}{du}\,\frac{du}{dy}$$

$$\frac{dz}{dx} = \left(\frac{dz}{dx}\right) + \frac{dz}{du}\,\frac{du}{dx},$$

which being substituted, we find

$$x^m z^n = \left(\frac{dz}{dx}\right) + \frac{dz}{du}\left(\frac{du}{dx} + \frac{y}{x}\,\frac{du}{dy}\right).$$

Suppose now u so determined, that

$$\frac{du}{dx} + \frac{y}{x}\,\frac{du}{dy} = 0,$$

which, treated in the same manner as the equation of Ex. 4, gives

$$u = \varphi\left(\frac{y}{x}\right).$$

The equation is now reduced to the more simple form

$$\left(\frac{dz}{dx}\right) = x^m z^n,$$

in which it must be recollected, that z is considered as a

function of u and x; and since $\frac{dz}{du}$ does not enter into it, u must be regarded as constant, and we have

$$\int z^{-n} dz = \int x^m dx,$$

or,

$$\frac{z^{-n+1}}{-n+1} = \frac{x^{m+1}}{m+1} + F(u)$$

$$= \frac{x^{m+1}}{m+1} + F\varphi\left(\frac{y}{x}\right);$$

but F and φ denoting arbitrary functions, their combination is equally so, and may therefore be represented by one character, φ, so that

$$\frac{x^{m+1}}{m+1} + \phi\left(\frac{y}{x}\right) = \frac{1}{(1-n)z^{n-1}}.$$

This very elegant process, delivered by Laplace, in the *Memoirs of the Academy of Sciences*, for 1773, is equally applicable to the equation

$$0 = \frac{dz}{dx} + \frac{dz}{dy} \cdot f(x, y) + F(x, y, z).$$

When the term, independent of the differential coefficients, contains y as well as x, the operation will require one additional step, which was not necessary in the above example: it consists in substituting for y its value, in terms of u and x, derived from the general expression of u, so as to arrive at a final equation between z, u, and x alone. Thus, suppose we have

Ex. 10.

$$y^2 \frac{dz}{dy} + yx\frac{dz}{dx} + xz = axy\sqrt{x^2+y^2}.$$

If this equation be treated in the same way as the last, we shall arrive at the same form of u,

$$u = \phi\left(\frac{y}{x}\right).$$

Suppose this equation resolved, and a value of $\frac{y}{x}$ deduced from it, in terms of u, we may denote this by $\psi(u)$, and we have

$$y = x \, . \, \psi(u),$$

and it is evident, that the form of the function ψ is equally arbitrary with that of ϕ: we shall then get

$$\left(\frac{dz}{dx}\right) = a\sqrt{x^2+y^2} - \frac{z}{y}$$

$$= a\,x\sqrt{1+\psi(u)^2} - \frac{z}{x\,.\,\psi(u)},$$

which, since $d\,u$ does not enter into it, may be treated as an equation of total differentials, between z and x, of the first order and degree (257), and gives

$$z = e^{-\int\frac{dx}{x\,.\,\psi(u)}}\left\{F(u) + \int e^{\int\frac{dx}{x\,\psi(u)}}\,a\,x\,d\,x\sqrt{1+\psi(u)^2}\right\},$$

since the constant may be any function of u, which is regarded as constant in the integration. Hence performing the integrations indicated, we get

$$z = x^{-\frac{1}{\psi(u)}} \, . \, F(u) + a\,x^2 \, . \, \frac{\psi(u)\sqrt{1+\psi(u)^2}}{1+2\,\psi(u)},$$

in which, if we replace $\psi(u)$ by its equal $\frac{y}{x}$, and instead of the arbitrary function $F(u)$, or $F\phi\left(\frac{y}{x}\right)$, write simply $\phi\left(\frac{y}{x}\right)$, we find at length

$$z = x^{-\frac{x}{y}} \, . \, \phi\left(\frac{y}{x}\right) + \frac{a\,x\,y\sqrt{x^2+y^2}}{x+2\,y}.$$

Ex 11.

$$\frac{dz}{dx}\cdot\frac{dz}{dy}=1, \text{ or } pq=1,$$

we have

$$dz=p\,dx+q\,dy=d\,(px+qy)-(x\,dp+y\,dq),$$

and therefore

$$d(px+qy-z)=x\,dp+y\,dq$$

$$=x\,dp-y\,\frac{dp}{p^2}$$

$$=\left(x-\frac{y}{p^2}\right)dp,$$

since $q=\frac{1}{p}$, and $dq=-\frac{dp}{p^2}$.

Now, the first member being an exact differential, the second must be so likewise; so that we must have

$$x-\frac{y}{p^2}=\phi'(p),$$

and $$px+qy-z=\phi(p),$$

that is, $$px+\frac{y}{p}-z=\phi(p);$$

by the help of these two equations, assigning any form we please to ϕ, we may eliminate p, and an equation will result, expressing the relation between z, x, and y: thus, if we take $\phi(p)=0$, $\phi'(p)=0$,

$$x=\frac{y}{p^2}, \quad z=px+\frac{y}{p};$$

whence,

$$z=(1+x^2)\sqrt{\frac{y}{x}}$$

Ex. 12.

$$\frac{dz}{dx}\cdot\frac{dz}{dy}=ax+b\left(\frac{dz}{dx}\right)^2,$$

or,
$$q = a\frac{x}{p} + bp;$$
whence
$$dz = p\,dx + q\,dy = d(px) - x\,dp + q\,dy$$
$$= d(px) - \frac{pq - bp^2}{a}\,dp + q\,dy,$$
and
$$dz - d(px) - \frac{b}{a}p^2dp = q\left(dy - \frac{p\,dp}{a}\right).$$
The first member is the complete differential of
$$z - px - \frac{b}{3a}p^3,$$
so that we must have
$$q = \phi'\left(y - \frac{p^2}{2a}\right),$$
and
$$z - px - \frac{bp^3}{3a} = \phi\left(y - \frac{p^2}{2a}\right);$$
but the proposed equation gives
$$q, \text{ or } \phi'\left(y - \frac{p^2}{2a}\right) = \frac{ax}{p} + bp,$$
eliminating p from these two equations, we obtain one expressing the relation between z, x, and y.

Ex. 13. Let us take the equation of the second order,
$$0 = z\frac{d^2z}{dx^2} - \left(\frac{dz}{dx}\right)^2 + a\left\{z\frac{d^2z}{dx\,dy} - \frac{dz}{dx}\frac{dz}{dy}\right\}$$
$$+ b\left\{z\frac{d^2z}{dy^2} - \left(\frac{dz}{dy}\right)^2\right\}$$
If we assume, as in Ex. 7, $z = e^u$, we shall have
$$\frac{dz}{dx} = e^u\frac{du}{dx}, \quad \frac{dz}{dy} = e^u\frac{du}{dy}$$
$$\frac{d^2z}{dx^2} = e^u\left(\frac{du}{dx}\right)^2 + e^u\frac{d^2u}{dx^2}$$

$$\frac{d^2 z}{dx\,dy}=e^u\frac{du}{dx}\,\frac{du}{dy}+e^u\frac{d^2 u}{dx\,dy}$$

$$\frac{d^2 z}{dy^2}=e^u\left(\frac{du}{dy}\right)^2+e^u\frac{d^2 u}{dy^2},$$

so that by this substitution our equation becomes

$$e^{2u}\left\{\frac{d^2 u}{dx^2}+a\,\frac{d^2 u}{dx\,dy}+b\frac{d^2 u}{dy^2}\right\}=0,$$

which, divided by e^{2u}, takes the same form as that of No. 319, where $A=1$, $B=a$, $C=b$, $V=0$; and gives, by the application of the process there explained,

$$u=\log\phi(\alpha x-y)+\log\psi(\beta x-y),$$

α and β being the two roots of

$$\alpha^2-a\alpha+b=0,$$

provided these be unequal. Thus we find

$$z=e^u=\phi(\alpha x-y)\,.\,\psi(\beta x-y).$$

NOTE (P).

The process delivered in the text is manifestly applicable to the determination of the arbitrary functions, whatever be their number, provided they enter under the same form as in the equation

$$1=M\,.\,\phi(V)+N\,.\,\psi(V)+O\,.\,\chi(V)+\text{\&c.}$$

in which M, N, O, ... V, represent any functions of x, y, z, the conditions being similar to those in the text. Instances, however, are frequent where the arbitrary functions enter into the integrals of equations under different forms, or combined with their differential coefficients, in which cases the task of determining them is of much greater difficulty. The case where one function only is to be determined, but which enters into the equations for determining it, under

two forms, should such occur, must be treated in the manner explained in (398), but most frequently we have to determine more than one arbitrary function, by elimination from equations into which each enters, under its own peculiar form; and in this enquiry great and unexpected difficulties are found to occur. The following cases, however, are sufficiently easy.

1. To determine the arbitrary functions φ and ψ, when

$$z=\varphi(U)+\psi(V),$$

U and V being functions of x, y, z, the conditions being that

$$F(x, y, z)=0, \text{ gives } f(x, y, z)=0,$$

and $$F'(x, y, z)=0, \text{ gives } f'(x, y, z)=0.$$

Make $U=t$, and from the three equations

$$U=t, \quad F(x, y, z)=0, \quad f(x, y, z)=0,$$

derive the values of x, y, z, in functions of t. If these be substituted in V, it will become a function of t, which we will call V', and denoting by Z' the value of z in terms of t, we have

$$Z'=\varphi(t)+\psi(V').$$

In like manner, if we combine the equation $U=t$ with the other conditions,

$$F'(x, y, z)=0, \quad f'(x, y, z)=0,$$

we shall get

$$z=Z'', \quad V=V'',$$

Z'' and V'' being certain other functions of t, and

$$Z''=\varphi(t)+\psi(V'').$$

Subtracting this from the former, we get

$$Z''-Z'=\psi(V'')-\psi(V'),$$

in which there is but one unknown function entering under two forms, and which may therefore be treated by Laplace's method (398).

Ex. Let $z=\phi(\alpha x-y)+\psi(\beta x-y)$.

Required the forms of ϕ and ψ, so that when

$y=Ax$, $z=Bx$, and when $y=ax$, $z=bx$.

1. Put $\alpha x-y=t$, $y=Ax$, $z=Bz$, and we have

$$x=\frac{t}{\alpha-A},\quad y=\frac{At}{\alpha-A},\quad z=\frac{Bt}{\alpha-A}.$$

$$\frac{Bt}{\alpha-A}=\phi(t)+\psi\left\{\frac{\beta-A}{\alpha-A}t\right\}.$$

In like manner, from the other condition we get

$$\frac{bt}{\alpha-a}=\phi(t)+\psi\left\{\frac{\beta-a}{\alpha-a}t\right\},$$

whence, subtracting

$$\psi\left\{\frac{\beta-A}{\alpha-A}t\right\}-\psi\left\{\frac{\beta-a}{\alpha-a}t\right\}=\left\{\frac{B}{\alpha-A}-\frac{b}{\alpha-a}\right\}t=mt;\ldots(a)$$

If now we take

$$\frac{\beta-A}{\alpha-A}t=u_{n+1},\quad \frac{\beta-a}{\alpha-a}t=u_n,$$

we have

$$\frac{u_{n+1}}{u_n}=\frac{\beta-A}{\alpha-A}\cdot\frac{\alpha-a}{\beta-a}=k,$$

and integrating (381),

$$u_n=Ck^n;$$

but, the equation (a) gives

$$\psi(u_{n+1})-\psi(u_n)=\Delta\psi u_n=mt$$

$$=m\cdot\frac{\alpha-a}{\beta-a}u_n=m\cdot\frac{\alpha-a}{\beta-a}\cdot Ck^n;$$

whence, integrating,

$$\psi(u_n)=\frac{m(\alpha-a)}{\beta-a}\,\frac{Ck^n}{k-1}+C'$$

$$=\frac{B(\alpha-a)-b(\alpha-A)}{(\beta-A)(\alpha-a)-(\beta-a)(\alpha-A)}u_n+C',$$

which expresses the form of ψ, and in like manner may that of ϕ be determined.

2. The equation

$$1 = P \cdot \phi(U) + Q \cdot \psi(V),$$

where P, Q, U, V, are any given functions of x, y, z, and ϕ, ψ are to be determined by conditions similar to the foregoing, presents no difficulties of a different kind from those of the more simple case just treated. We have only to make

$$U = t, \quad F(x, y, z) = 0, \quad f(x, y, z) = 0,$$

and P, Q, V, become functions of t, which we will call P', Q', V', and thus we have

$$1 = P'\phi(t) + Q'\psi(V').$$

In like manner, the other conditions afford a similar equation

$$1 = P''\phi(t) + Q''\psi(V''),$$

and if $\phi(t)$ be eliminated from these, the resulting equation suffices to determine the form of ψ.

3. Let $$z = \phi\{U + \psi(V)\},$$

the conditions being as before. This case, in reality, differs only in appearance from the last; for if we conceive the inverse operation of ϕ (or that which exactly counteracts the operation denoted by ϕ) to be represented by ϕ_1, we shall have

$$\phi_1\,\phi(t) = t;$$

taking therefore the function ϕ_1 of both members of the proposed equation, we have

$$\phi_1(z) = \phi_1\,\phi\{U + \psi(V)\}$$
$$= U + \psi(V),$$

and

$$1 = \frac{1}{U}\,\phi_1(z) - \frac{1}{U}\,\psi(V),$$

which is of the form before treated. Let the functions ϕ_2

and ψ be determined, and ϕ becomes known by resolving the equation

$$\phi_1\{\phi(t)\}=t.$$

The reader may apply this general reasoning to the particular equation,

$$z=\phi\{ax-y+\psi(bx-y)\},$$

the conditions being, that when $y=Ax$, $z=Bx$; and when $y=Cx$, $z=Dx$.

The above is the integral of the following partial differential equation,

$$\frac{d^2z}{dx^2}\cdot\frac{dz}{dy}-\frac{dz}{dx}\cdot\frac{d^2z}{dx\,dy}+b\left\{\frac{dz}{dy}\cdot\frac{d^2y}{dx\,dy}-\frac{dz}{dx}\cdot\frac{d^2z}{dy^2}\right\}=0.$$

into which it is worthy of remark, that the constant a does not enter, and therefore its integral, besides the two arbitrary functions ψ and ϕ, contains an arbitrary constant a, which can only be determined by assigning some particular value of z, corresponding to certain given values of x and y. In other words, determining some point through which the curve surface, expressed by the equation, must pass.

If there be more than two functions, as in the equation

$$1=M\,.\,\phi(U)+N\,.\,\psi(V)+O\,.\,\chi(W),$$

we must have as many conditions as there are functions, and from these, in the same manner as above, we obtain the equations

$$1=M'\,.\,\phi(t)+N'\,.\,\psi(V')+O'\,.\,\chi(W')$$
$$1=M''\,.\,\phi(t)+N''\,.\,\psi(V'')+O''\,.\,\chi(W'')$$
$$1=M'''\,.\,\phi(t)+N'''\,.\,\psi(V''')+O'''\,.\,\chi(W'''),$$

from which it does not appear practicable to eliminate at once $\phi(t)$, and the three forms of χ, viz. $\chi(W')$, $\chi(W'')$, and $\chi(W''')$, so as to arrive at a final equation, containing no other unknown function than ψ, and accordingly the extent of the process seems limited at this point, by an insurmountable obstacle.

NOTE (Q).

The proposition contained in this article is too important to be passed over without demonstration, as it is extremely easy of proof, and not only the whole theory of relative *maxima* and *minima* depends on it, but the reader will find the same principle employed in almost every other application of the Calculus of Variations, and in none more frequently than in the investigation of the general laws of mechanical action and equilibrium.

In any equation

$$P+Q\,a+R\,b+\text{\&c.}$$

if the quantities $a, b, c, \ldots$ be supposed absolutely indeterminate and arbitrary, having no necessary dependence whatever upon any quantities, such as x, y, z, p, q, &c. of which P, Q, R, &c. may happen to be composed, then this equation is in fact equivalent to the several equations,

$$P=0, \quad Q=0, \quad R=0, \text{ \&c.}$$

for, since a, b, c, &c. are independent of x, y, &c. any change we may make in the former can produce no alteration in the values of the latter, or the relations subsisting between them; consequently any number of equations,

$$P+Q\,a+R\,b+\text{\&c.}=0,$$

$$P+Q\,a'+R\,b'+\text{\&c.}=0,$$

$$P+Q\,a''+R\,b''+\text{\&c.}=0, \text{ \&c.}$$

must hold good at once, a, a', a'', b, b', b'', &c. being indeterminate and independent. Suppose the number of these equations equal to that of the quantities P, Q, &c. and by eliminating all but one of them (suppose P), we arrive at an equation of the form

$$P.f(a, a', b, b', \ldots)=0,$$

in which, since the factor $f(a,$ &c.) may have any indeterminate value depending on those of $a, b,$ &c. we must have $P=0$; and in like manner it may be shewn, that $Q=0$, &c. The same conclusion may be at once obtained by considering, that if the equation

$$P+Qa+Rb+\&c.=0$$

be not identically true, by the vanishing of each separate coefficient $P, Q,$ &c. it is in fact the expression of a relation subsisting between $a, b,$ &c. and $P, Q, R,$ &c. that is, $a, b,$ &c. are not, as was supposed, independent of $x, y, z,$ &c.

Hence it appears, that if any equations

$$P=0, \quad Q=0, \quad R=0, \quad \&c.$$

be proposed, we may include them all in one, by adding to any one of them the sum of the rest, each multiplied by an indeterminate quantity independent on $x, y, z,$ &c. Suppose now,

$$P=\int\delta U, \quad Q=\int\delta U_1, \quad R=\int\delta U_2, \ \&c.$$

then will the equation

$$0=\int\delta(U+aU_1+bU_2+\&c.)$$

be equivalent to all the separate equations,

$$\int\delta U=0, \quad \int\delta U_1=0, \ \&c.$$

provided $a, b,$ &c. be perfectly arbitrary and independent on $x, y, z,$ that is, provided these quantities do not vary by the variation of x, y, z; whence it follows, that $da=0$, $\delta a=0$, or $a, b,$ &c. are to be regarded as arbitrary constants in the final result. Now, if $\int U$ is to be a *maximum* or *minimum*, we have $\int\delta U=0$; and if at the same time $\int U_1$ taken between the same limits, is restricted to a given value, its variation is zero, or $\int\delta U_1=0$, and so on; and from what has been said, it appears that

$$0=\int\delta(U+aU_1+bU_2+\&c.)$$

is equivalent to all these equations, and therefore embraces all the conditions of the problem.

Suppose $a=\frac{\delta y}{\delta x}$, $b=\frac{\delta z}{\delta x}$, &c. the variations δx, δy, δz, &c. being arbitrary and independent, then will the equation

$$P\delta x+Q\delta y+R\delta z+\&c.=0$$

be equivalent to all the separate equations

$$P=0,\quad Q=0,\quad R=0,\quad \&c.$$

but if any relations subsist between δx, δy, δz, &c. and, by means of these relations, and the above equation, so many of the variations δx, &c. be eliminated, those which remain may be regarded as arbitrary and independent, and their coefficients in the resulting equation must vanish. The proposition in this form, simple as it appears, is the foundation of the whole analytical part of the theory of Statics and Dynamics.

We have considered it necessary to explain this point at some length, because we have known it regarded as a difficulty in principle, while in reality it is nothing more than an artifice of analysis, enabling us to dispense with the consideration of *relative*, and confine our views solely to questions of *absolute maxima and minima*.*

For examples of the application of the method of variations to the solution of problems of *maxima* and *minima*,

---

* This demonstration of the theory of relative *maxima* and *minima* is the same in spirit with that given by Lagrange, in the 22d Lesson of the *Calcul des Fonctions*, and seems more simple, as well as more natural than the proof previously delivered by Euler, by resolving each integral into its component infinitely small elements.

the reader is referred to Mr. Woodhouse's *Treatise on Isoperimetrical Problems*, in which he will meet with a great variety of very select and interesting cases, as well as a more extended account of the method itself, than the necessary limits of this work will admit.

That the equation

$$d\,x'' + p''\,d\,z'' + (d\,y'' + q''\,d\,z'')\,n'' = 0,$$

of (337), expresses the condition of the two curves intersecting each other at right angles, may be shewn as follows. TM, fig. 33, being a tangent to any curve of double curvature $M\,X$, the trigonometrical tangent of the angle TMM' will be $\frac{TM'}{M\,M'} = \frac{\sqrt{d\,x^2 + d\,y^2}}{d\,z}$, or if we represent by $d\,z = p\,d\,x + q\,d\,y$, the differential equation of the surface, in which both curves lie, and by $d\,y = n\,d\,x$, the equation of the projection $M'X'$, $\frac{T\,M'}{MM'} = \frac{\sqrt{1+n^2}}{p+q\,n}$; the sine and cosine of the angle $TM'P$ will also be respectively represented by

$$\frac{d\,x}{\sqrt{dx^2+dy^2}} = \frac{1}{\sqrt{1+n^2}}, \text{ and } \frac{d\,y}{\sqrt{d\,x^2+d\,y^2}} = \frac{n}{\sqrt{1+n^2}}.$$

If we now conceive another curve lying on the same surface, having MT_1' for its tangent,* and represented by two equations between x, y_1, and z_1; one of these will be the equation of the surface

$$d\,z_1 = p_1\,d\,x + q_1\,d\,y_1,$$

p_1 and q_1 being the same functions of x and y_1 that p and q

* The lines MT_1', &c. relative to this second curve, are not represented in the figure, as they are easy of conception.

are of x and y, and therefore at the point M, common to both curves, $y_1=y$, $z_1=z$, $p_1=p$, $q_1=q$. Let $n_1\,dx=dy_1$ be the equation of the projection MX_1' of this curve on the plane of the x and y, and we shall have, in like manner,

$$\tan T_1'MM'=\frac{\sqrt{1+n_1^2}}{p+q\,n_1},\quad \cos T_1'M'P=\frac{n_1}{\sqrt{1+n_1^2}},$$

and

$$\sin T_1'M'P=\frac{1}{\sqrt{1+n_1^2}},\ \text{whence}$$

$$\cos T'M'T_1'=\cos\,(T'M'P-T_1'M'P)=\frac{n\,n_1+1}{\sqrt{1+n^2}\,.\,\sqrt{1+n_1^2}}.$$

Now, if we conceive a sphere, whose center is M, and radius unity, the intersections of its surface with the three planes $T'MM'$, $T_1'MM'$, $T'MT_1'$, will form a spherical triangle, of which if we call the sides (in the above order) a, b, c, and the opposite angles A, B, C, we have

$$a=\angle T'MM',\ b=\angle T_1'MM',\ c=\angle T'MT_1'=90^\circ;$$

also the angle C measures the inclination of the planes $T'MM'$ and $T_1'MM'$ to each other, and is therefore equal to the angle $T'MT_1'$. Now (Woodhouse's Trigonometry, 1st edit. p. 100),

$$\cos C=\frac{\cos c-\cos a\,.\,\cos b}{\sin a\,.\,\sin b},$$

which, since $\cos c=0$, gives

$$0=1+\tan a\,.\,\tan b\,.\,\cos C,$$

and substituting for these quantities their values, above found,

$$0=1+\frac{n\,n_1+1}{(p+q\,n_1)\,(p+qn)},$$

or, writing for $p+q\,m$ its value, $\frac{dz}{dx}$, and for n, $\frac{dy}{dx}$;

$$o=1+\frac{dz}{dx}\cdot p+n_1\left(q\frac{dz}{dx}+\frac{dy}{dx}\right);$$

whence

$$dx+p\,dz+(dy+q\,dz)\,.\,n_1=o,$$

which (making the requisite change in the notation) is manifestly the equation in question.

ERRATA.

The most important of the Errata are marked with an Asterisk.

ge	Line	Error.	Correction.	
	25	$v' - u$ *in denominator*	$v' - v$	
	10	$du + du - dw$	$du + dv - dw$	
	21	$\frac{a}{dx}$	$v\frac{a}{dx}$	*
	17	$\frac{-d.x}{x^{2n}}$	$\frac{-d.x^n}{x^{2n}}$	
	16		*dele* (7)	
	12	$\frac{h^3}{1.2}$	$\frac{h^3}{1.2.3}$	
	24	A + &c.	$x = A$ + &c.	
	7	$n = 1. z = 1$	$n = 1$ and $z = 1$	
	14	$l\left(\frac{y}{2}\right)$	$l\left(\frac{y}{z}\right)$	
	21	$-\sqrt{1-x}$	$+\sqrt{1-x}$	*
	13	$\sqrt{1+x^2}$ *in denominator*	$\sqrt{1-x^2}$	*
	14	$du = dx\sqrt{1-u^2}$	$du = -dx\sqrt{1-u^2}$	
	17	$du = dy$	$du = -dy$	
	19	$d =$	$dx =$	
	9	$\frac{d^3y}{dx^2}$	$\frac{d^3y}{dx^3}$	
	12	$2xydx$	$2xydy$	*
	8	Dx^4	$D.x^3$	

Page	Line	Error.	Correction.
57	7	$\frac{d^2 y}{d x^2} \frac{h^2}{1.2} \quad \frac{d^3 y}{d x^3}$	$\frac{d^2 y}{d x^2} \frac{h^2}{1.2} - \frac{d^3 y}{d x^3}$
58	2	$\frac{d_3 y}{d x^3}$	$\frac{d^3 y}{d x^3}$
—	24	the one o	the one or
62	12	$-x^3$ *in the numerator*	$-2x^3$
67	8	a	b
69	3	$(x-a)^{\frac{3}{2}}$ *in the numerator*	$(x^2-a^2)^{\frac{3}{2}}$
73	25	functions ;	functions,
79	34	$\overline{PM}$	$\overline{PM^2}$
88	2	$h\sqrt{1+p^2} - Ph$	$h\sqrt{1+p^2} - Ph^2$
—	14	PP *in denominator*	PP'
101	19	not, therefore, follow	not therefore follow
103	3	succeed	precede
134	16	$T' = M\sqrt{\overline{AM}^2 + \overline{AT}^2}$	$TM = \sqrt{\overline{AM}^2 + \overline{AT}^2}$
140	3	of x	of t
141	1	$-3\, dx\, d^2 y$	$-3\, dx\, d^2 x\, d^2 y$
149	10	$(x+h,\ y+k)$	$f(x+h,\ y+k)$
165	6	$\frac{du}{dy} = 0.$	$\frac{du}{dx} = 0.$
180	19		dele index
185	3	$C x_q dx$	$C x^q dx$
187	10	A'	A
—	24	$1\{(x+a)N(x+a')N'(x+a'')N''\}$	$1\{(x+a)^N(x+a')^{N'}(x+a'')$
188	14	$+T$	$+T'$
189	3	$z^{-}p+1$	z^{-p+1}
193	19	$\int^{\infty} \frac{dz}{z^2+\beta^2)^{q-3}}$	$\int^{\infty} \frac{dz}{(z^2+\beta^2)^{q-3}}$
200	21	$x = 0$	$R = 0$
208	4	$\sqrt{\alpha + \beta x + \gamma x^2}$	$\gamma\sqrt{\alpha + \beta x + \gamma x^2}$
213	6	$\cos z \pm \sqrt{-1}\sin z)^n$	$(\cos z \pm \sqrt{-1}\sin z)^n$
214	8	ea chof	each of
218	14	$\pm\sqrt{-1}\sin^{5\pi}_{6}$	$\pm\sqrt{-1}\sin\frac{5\pi}{6}$
219	28	composing	comparing

ge	Line	Error.	Correction.
0	3	sin $(m\pi + \delta)$	sin $(2m\pi + \delta)$
1	16	$x^n = \frac{z_q - a}{b}$	$x^n = \frac{z^q - a}{b}$
2	2	$(a + bx^n)^{\frac{p}{q}}$	$(a + bx^n)^{\frac{p}{q}}$
3	2	one	we
-	22	number,	number;
4	20	$(b + ax^{-n})^{\frac{np}{q}}$	$(b + ax^{-n})^{\frac{p}{q}}$
-	21	$x^{m + np - 1}dx$	$x^{m + \frac{np}{q} - 1}dx$
6	15	$-\frac{1}{2}\frac{x^3}{5}$	$-\frac{1}{2}\frac{x^3}{3}$
-	16	$-\frac{1}{1 \cdot 2x^2}$	$-\frac{1}{1 \cdot 2 \cdot 2 \cdot x^4}$
7	5	$2u^{\frac{1}{2}} =$	$2u^{\frac{1}{2}} -$
3	7	$-A$	$-\frac{1}{2}A$
-	9	$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} = A$	$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}A$
)	18	$+\frac{1}{(m+1)^2}$	$+\frac{1 \cdot 2}{(m+1)^3}$
(	4	in *infinitum* by	in *infinitum*; by
.	6	$+\frac{1 \cdot 2}{2(m+1)(l\,x)}$	$+\frac{1 \cdot 1}{2(m+1)(l\,x)^{\frac{3}{2}}}$
!	16	$m = 1$	$m = -1$
;	12	$dz = e^u$	$dz = e^u du$
.	20	$a^x = a$	$a^x = u$
	21	preceding will	preceding Article will
;	9	sines of	sines and cosines of
.	10	the last	the first and last
	17	$(u^{n-3} + v^{n-6})$	$(u^{n-6} + v^{n-6})$
.	18	the question	the equation
	2	cos x	cos nx
	21	would	should
	24	compliment	complement

Page	Line	Error.	Correction.
264	16	$+\int dz \sin z^2 \cos z^2$	$+\frac{1}{4}\int dz \sin z^2 \cos z^0$
274	18	$+A^n{}_{-1}$	A'_{n-1}
—	25	$+A^n{}_{-1}\alpha$	$A'_{n-1}\alpha$
277	11	principle	principal
—	14	and be	and A be
279	15	reciprocally	precisely
280	5	$\int \frac{dx}{\sqrt{1=x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}}$
—	12	$2\sqrt{1-2}$	$2\sqrt{1-x}$
281	15	$x = a - \delta$, the	$x = a - \delta$. The
282	3	$e^{-\frac{1}{z}}$	$\int e^{-\frac{1}{z}} dx$
283	1	e^{-1}	$e^{-\frac{n}{1}}$
—	13	in several	in several parts
285	24	$x\int X dy$	$x\int X dx$
288	14	curvature	cubature
289	8	forms	form
291	19	abscissa $AB = b$	abscissa $AP = b$
293	11	for x	for $1\,x$
—	32	$P, M,$	$P' M'$
294	—		dele · See Note (L)
295	21	or	as
298	28	a constant	no constant
300	4	and, we	and we
301	1	case in	case as in
302	19	$\frac{1}{2p}(1\,2px + \sqrt{1+4p^2x^2})$	$\frac{1}{2p}\,1\,(2px + \sqrt{1+4p^2x^2})$
—	21	$\frac{1}{2p}\,1\,(2px + \sqrt{1+4p^2x^2})$	$\frac{1}{4p}\,1(2px + \sqrt{1+4p^2x^2})$
303	17	$-\frac{1}{2}e^2$	$-\frac{1\,.\,1}{2\,.\,2}e^2$
305	13	an infinite	a finite
306	13	united	unity
307	26	Element	Elements
308	9	$\sqrt{a^4 - (a^2 - b)x^2}$	$\sqrt{a^4 - (a^2 - b^2)x^2}$

Page	Line	Error.	Correction.	
308	20	$M'M + z$	$M'M = z$	
311	23	to $\acute{x}$; and	to x and	
312	1	in series	in a series	
——	5	being	may be	
313	9	connected	contained	
314	10	$\frac{\pi}{4}\left(r^2 - \frac{y^3}{3}\right)$	$\frac{\pi}{4}\left(r^2 y - \frac{y^3}{3}\right)$	*
316	23	observe the	observe that the	
317	Note	Géometre	Géométrie	
318	30	representing	including	*
321	21	polynomist	polynomial	
323	20	$\frac{dz}{x}$	$\frac{dx}{x}$	
328	23	$z = \frac{1}{y}$	$z = \frac{1}{y'}$	
341	24	commonly	only	*
347	8	resolvible	resolvable	
349	4	xpd	xdp	
——	12	$y =$	$p =$	*
353	25	$\int \frac{dy}{Q}$	$\int \frac{dq}{Q}$	
361	8	is	is not	*
——	15	made	make	
——	24	$e^{\int dtx} t$	$e^{\int tdx} t$	
——	26	$e^{\int dtx}$	$e^{\int tdx}$	
385	12	of abscissæ	of the abscissæ	
386	23	z	x	
——	30	$n\sqrt{dx^2 + dy^2} + dy^2$	$n\sqrt{dx^2 + dy^2}$	*
390	14	this	an	
——	20	$\frac{dC}{xc}$	$\frac{dC}{dc}$	
391	20	$= \frac{dy\,du}{\sqrt{u}} - 0$	$- \frac{dy\,du}{\sqrt{u}} = 0$	*
393	13	$(C = C')^{\mu}$	$(C - C')^{\mu}$	
403	15		*dele* would	
407	19	coefficient	coefficients	

Page	Line	Error.	Correction.
407	25	satisfy	satisfied
416	1	$\frac{x^2-y}{x^2+y}$	$\frac{x^2-y^2}{x^2+y^2}$
418	17	$Pp+Qq$	$Pp+Qq=R$
419	5	$dN=\phi(M)\,dM$	$dN=\phi'(M)\,dM$
—	12	$Q\,(dx)$	$Q\,dx)$
—	19	$-\frac{dz}{dx}$	$\frac{dz}{dx}$
420	3	a	b
421	21	to q	to $\phi\,(q)$
424	13	$\phi\,(x)$	$\phi\,(y)$
428	9	$q\frac{dM}{dx}$	$q\frac{dM}{dz}$
430	5	any one	either
434	29	$U=c$	$U=C$
453	25	consequently to	consequently it will be necessary to
456	1	$m\,\delta x$	$m'\,\delta x'$
—	6	$dx+m'dy'$	$dx'+m'dy'$
457	5	$(y-y'')^2$	$(y'-y'')^2$
458	7	curve	curve surface
460	15	equation	equations
461	10	$\frac{dx'}{u'ds'}\,dx'$	$\frac{dx'}{u'ds'}\,\delta x'$
466	16	this	their
—	24	x, in these expressions we	x in these expressions, we
467	24	x_{x+1}	u_{x+1}
468	27		*insert* $-u_x v_x$ *at end of the line*
471	10	,	*dele* ,
476	2	function $\Delta^n u_x$	function of $\Delta^n u_x$
478	6	$\Delta^n\,.\,o^m$	$\Delta^n\,o^m$
—	22	$u^m\,x$	u^m_x
479	9	$\overset{n}{\underset{1}{}}\Delta$	$\frac{n}{1}\Delta$
481	14	$(A+a)-$	$(A+a)+$

Page	Line	Error.	Correction.
483	13	$\Delta'\Delta\Delta'$	$\Delta'+\Delta\Delta'$
——	27		*insert* . &c. *at end of the line*
484	18	$+$ &c. $\Delta\Delta'+$	$+$ &c. $+\Delta\Delta'+$
486	1	Δ^n	$\Delta^n u_x$
494	3	$\Delta u_{x,\ y+n}$	$\Delta u_{x,\ y+1}$
496	10	,	*dele* , *after unchanged*
——	15	,	ditto ditto
497	22	$a\,\Sigma\,\phi\,(x)$	$a\,\Sigma f\,(x)$
501	17	$\frac{(2x+1)\,(2x+1)\,(2x+3)}{6}$	$\frac{(2x-1)\,(2x+1)\,(2x+3)}{6}$
502	21	$(2+3)$	$(x+3)$
504	3	u *in denominator*	u_x
——	13	$(x+1)\,x\,(x+1)$ *in denom.*	$(x-1)\,x\,(x+1)$
——	17	higher	lower
505	13	(f)	$f\,(x)$
508	10	$\Sigma^2\,(u_x,\ v_x)$	$\Sigma^2\,(u_x\,.\,v_x)$
512	20	e^{x}	e^{sx}
519	25	;	*dele* ;
522	21	$0={}_{x+n}-$	$0=u_{x+n}-$
528	13	figurative	figurate
536	28	δ_x	δ_y
538	1		*at end of line annex* $+$ ......
——	10	coefficients	coefficient
——	13	u_1+	$u_1 t+$
540	26	$\frac{\Delta f(t+h)}{dt}-\frac{\Delta f(t)}{dt}$	$\frac{df(t+h)}{dt}-\frac{df(t)}{dt}$
541	22	$\frac{d^n\,(uu'\ \&c.}{dt^n}$	$\frac{d^n\,(uu'\ \&c.)}{dt^n}$
543	1	$\Delta^1 u_{x+1},\ y$	$\Delta' u_{x+1},\ y$
——	5	, which	*dele* , which
550	13	$A'B'-u_1$	$A'B'=u_1$
——	19	$F\,(x_1\ u_x)$	$F\,(x,\ u_x)$
553	4	$\left(M z^{n-1}_{n-1}\right)$	$M z^{n-1}_{n-1}$
554	2	$U^1{}_0 z_0 z_1$	$U_0'\,z_0\,z_1$
555	5	$(z_1-z_n$ *in denominator*	(z_1-z_n)

Page	Line	Error.	Correction.
557	8	x	$z_,$
558	14	z^0	z_0
560	6	$e^{\frac{t}{2}+1}$ *in denominator*	$e^{\frac{t}{2}}+1$
—	8	$f\left(\frac{t}{2}\right)f-(t)$	$f\left(\frac{t}{2}\right)-f(t)$
—	9	$e^{\frac{t}{2}+1}$	$e^{\frac{t}{2}}+1$
563	11	x	∞
—	12	B^5	B_5
—	17	$+e^{\theta\sqrt{-1}}$ *in numerator*	$+e^{-\theta\sqrt{-1}}$
567	5	$\frac{1}{x^2}$	$\frac{1}{x^3}$
571	20	$\frac{a}{b}$ *in the exponent*	$\frac{a}{b}$
576	3	$\left(\frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}\right)$
—	16	$\frac{d\,u'}{h}$	$\frac{d\,u'}{d\,h}$
612	32	Infinitisimals	Infinitesimals
619	2	$\left(z^3\frac{\dot{u}}{\dot{y}}\right)^{\cdot\cdot}$	$\left(z^3\frac{\dot{u}}{\dot{y}}\right)^{\cdot\cdot}$
620	7	$\left(e\frac{d}{dx}-1\right)^n u_x$	$\left(e^{\frac{d}{dx}}-1\right)^n u_x$
634	5	$\frac{3\,d^2v}{dy^2}\cdot\frac{dy\,d^2y}{dx^3}\;\frac{dv}{dy}\cdot\frac{d^3y}{dx^3}$	$\frac{3\,d^2v}{dy^2}\cdot\frac{dy\,d^2y}{dx^3}+\frac{dv}{dy}\cdot\frac{d^3}{dx}$
640	23	$\frac{y^2}{z^3}$	$\frac{2y}{z^3}$
661	17	formula	formulæ
666	9	$u_1 =$	$u_1' =$
677	7	$1-\sqrt{(1e^{-2})}$ *in the num^r*	$1-\sqrt{(1-e^2)}$
631	12	$3^2 \cos 2x$	$3^2 \cos 3x$

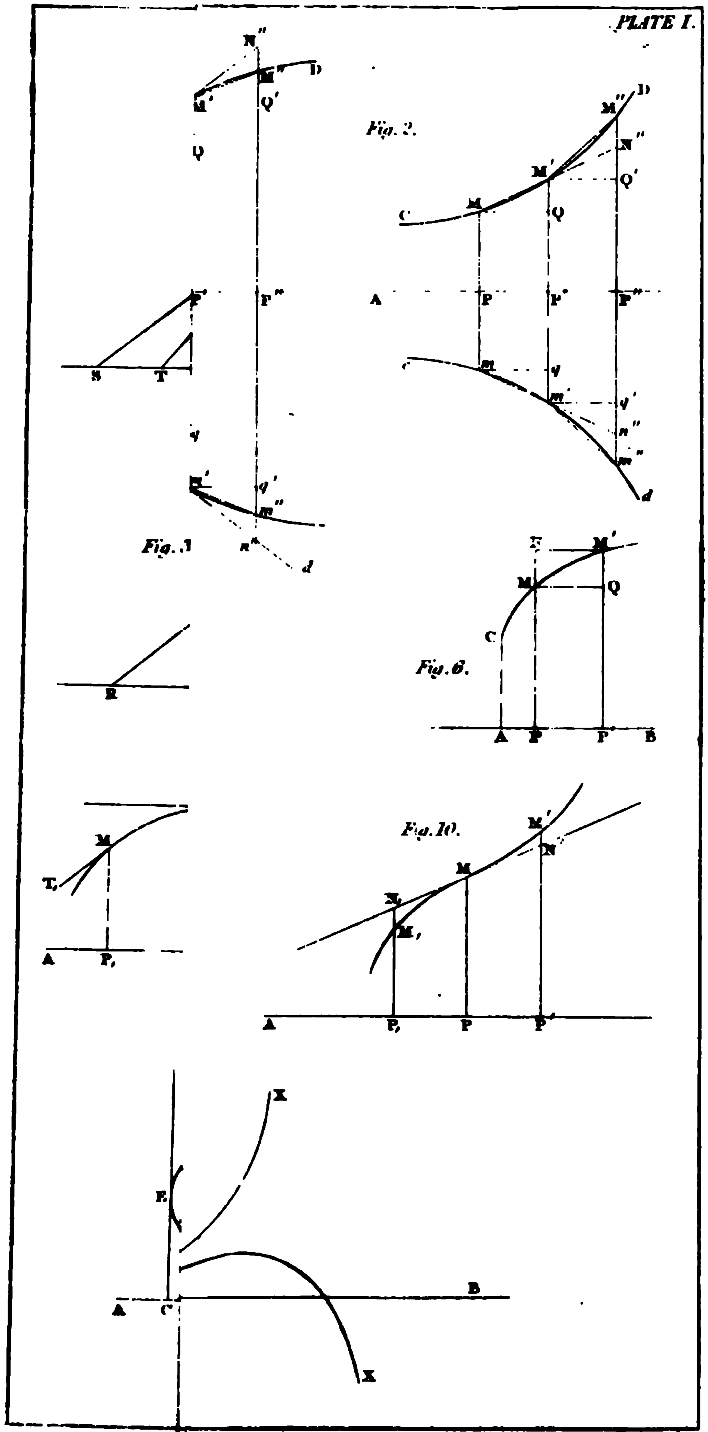


PLATE I.
Fig. 1
Fig. 2.
Fig. 6.
Fig. 10.
Lowry sculp.

Page	Line	Error.	Correction.
557	8	x	z_x
558	14	z^0	z_0
560	6	$e^{\frac{t}{2}+1}$ *in denominator*	$e^{\frac{t}{2}}+1$
—	8	$f\left(\frac{t}{2}\right)f-(t)$	$f\left(\frac{t}{2}\right)-f(t)$
—	9	$e^{\frac{t}{2}+1}$	$e^{\frac{t}{2}}+1$
563	11	x	∞
—	12	B^5	B_5
—	17	$+e^{\theta\sqrt{-1}}$ *in numerator*	$+e^{-\theta\sqrt{-1}}$
567	5	$\frac{1}{x^2}$	$\frac{1}{x^2}$
571	20	$\frac{a}{b}$ *in the exponent*	$\frac{a}{b}$
576	3	$\left(\frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}\right)$
—	16	$\frac{du'}{h}$	$\frac{du'}{dh}$
612	32	Infinitisimals	Infinitesimals
619	2	$\left(z^3 \dot{u} \atop \dot{y}\right)^{\cdot\cdot}$	$\left(z^3 \frac{\dot{u}}{\dot{y}}\right)^{\cdot\cdot}$
620	7	$\left(e\frac{d}{dx}-1\right)^n u_x$	$\left(e^{\frac{d}{dx}}-1\right)^n u_x$
634	5	$\frac{3d^2v}{dy^2}\cdot\frac{dy\,d^2y}{dx^3}\,\frac{dv}{dy}\cdot\frac{d^3y}{dx^3}$	$\frac{3d^2v}{dy^2}\cdot\frac{dy\,d^2y}{dx^3}+\frac{dv}{dy}\cdot\frac{d^3y}{dx^3}$
640	23	$\frac{y^2}{z^3}$	$\frac{2y}{z^3}$
661	17	formula	formulæ
666	9	$u_1=$	$u_1'=$
677	7	$1-\sqrt{(1e^{-2})}$ *in the num^r*	$1-\sqrt{(1-e^2)}$
631	12	$3^2 \cos 2x$ ———	$3^2 \cos 3x$

PLATE I.

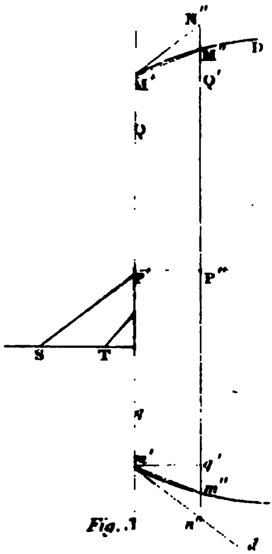


Fig. 1

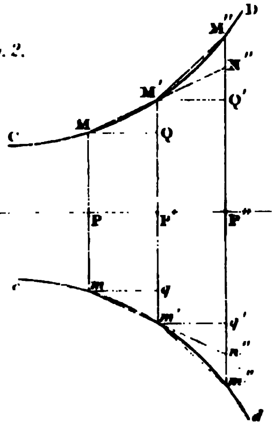
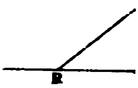


Fig. 2.



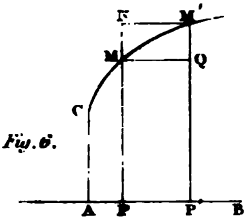
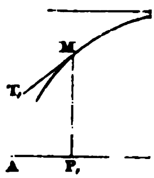


Fig. 6.



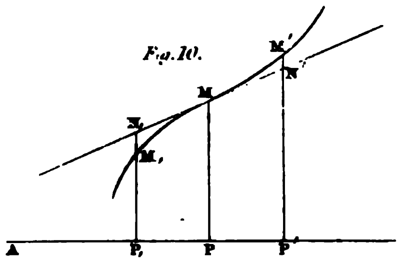
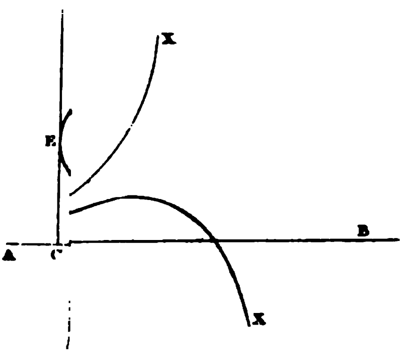


Fig. 10.



Lowry sculp.

PLATE II.

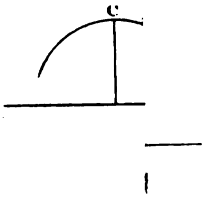


Fig. 18.

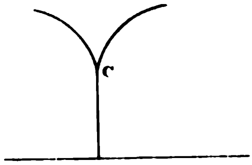


Fig. 19.

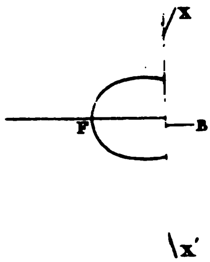


Fig. 22.

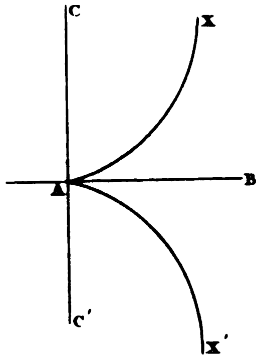


Fig. 25.

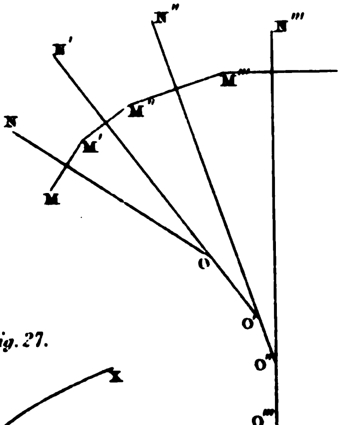


Fig. 23.

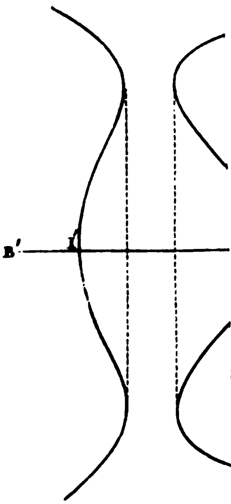
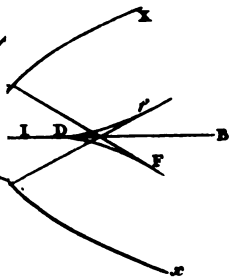


Fig. 27.



Lowry sculp.

PLATE III.

Fig. 29.

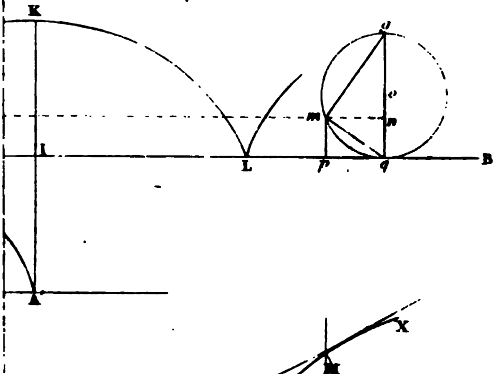


Fig. 32.

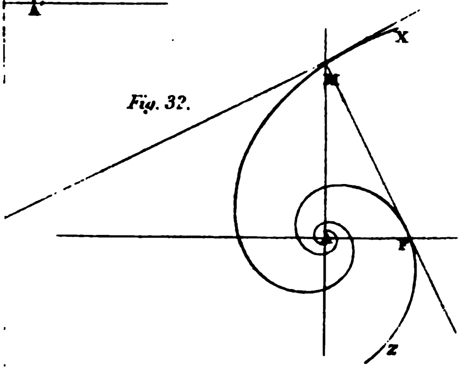
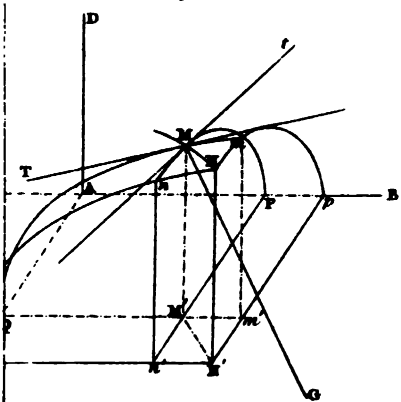


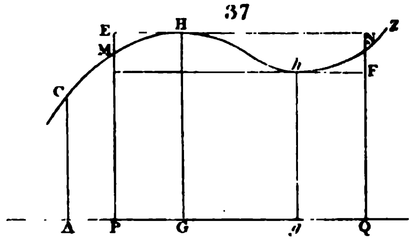
Fig. 34.



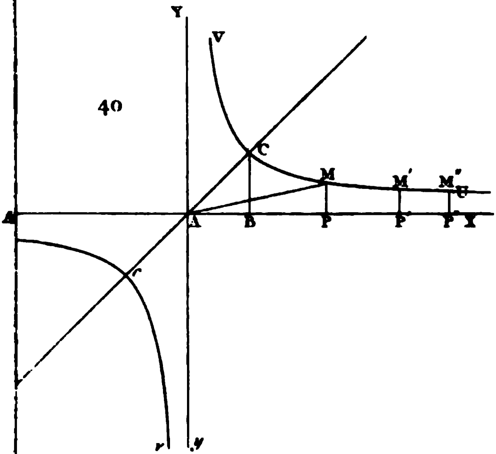
Lowry sculp.

PL. I

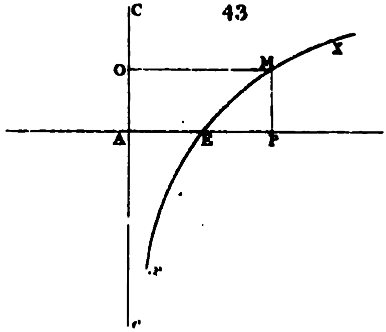
37



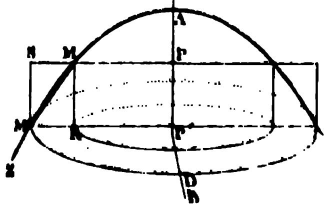
40



43

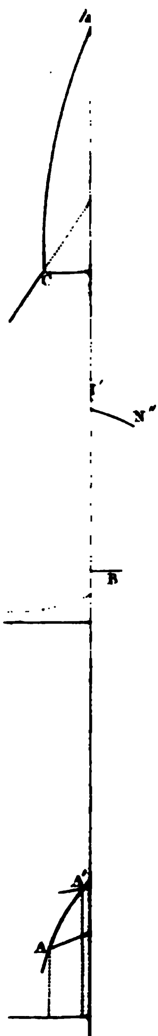


46

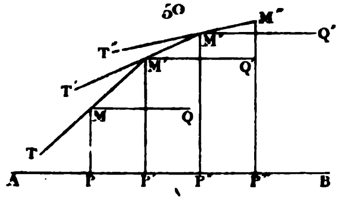


PL. II.

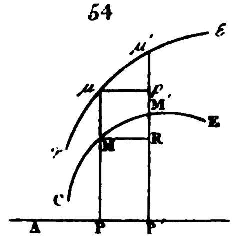
Fig.



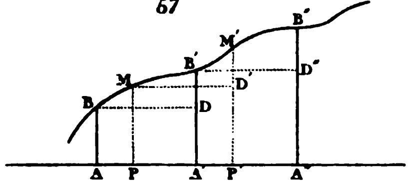
50



54



57



Lowry Sc.